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NON-SEPARATING SUBCONTINUA OF PLANAR CONTINUA

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Abstract. We revisit an old question of Knaster by demonstrating that each non-degenerate plane hereditarily unicoherent continuum X contains a proper, non-degenerate subcontinuum which does not separate X.

1. Introduction. It is an important result of R. L. Moore [5] that each non-degenerate continuum X contains at least two points neither of which disconnects X. A *cut continuum* is a continuum which is disconnected by each of its proper, non-degenerate subcontinua. Knaster [2] constructed a dendroid (i.e. an arc connected, hereditarily unicoherent continuum) in euclidean 3-space which is a cut continuum. Nadler and Seldomridge [7] constructed a similar example in response to a question of Bellamy. Earlier, Roberts [8] constructed a plane cut continuum in answer to a question of Knaster [2]. Roberts' example contains simple closed curves, so it is not hereditarily unicoherent.

In this note we prove there is no hereditarily unicoherent, planar, cut continuum.

Both the Knaster and Roberts examples are Suslinian, i.e. they do not contain uncountable collections of pairwise disjoint non-degenerate subcontinua. We show that all metric cut continua are Suslinian.

2. Preliminaries. A *continuum* is a non-degenerate, compact, connected, metric space. If X is a continuum and S is a subcontinuum of X then S is said to be *non-separating* if $X - S$ is connected. Otherwise, S *separates* X. As a special case, if X is a continuum then $p \in X$ is a *cut point* or *separating point* of X if $X - \{p\}$ is not connected.

A continuum M is *irreducible* if there is a two-element subset of M that is a subset of no proper subcontinuum of M. A point p is a *point of irreducibility* of a continuum M if there exists a point $q \in M$ such that M is irreducible between p and q. A continuum is *hereditarily unicoherent* if the intersection

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of each pair of its subcontinua is connected. A point x of a plane continuum X is *accessible* from a component U of $\mathbb{R}^2 - X$ if there is an arc $A \subset U \cup \{x\}$ with $A \cap X = \{x\}$.

For other notions, the reader is referred to a standard reference such as Kuratowski [3] or Whyburn [9].

3. Main results. We will show that there is no hereditarily unicoherent cut continuum in the plane. We do so by a sequence of results. In Theorems 3.1, 3.2 and 3.3, X will denote a cut continuum.

THEOREM 3.1. *X is Suslinian.*

Proof. Assume that X contains an uncountable family $\mathcal{C}$ of pairwise disjoint non-degenerate subcontinua. Then $\mathcal{C}$ is a non-separated collection in the sense of [9, p. 45]. By [9, III.2.2], there is C in $\mathcal{C}$ which separates X into exactly two complementary components A_C and B_C. Then $C \cup A_C$ is a continuum and it does not separate X, a contradiction. ∎

It follows by [3, p. 212] that X contains no indecomposable continuum.

THEOREM 3.2. *If K is either a cut point of X or a proper non-degenerate subcontinuum of X then no component of $X - K$ is open in X. In particular, $X - K$ has infinitely many components.*

Proof. If a component C of $X - K$ were open in X then $X - C$ would be a proper non-degenerate subcontinuum of X with connected complement. ∎

THEOREM 3.3. *The set of points at which X is locally connected has empty interior.*

Proof. Suppose that X is locally connected at each point of a non-empty open subset U of X. Let K be a non-degenerate proper subcontinuum of X such that $K \subseteq U$, and let L be a component of $X - K$. Of course $L \cap U$ is open. Let W be an open subset of X such that $K \subseteq W \subseteq \mathrm{Cl}(W) \subseteq U$. Its boundary $\mathrm{Cl}(W) - W$ may be covered by finitely many connected sets not intersecting K. Therefore, there are only finitely many components of $X - K$ intersecting $X - \mathrm{Cl}(W)$. Hence, each point of $L - \mathrm{Cl}(W)$ has a neighborhood contained in L. It follows that L is open, a contradiction to Theorem 3.2. ∎

LEMMA 3.4. *Let X be a hereditarily unicoherent cut continuum in $\mathbb{R}^2$ such that $X \cap (\mathbb{R} \times \{0\}) = [0, 1] \times \{0\}$. Then there exists $b \in (1/2, 1)$ and irreducible continua $Z_i \subset X$ such that $\phi \neq Z_i \cap (\mathbb{R} \times \{0\}) \subset [0, 1/2] \times \{0\}$, $(b, 0) \in \lim Z_i$, and either $Z_i \subset \mathbb{R} \times [0, \infty)$ or $Z_i \subset \mathbb{R} \times (-\infty, 0]$. If infinitely many Z_i are in $\mathbb{R} \times [0, \infty)$ and A is a continuum in $X \cap (\mathbb{R} \times [0, \infty))$ such that $\phi \neq A \cap (\mathbb{R} \times \{0\}) \subset (1/2, b) \times \{0\}$ then $A \subset [0, 1] \times \{0\}$. Moreover, it is not possible that infinitely many Z_i lie in the closed upper half-plane, and that infinitely many others lie in the closed lower half-plane.*

Proof. Let $A_1 = [0, 1/2] \times \{0\}$ and $A_2 = (1/2, 1] \times \{0\}$. Let $D_1 = A_1 \cup \bigcup\{D : D \text{ is a component of } X - ([0, 1] \times \{0\}) \text{ such that } \mathrm{Cl}(D \cup A_1) \subset A_1 \cup D\}$. Let $D_2 = A_2 \cup \bigcup\{D : D \text{ is a component of } X - ([0, 1] \times \{0\}) \text{ such that } \mathrm{Cl}(D) \text{ meets } A_2\}$. Note that $X = D_1 \cup D_2$, a disjoint union, and that D_i is connected for each of $i = 1, 2$. Also, note that D_1 is not closed since X is a cut continuum and $X - D_1 = D_2$ is connected. Let $z \in \mathrm{Cl}(D_1) - D_1$. Let $y = (b, 0) \in A_2$ be such that the irreducible continuum yz from z to $A_1 \cup A_2$ contains y. Let $\{z_j : j = 1, 2, \ldots\}$ be a sequence in D_1 converging to z. Let Z_j be the irreducible continuum in X from z_j to A_1. We may suppose Z_i is in the closed upper half-plane. By compactness of the hyperspace of subcontinua of X we may assume by passing to a subsequence if necessary that $\lim Z_i = Z$ is a continuum from A_1 to z. By hereditary unicoherence, Z contains the irreducible continuum in X from A_1 to z. Hence, $[1/2, b] \times \{0\} \subset Z$.

Now suppose A is a continuum in $X \cap (\mathbb{R} \times [0, \infty))$ with $A \cap (\mathbb{R} \times \{0\})$ non-empty and contained in $(1/2, b) \times \{0\}$. By hereditary unicoherence A misses Z_i. Moreover, $z \notin A$. Let $z_i z$ be the line segment from z_i to z. For large i, A is contained in the topological hull of $Z_i \cup yz \cup ([0, 1] \times \{0\}) \cup z_i z = Y_i$. Since the limit of the boundaries of the Y_i is $Z \cup [0, 1] \times \{0\}$, which is in X, and no subcontinuum of X separates the plane, we have $A \subset Z$.

If A meets the open upper half-plane, let B be a non-degenerate subcontinuum of A contained in the open upper half-plane. Since X is a cut continuum, $X - B = Y_1 \cup Y_2$ where Y_1 and Y_2 are separated sets. Without loss of generality $[0, 1] \times \{0\} \subset Y_1$. Hence, $Z_i \subset Y_1$ for each i, and $Z - B \subset Y_1$. Now, $Y_2 \cup B \subset Y_2 \cup A$ and $Y_2 \cup A$ is a continuum contained in Z as in the preceding paragraph. So $Y_1 \supset Y_2 = \emptyset$. This is a contradiction, so $A \subset [0, 1] \times \{0\}$.

If there were infinitely many of the Z_i in the closed upper half-plane and infinitely many of them in the closed lower half-plane then the interval $\left[\frac{3}{8} + \frac{1}{4}b, \frac{1}{8} + \frac{3}{4}b\right] \times \{0\}$ would be a non-degenerate subcontinuum of X which does not separate X. ∎

Proposition 3.5. *A monotone non-degenerate continuous image of a cut continuum is a cut continuum.*

Proof. Let $f : X \to Y$ be a monotone continuous mapping of a cut continuum X onto a non-degenerate continuum Y. Let A be a non-degenerate proper subcontinuum of Y. Then $f^{-1}(A)$ is a non-degenerate proper subcontinuum of X. Since X is a cut continuum, $X - f^{-1}(A) = U \cup V$ with U and V non-empty disjoint open sets in X. Since f is monotone, it follows that $f(U)$ and $f(V)$ are disjoint. Since $f^{-1}(A) \cup V$ is closed in X, its image under f is compact and closed, and $f(U) = Y - f(f^{-1}(A) \cup V)$ is therefore open. ∎

LEMMA 3.6. *Let M be a Suslinian hereditarily unicoherent continuum in the closed lower half-plane such that $M \cap (\mathbb{R} \times \{0\}) = [0,1] \times \{0\}$. If no non-degenerate subset of $[1/2, b] \times \{0\}$, where $1/2 < b < 1$, is contained in a convergence continuum of M, then there exists a dense set of points of $[1/2, b] \times \{0\}$ each of which is accessible from ∞ in the lower half-plane.*

Proof. For $w \in M - ([0,1] \times \{0\})$ let K_w be the unique irreducible continuum from w to $[0,1] \times \{0\}$. Let $S = \{x \in [1/2, b] : (x, 0) \in K_w \text{ for some } w\}$. Let $w \in M - ([0,1] \times \{0\})$. Since M is Suslinian, it is hereditarily decomposable, so K_w is a continuum of type λ; see [3, p. 197]. There is a finest monotone decomposition of K_w into layers (often called tranches in the literature) which are continua. The end-layer E_w of K_w which does not contain w is a continuum of convergence, so by hypothesis it can meet $[1/2, b] \times \{0\}$ in at most one point. Hence, K_w meets $[1/2, b] \times \{0\}$ in at most a single point. Since M is Suslinian, S is at most countable. If $x \in [1/2, b) - S$ then by hereditary unicoherence, and by the fact that $[1/2, b] \times \{0\}$ contains no non-degenerate subset of a continuum of convergence, $M - \{(x, 0)\}$ is the union of the following two disjoint open sets: $([0, x) \times \{0\}) \cup \{K_w : K_w \cap ([0, x) \times \{0\}) \neq \emptyset\}$ and $((x, 1] \times \{0\}) \cup \{K_w : K_w \cap ((x, 1] \times \{0\}) \neq \emptyset\}$. By hereditary normality of the plane, a bounded closed set $L \subset \mathbb{R}^2 - (M - \{(x, 0)\})$ separates $M - \{(x, 0)\}$ in the plane. Since the plane is locally connected, we may assume by [3, 49.V.3] that L is an irreducible separator. Since the plane is unicoherent, L is connected. We can fatten L to a continuum K consisting of $(x, 0)$ together with a locally finite null family of closed disks each of which meets L but misses M. Since K is locally connected at all but one point, it is locally connected because a continuum cannot fail to be locally connected only on a 0-dimensional set. Hence, K contains a simple closed curve which separates $M - \{(x, 0)\}$. ∎

THEOREM 3.7. *If X is a hereditarily unicoherent continuum in the plane then X is not a cut continuum.*

Proof. Assume by way of contradiction that X is a cut continuum. Let a, b be points of X that are accessible from ∞ in $\mathbb{R}^2$. Without loss of generality, $a = (0, 0)$, $b = (1, 0)$, and $X \subseteq \mathbb{R}^2 - (((-\infty, 0) \times \{0\}) \cup ((1, \infty) \times \{0\}))$ (since all arcs in the 2-sphere are tame [3, 61.V.1]).

Let ab denote the unique continuum in X that is irreducible between a and b. As in Lemma 3.6, ab is a continuum of type λ. Let $\pi : \mathbb{R}^2 \to \pi(\mathbb{R}^2)$ be the quotient mapping that collapses non-degenerate layers in ab to points (see [3, 48.VII.3]). The above decomposition is upper semi-continuous into non-separating subcontinua of $\mathbb{R}^2$, so $\pi(\mathbb{R}^2)$ is homeomorphic to $\mathbb{R}^2$ by Moore's Plane Decomposition Theorem. By Proposition 3.5, $\pi(X)$ is a cut continuum. By abuse of notation we suppose $\pi(X) = X$. We also suppose that all of the conditions and notation of Lemma 3.4 hold, and that each Z_i is in the upper half-plane, so that $[0,1]\times\{0\}\subseteq X$ and $X\subseteq\mathbb{R}^2-(((-\infty,0)\times\{0\})\cup((1,\infty)\times\{0\}))$.

For each $w\in X\cap(\mathbb{R}\times(-\infty,0))$ let K_w be the irreducible continuum in X from w to $[0,1]\times\{0\}$. Suppose that for some such w, we have $L=K_w\cap([0,1]\times\{0\})\subset(1/2,b)\times\{0\}$. Then L is contained in an end-layer of K_w, so L is a connected set $[a,c]\times\{0\}$ by hereditary unicoherence of X. Just suppose $a\neq c$. Then there exist pairwise disjoint continua X_i in K_w such that $\lim X_i\supset[a,c]\times\{0\}$.

Let $C=\left[\frac{3}{4}a+\frac{1}{4}c,\frac{1}{4}a+\frac{3}{4}c\right]$. Then $(0,0)$ and $(b,0)$ are in the same component of $X-C$, so $[0,1]\times\{0\}-C$ is contained in one component of $X-C$, and no component of $X-C$ lies entirely in the open upper half-plane by Lemma 3.4. If A is a subcontinuum of X with $A\subset\mathbb{R}\times(-\infty,0]$, $\emptyset\neq A\cap(\mathbb{R}\times 0)\subset(a,c)\times\{0\}$, then $A\subset K_w$ and in fact $A\subset(a,c)\times\{0\}$ by the same proof as in Lemma 3.4. Thus C does not disconnect X. This contradiction implies $a=c$. A similar argument shows that, in fact, $K_w\cap([0,1]\times\{0\})$ does not contain a non-degenerate subinterval of $[1/2,b]\times\{0\}$. By the argument in Lemma 3.4, $[1/2,b]\times\{0\}$ contains no non-degenerate subset of a continuum of convergence of $X\cap(\mathbb{R}\times(-\infty,0])$.

Let $1/2<d<e<b$ be such that neither d nor e is in any K_w. This is possible because X is Suslinian and because each K_w meets $[1/2,b]\times\{0\}$ in at most one point. By Lemma 3.6 each of $(d,0)$ and $(e,0)$ is accessible from ∞ in the lower half-plane. Thus, $P=([d,e]\times\{0\})\cup\bigcup\{K_w:K_w\cap([d,e]\times\{0\})\neq\emptyset\}$ is a continuum which does not disconnect X since, clearly, the points $(0,0)$ and $(b,0)$ lie in the same component of $X-P$. This contradiction completes the proof of the theorem. ∎

COROLLARY 3.8. *No plane dendroid is a cut continuum.*

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