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THE EXISTENCE OF c-COVERS OF LIE ALGEBRAS

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Abstract. The aim of this work is to obtain the structure of c-covers of c-capable Lie algebras. We also obtain some results on the existence of c-covers and, under some assumptions, we prove the absence of c-covers of Lie algebras.

1. Introduction. Interaction between Schur multipliers and other mathematical concepts has a long history. This basic notion was introduced by I. Schur [S] in 1904 to study projective representations of groups. In 1942, H. Hopf [H2] proved that $M(G) \cong (R \cap F')/[R, F]$, where $M(G)$ is the Schur multiplier of G and $G = F/R$ is a free presentation of G. He also proved that the Schur multiplier of G is independent of the free presentation of G. The first to generalize the Schur multiplier to any variety of groups was R. Baer [B1]. It is well known that his concept is useful in classifying groups into isologism classes. Now it is clear that, if $\mathcal{A}$ is the variety of abelian groups, then the Baer invariant of G with respect to $\mathcal{A}$ is the Schur multiplier $M(G)$, and if $\mathcal{N}_c$ is the variety of nilpotent groups of class at most $c \geq 1$, then the Baer invariant of G with respect to $\mathcal{N}_c$ is $\mathcal{N}_c M(G) = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R, F)$. Following J. Burns and G. Ellis' paper [BE], we shall call $\mathcal{N}_c M(G)$ the *c-nilpotent multiplier* of G and denote it by $M^{(c)}(G)$. It is easy to see that the 1-nilpotent multiplier is actually the Schur multiplier.

By a *Lie algebra* we mean a Lie k-algebra, where k is a field. The finite-dimensional Lie algebra analogous to the Schur multiplier was introduced in [B2, BMS2], and has been studied in various places: [BMS1, H1, M, Y]. Let L be a finite-dimensional Lie algebra. Its Schur multiplier, $M(L)$, can be defined as a second cohomology group, a quotient of a free Lie algebra, and as the second number of a maximal defining pair. The first two authors of [SEA] generalized the notion of the Schur multiplier to c-nilpotent multiplier as follows. Let L be a Lie algebra presented as a quotient $L = F/R$ of a free Lie algebra F and an ideal R. Then the *c-nilpotent multiplier* of L, $c \geq 1$, is

$$M^{(c)}(L) = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R, F),$$

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where $\gamma_{c+1}(F)$ is the $(c+1)$th term of the lower central series of F, $\gamma_1(R,F) = R$ and $\gamma_{c+1}(R,F) = [\gamma_c(R,F),F]$. This is analogous to the definition of the Baer invariant of a group with respect to the variety of nilpotent groups. The Lie algebra $M^{(1)}(L) = M(L)$ is the most studied Schur multiplier of L. It is readily verified that the Lie algebra $M^{(c)}(L)$ is abelian and independent of the choice of the free presentation of L (see [SEA]).

Notation. Let L be an arbitrary Lie algebra, and let L^n denote the nth term of the *lower central series* of L defined inductively by $L^1 = L$ and $L^{n+1} = [L^n, L]$ for $n \geq 1$. Let $Z_n(L)$ denote the nth term of the *upper central series* of L defined inductively by $Z_0(L) = \{0\}$ and requiring $Z_{n+1}(L)/Z_n(L)$ to be the center of $L/Z_n(L)$ for $n \geq 0$. An exact sequence $0 \to M \to K \to L \to 0$ $(*)$ of Lie algebras is a *c-central extension* of L if M is a c-central subalgebra of K, i.e. $\gamma_{c+1}(M,K) = 1$ or equivalently $M \subseteq Z_c(L)$. The c-central extension $(*)$ is said to be a *c-stem extension* of L whenever $M \subseteq \gamma_{c+1}(K)$. In addition, if M is isomorphic to $M^{(c)}(L)$, then the extension e is called a *c-stem cover* of L. In this case, K is said to be a *c-cover* of L.

Definition 1.1. A Lie algebra L is said to be *c-capable* if there exists a Lie algebra K such that $L \cong K/Z_c(K)$.

2. Main results. In this section, we present the structure of c-covers of c-capable Lie algebras and state conditions which guarantee the absence of c-covers of Lie algebras. Batten et al. [BMS1] showed the existence of covers for finite-dimensional Lie algebras. Salemkar et al. [SEA] proved that any c-perfect Lie algebra admits at least one c-cover (recall that a Lie algebra L is *c-perfect* if $L^{c+1} = L$). The following lemma, determines the structure of the c-cover of the Lie algebra L for which $M^{(c)}(L)$ is Hopfian.

Lemma 2.1 ([H1]). *Let L be a Lie algebra whose c-nilpotent multiplier has the Hopfian property, and let $0 \to R \to F \xrightarrow{\pi} L \to 0$ be a free presentation of L. Then the extension $0 \to M \to L^* \xrightarrow{\psi} L \to 0$ is a c-stem cover of L if and only if there exists an ideal S in F such that*

(i) $L^* \cong F/S$ *and* $M \cong R/S$;
(ii) $R/\gamma_{c+1}(R,F) = M^{(c)}(L) \oplus (S/\gamma_{c+1}(R,F))$.

The next corollary states the existence of c-covering algebras of a finite-dimensional Lie algebra.

Corollary 2.2. *Any finite-dimensional Lie algebra L has at least one c-covering algebra.*

Proof. Let $F/R \cong L$ be a free presentation of L, and $S/\gamma_{c+1}(R,F)$ be a complement of $M^{(c)}(L)$ in $R/\gamma_{c+1}(R,F)$ for a suitable ideal S in F. Then by Lemma 2.1, the Lie algebra F/S is a c-covering algebra of L. ∎

The following result indicates the existence of c-covers for c-capable Lie algebras.

THEOREM 2.3. *Let L be a c-capable Lie algebra. Then there exists a Lie algebra K such that*

(i) $Z_c(K) \subseteq \gamma_{c+1}(K)$;
(ii) $K/Z_c(K) \cong L$.

Proof. Since L is c-capable, there exists a Lie algebra T such that $L \cong T/Z_c(T)$. Let $0 \to S \to F \to T \to 0$ be a free presentation of T. There exists an ideal R of F such that $Z_c(T) \cong R/S$. Then $0 \to R \to F \to L \to 0$ is a free presentation of L. Let $E = H/\gamma_{c+1}(R,F)$ be a vector complement of $D = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R,F)$ in $B = R/\gamma_{c+1}(R,F)$. As $E \subseteq B \subseteq Z(C)$ ($C = \gamma_{c+1}(F)/\gamma_{c+1}(R,F)$), E is an ideal of C. Hence $B = D \oplus E$. Set $P = F/\gamma_{c+1}(R,F)$. Let $K = P/E$ and $M = B/E$. One may observe that $M \subseteq Z_c(K)$. We also have $Z_c(F/S) = R/S$, which yields $\gamma_{c+1}(R,F) \subseteq S$. Suppose $f + \gamma_{c+1}(R,F) + E \in Z_c(K)$. Then $[f, x_1, \ldots, x_c] + \gamma_{c+1}(R,F) \in E$ for all $x_i \in F$, $(1 \le i \le c)$. Now

$$[f, x_1, \ldots, x_c] + \gamma_{c+1}(R,F) \in \gamma_{c+1}(F)/\gamma_{c+1}(R,F) \cap H/\gamma_{c+1}(R,F) = 0.$$

Thus, $[f, x_1, \ldots, x_c] \in \gamma_{c+1}(R,F) \subseteq S$. Hence $f + S \in Z_c(F/S) = R/S$, and so $f \in R$. Therefore, $Z_c(K) = M$. Also,

$$Z_c(K) = B/E = (D \oplus E)/E \cong D = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R,F). \ \blacksquare$$

COROLLARY 2.4. *Any c-capable Lie algebra L has at least one c-covering algebra.*

One should note that, in general, Lie algebras might not have a c-cover. For example, abelian Lie algebras of dimension at least 2 admit no c-cover with $c \ge 2$ (Corollary 2.8). Now, we prove the absence of c-covers for finite-dimensional nilpotent Lie algebras.

THEOREM 2.5. *Let L be a nilpotent Lie algebra of class at most $c \ge 1$ with $M^{(n)}(L) \neq \langle 1 \rangle$ for some $n > c$. Then L does not have any n-cover.*

Proof. Towards a contradiction, suppose $0 \to M \to L^* \to L \to 0$ is an n-stem cover of the nilpotent Lie algebra L. Then $L^*/M \cong L$, $M \subseteq Z_n(L^*) \cap \gamma_{n+1}(L^*)$ and $M \cong M^{(n)}(L)$. Since L is nilpotent of class c, this implies that $\gamma_{c+1}(L) = 1$, and hence $\gamma_{c+1}(L^*) \subseteq M$. Using the fact that $M \subseteq Z_n(L^*) \cap \gamma_{n+1}(L^*)$ we deduce that $\gamma_{n+c+1}(L^*)$ is trivial. Thus we get $\gamma_{2c+1}(L^*) \subseteq \gamma_{n+c+1}(L^*) = 1$. Now, if $n \ge 2c$ then $\gamma_{n+1}(L^*) \subseteq \gamma_{2c+1}(L^*) = 1$, and if $n < 2c$ then

$$\langle 1 \rangle = \gamma_{2c+1}(L^*) = \gamma_{n+2c-n+1}(L^*) \supseteq \gamma_{c+2c-n+1}(L^*) = \gamma_{3c-n+1}(L^*).$$

Since $n > c$, we have $3c - n < 2c$. Now continuing the above procedure we can show that $\gamma_{3c-3c+n+1}(L^*) = \gamma_{n+1}(L^*) = \langle 1 \rangle$. Thus $M^{(n)}(L) \cong M \subseteq \gamma_{n+1}(L^*) = \langle 1 \rangle$, a contradiction. $\blacksquare$

THEOREM 2.6. *If L is a finite-dimensional nilpotent Lie algebra of dimension greater than 1 and class $c \geq 1$, then $M^{(n)}(L) \neq 0$ for all $n \geq c$.*

Proof. Let L be a nilpotent Lie algebra generated by $m > 1$ elements. Hence, $\dim(L/L^2) = m$. Let F be a free Lie algebra generated by $m > 1$ elements with $L \cong F/R$. Since L has class c, we have $\gamma_{c+2}(F) \subsetneq \gamma_{c+1}(F)$ and $M^{(n)}(L) \cong \gamma_{n+1}(F)/\gamma_{n+1}(R, F)$. Also

$$m = \dim(L/L^2) = \dim\left(\frac{F/R}{(F^2+R)/R}\right) = \dim(F/F^2+R) \leq \dim(F/F^2) = m.$$

Hence, $R \subseteq F^2$.

For contradiction, suppose $M^{(n)}(L) = 0$. Then

$$\begin{aligned}\gamma_{c+1}(F) \supseteq \gamma_{c+2}(F) \supseteq \gamma_{n+2}(F) &= \gamma_{n+1}(F^2, F)\\ &\supseteq \gamma_{n+1}(R, F) = \gamma_{n+1}(F) = \gamma_n(F^2, F) \supseteq \gamma_n(R, F)\\ &= \gamma_n(F) \supseteq \cdots \supseteq \gamma_{c+1}(R, F) = \gamma_{c+1}(F),\end{aligned}$$

which contradicts $\gamma_{c+2}(F) \subsetneq \gamma_{c+1}(F)$. ∎

COROLLARY 2.7. *Let L be a finite-dimensional nilpotent Lie algebra of dimension greater than 1 and class $c \geq 1$. Then L has no n-cover with $n > c$.*

COROLLARY 2.8. *Let L be an abelian Lie algebra of dimension greater than 1. Then L has no n-cover with $n \geq 2$.*

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