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A SEQUENCE OF SHARP TRIGONOMETRIC INEQUALITIES

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Abstract. The purpose of this paper is to prove the following sequence of sharp trigonometric inequalities. Let $n \geq 5$ and $0 < x < \pi$. Then

$$\left(\cos \frac{x}{\sqrt{n-1}}\right)^{n-1} < \left(\frac{\sin \sqrt{5/n}\,x}{\sqrt{5/n}\,x}\right)^{\frac{3}{5}n} < \left(\cos \frac{x}{\sqrt{n}}\right)^{n}.$$

1. Sharp trigonometric inequalities. In [1, Proposition 1] and [2] the following inequalities were stated (without proof): Let $0 < x < \pi$. Then

$$\left(\cos \frac{x}{2}\right)^4 < \left(\frac{\sin x}{x}\right)^3 < \left(\cos \frac{x}{\sqrt{5}}\right)^5.$$

Although they may look deceptively simple, these inequalities are remarkably sharp. The graphs of the three functions almost coincide. The absolute errors in the first and the second inequalities are at most about 0.0091 and 0.0025, respectively.

As no proofs have been published so far, we will start by proving these inequalities:

Theorem 1.1. *Let $0 < x < \pi$. Then*

$$\left(\cos \frac{x}{2}\right)^4 < \left(\frac{\sin x}{x}\right)^3 < \left(\cos \frac{x}{\sqrt{5}}\right)^5. \tag{1.1}$$

Proof. The left-hand inequality in (1.1) is actually a substantial improvement of inequality (3.4.18) from Mitrinović [3],

$$\cos x < \left(\frac{\sin x}{x}\right)^3,$$

which holds for $0 < x < \pi/2$. It can, in fact, be easily derived from it, as is clear from

$$\left(\frac{\sin x}{x}\right)^3 = \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \cos \frac{x}{2}\right)^3 > \left(\cos \frac{x}{2}\right)^4.$$

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We will now prove the right-hand inequality. For $0 < y < \pi/2$ we have

$$\begin{aligned}
&\frac{d^7}{dy^7}(\cos y)^{5/3}\\
&\quad= \frac{5}{2187}(15625\cos^6 y + 1302\cos^4 y - 4200\cos^2 y + 7280)\frac{\sin y}{\cos^{16/3} y}\\
&\quad> \frac{5}{2187}(-4200 + 7280)\sin y = \frac{15400}{2187}\sin y > 0.
\end{aligned}$$

This implies that

$$\frac{d^7}{d\xi^7}\left(\cos\frac{\xi}{\sqrt{5}}\right)^{5/3} > 0$$

for all $0 < \xi < \pi$. Now we use Taylor's expansion with Lagrange's remainder term (see [4]),

$$\left(\cos\frac{x}{\sqrt{5}}\right)^{5/3} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{23}{162000}x^6 + \frac{1}{7!}x^7\frac{d^7}{d\xi^7}\left(\cos\frac{\xi}{\sqrt{5}}\right)^{5/3}$$

for some $\xi \in (0, \pi)$. Since also

$$-\frac{23}{162000}x^7 - \left(-\frac{1}{5040}x^7 + \frac{1}{362880}x^9\right) = \frac{x^7(512 - 25x^2)}{9072000} > 0$$

for all $0 < x < \pi$, it is clear that

$$\begin{aligned}
x\left(\cos\frac{x}{\sqrt{5}}\right)^{5/3} &> x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{23}{162000}x^7\\
&> x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 > \sin x
\end{aligned}$$

for $0 < x < \pi$, by using the well-known fact that Taylor sums for $\sin x$ are actually upper and lower bounds, alternately (see [4]). This proves the desired inequality. ∎

2. Generalization of the inequalities. Noticing that the left-hand and right-hand expressions in (1.1) are both of the form $\left(\cos\frac{x}{\sqrt{k}}\right)^k$, one may wonder whether these functions form an increasing sequence, and whether $\left(\frac{\sin x}{x}\right)^3$ could be generalized to a sequence of interlacing functions. It turns out that this is indeed possible. In this section we will prove the following generalization of (1.1).

THEOREM 2.1. *Let $n \geq 5$ and $0 < x < \pi$. Then*

$$\left(\cos\frac{x}{\sqrt{n-1}}\right)^{n-1} < \left(\frac{\sin\sqrt{5/n}\,x}{\sqrt{5/n}\,x}\right)^{\frac{3}{5}n} < \left(\cos\frac{x}{\sqrt{n}}\right)^n < e^{-\frac{1}{2}x^2}. \tag{2.1}$$

Proof. The second inequality in (2.1) follows easily from the right-hand inequality in Theorem 1 by changing the argument to $\sqrt{5/n}\,x$ and by taking the $n/5$th power of the resulting inequality.

The first inequality in (2.1) is more difficult to prove, however. For $0 < y < \pi/2$ we consider the function

$$g(\alpha) := \left(\frac{\sin \alpha y}{\alpha y}\right)^{3/\alpha^2}.$$

Its derivative is given by

$$g'(\alpha) = \left(\frac{\sin \alpha y}{\alpha y}\right)^{3/\alpha^2-1} \frac{3}{\alpha^3}\left(\cos \alpha y - \frac{\sin \alpha y}{\alpha y}\left(1 + 2\log\frac{\sin \alpha y}{\alpha y}\right)\right).$$

We will show that this derivative is negative. Let $h(t) := t(1 + 2\log t)$. By using Taylor's expansion with Lagrange's remainder term we find that

$$h(t) = 1 + 3(t-1) + (t-1)^2 + \frac{1}{3}(1-t)^3\frac{1}{\tau^3}$$

for $0 < t < 1$ and some $\tau(t) \in [t, 1]$. Therefore,

$$t(1 + 2\log t) \geq 1 + 3(t-1) + (t-1)^2 = t^2 + t - 1$$

for all $t \in [0, 1]$. From the left-hand inequality in (1.1) we see that

$$\cos s = 2\left(\cos\frac{s}{2}\right)^2 - 1 < 2\left(\frac{\sin s}{s}\right)^{3/2} - 1$$

for all $s \in (0, \pi)$. Therefore,

$$\begin{aligned}\frac{\sin s}{s}\left(1 + 2\log\frac{\sin s}{s}\right) - \cos s &\geq \left(\frac{\sin s}{s}\right)^2 + \frac{\sin s}{s} - 2\left(\frac{\sin s}{s}\right)^{3/2} \\ &= \frac{\sin s}{s}\left(\sqrt{\frac{\sin s}{s}} - 1\right)^2 > 0\end{aligned}$$

for all $s \in (0, \pi)$. It follows that g is strictly decreasing if $\alpha y \in (0, \pi)$. Now let $n \geq 5$, $\alpha = \sqrt{5(n-1)/n}$, $x \in (0, \pi)$, and $y = x/\sqrt{n-1}$. Then $\alpha \in [2, \sqrt{5})$, $y \in \big(0, \pi/2\big)$, and $\alpha y \in (0, \pi)$. Furthermore, by inequality (1.1), we see that

$$\left(\frac{\sin \alpha y}{\alpha y}\right)^{3/\alpha^2} = g(\alpha) \geq g(2) = \left(\frac{\sin 2y}{2y}\right)^{3/4} > \cos y.$$

This means that

$$\cos\frac{x}{\sqrt{n-1}} < \left(\frac{\sin\sqrt{5/n}x}{\sqrt{5/n}x}\right)^{\frac{3}{5}\frac{n}{n-1}}$$

for all $x \in (0, \pi)$, which proves the left-hand inequality in (2.1).

We have now proved the first two inequalities in (2.1), and from these we conclude that $n \mapsto \cos^n \frac{x}{\sqrt{n}}$ is a strictly increasing sequence of functions. As its limit is $e^{-\frac{1}{2}x^2}$, the rightmost inequality in (2.1) follows. ∎

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