COLLOQUIUM MATHEMATICUM

VOL. 103 2005 NO. 2

REGULAR SETS AND CONDITIONAL DENSITY: AN EXTENSION OF BENFORD'S LAW

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Abstract. We give an extension of Benford's law (first digit problem) by using the concept of conditional density, introduced by Fuchs and Letta. The main tool is the notion of *regular* subset of integers.

1. Introduction. The authors of [3] introduced the notion of conditional density with respect to a subset $\mathbb{H}$ of $\mathbb{N}^*$ (here and in what follows, the symbol $\mathbb{N}^*$ denotes the set of strictly positive integers). This notion had been previously used, though not explicitly stated, in the paper [9], where the so-called *Benford's law* is discussed and extended. In terms of conditional density, Benford's law can be stated as follows. Let $\mathbb{H}$ be a subset of integers, and A_q the set of integers whose first digit is q. We say that $\mathbb{H}$ *obeys Benford's law* if the conditional logarithmic density of A_q, given $\mathbb{H}$, is equal to its (non-conditional) logarithmic density, i.e. to the number $\log\left(\frac{q+1}{q}\right)$ (see [9] and [11] for a historical discussion on this topic). It is a known result that the set $\mathbb{P}$ of prime numbers obeys Benford's law (see [12]). In [3] it is shown that, for a large class of subsets A of $\mathbb{N}^*$, to which A_q belongs, the upper and lower arithmetic and logarithmic densities coincide with the corresponding conditional densities with respect to the set $\mathbb{P}$ (this result has been generalized by the first named author in [6], in connection also with her previous works on the comparison of densities [4], [5]). Such results show that $\mathbb{P}$ satisfies an "extended" Benford's law, inasmuch as it can be stated not only for A_q, but also for other sets A.

On the other hand, in [9] it is shown that Benford's law holds for some sets $\mathbb{H}$ other than $\mathbb{P}$; hence [9] extends Benford's law in another sense.

In the present paper we propose an extension in both directions, i.e. we allow A to belong to a rather large class of sets and show that, in conditioning, $\mathbb{P}$ can be replaced by any *regular* set $\mathbb{H}$. This is the object of our main Theorems (2.10) and (2.12), which we state in Section 2.

2000 *Mathematics Subject Classification*: Primary 11K99; Secondary 11B05.

Key words and phrases: asymptotic density, logarithmic density, conditional density, regular set, regularly varying function, Benford's law, first digit problem.

The proof is split into two steps, described in Sections 3 and 6, which contain some results that are also relevant in themselves. Our theorems can be applied to a large variety of situations, as we show in Section 9.

The authors wish to thank Prof. K. Nagasaka for helpful suggestions.

2. Definitions and main results. Let $\mathbb{H}$ be an infinite subset of $\mathbb{N}^*$, which will be fixed throughout. The *counting function* H of $\mathbb{H}$ is defined for $x \geq 1$ as

$$H(x) = \text{card}(\mathbb{H} \cap [1, x]) = \sum_{\substack{n \in \mathbb{H} \\ n \leq x}} 1.$$

Recall the following

(2.1) Definition. A strictly positive function L (not necessarily monotone), defined on the half line (a, ∞), is said to be *slowly varying* as $x \to \infty$ if, for every $t > 0$, it satisfies the condition

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1$$

(see [1, p. 276]).

(2.2) Remark. Since the property of slow variation depends only on the behaviour at infinity, we can (and shall) assume that L is defined on $[1, \infty)$ at least.

Following [10, p. 86], we give another

(2.3) Definition. Let λ be a number, with $0 < \lambda \leq 1$. The set $\mathbb{H}$ is said to be *regular with exponent* λ if the function L defined as

$$L(x) = \frac{H(x)}{x^\lambda}, \qquad x \geq 1,$$

is slowly varying as $x \to \infty$ (i.e., according to the terminology in [1], H is a *regularly varying function with exponent* λ).

(2.4) Remark. If $\mathbb{H}$ is regular, its exponent of regularity is obviously unique.

(2.5) Remark. If $\mathbb{H}$ is regular with exponent λ, its counting function H satisfies the relation

$$\lim_{x \to \infty} \frac{H(tx)}{H(x)} = t^\lambda \tag{2.6}$$

for every $t > 0$.

Let now $\mathbb{H}$ be any subset of $\mathbb{N}^*$, neither finite nor cofinite. Then $\mathbb{H}$ can be uniquely represented in the form

$$\mathbb{H} = \mathbb{N}^* \cap \bigcup_{n \geq 1} [r_n, s_n[,$$

where $(r_n)_n$ and $(s_n)_n$ are two sequences of integers with $1 \le r_n < s_n < r_{n+1}$ for every n. Every set of the form $\mathbb{N}^* \cap [r_n, s_n[$ (resp. $\mathbb{N}^* \cap [s_n, r_{n+1}[$) is called a *connected component* (resp. a *gap*) of $\mathbb{H}$.

Intuitively, the regularity assumption on $\mathbb{H}$ means that $\mathbb{H}$ cannot have "too large" both connected components and gaps. Indeed, consider for instance the set

$$\mathbb{H} = \mathbb{N}^* \cap \bigcup_{n \ge 1} [3^{2n}, 3^{2n+1}[.$$

It is not difficult to see that the counting function of this set is given by

$$H(x) = \begin{cases} \lfloor x \rfloor + \frac{1}{4} \cdot 3^{2k} + k + \frac{3}{4} & \text{if } 3^{2k} \le x \le 3^{2k+1}, \\ \frac{13}{4} \cdot 3^{2k} + k + \frac{3}{4} & \text{if } 3^{2k+1} < x < 3^{2k+2}. \end{cases}$$

In particular, if $x_k = 3^{2k+1}$ $(k \in \mathbb{N})$ and t is fixed with $1 < t < 3$, we have $tx_k = t \cdot 3^{2k+1} < 3^{2k+2}$, so that

$$\frac{H(tx_k)}{H(x_k)} = 1,$$

thus (2.6) does not hold, since $t > 1$.

(2.7) Remark. The regularity assumption on $\mathbb{H}$ does not imply that $\mathbb{H}$ has an arithmetic density. In order to see this, we shall build in the Appendix a bounded slowly varying function M which has no limit as $x \to \infty$, and a set $\mathbb{H}$ such that

$$L(x) = \frac{H(x)}{x} \sim M(x)$$

as $x \to \infty$.

Let now $(\mu(n))$ be a sequence of non-negative real numbers such that $\sum_{n \in \mathbb{N}^*} \mu(n) = \infty$; consider the measure μ (concentrated on $\mathbb{N}^*$) defined by the formula

$$\mu = \sum_{n \in \mathbb{N}^*} \mu(n) \varepsilon_n,$$

where ε_n denotes the measure of mass 1 concentrated at the integer n; in particular, if $\mathbb{H}$ is a subset of $\mathbb{N}^*$ and $\mu(n) = 0$ for $n \notin \mathbb{H}$, we obtain a measure concentrated on $\mathbb{H}$.

Put $F_\mu(1) = 0$ and, for every $n \ge 2$,

$$F_\mu(n) = \mu([1, n[) = \sum_{k=1}^{n-1} \mu(k).$$

Let f be a non-negative bounded function, defined on $\mathbb{N}^*$. The *lower μ-asymptotic density* of f (or simply *lower μ-density*), denoted by $\underline{\delta}_\mu(f)$, is defined as

$$\underline{\delta}_\mu(f) = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n-1}\mu(k)f(k)}{F_\mu(n)}.$$

The *upper μ-density* of f, denoted by $\overline{\delta}_\mu(f)$, is defined analogously.

When $\mu(n) = 1$ (resp. $\mu(n) = 1/n$) for all n, the corresponding density will be called the *arithmetic* (resp. *logarithmic*) density, and will be denoted by d (resp. ∂) (i.e. we shall not use the generic symbol δ_μ).

If A is a subset of $\mathbb{N}^*$, we denote by $\underline{\delta}_\mu(A)$ the lower μ-density of 1_A; it is immediate that

$$\underline{\delta}_\mu(A) = \liminf_{n\to\infty} \frac{\mu(A\cap[1,n[)}{F_\mu(n)};$$

the upper μ-density of 1_A, which will be denoted by $\overline{\delta}_\mu(A)$, satisfies

$$\overline{\delta}_\mu(A) = \limsup_{n\to\infty} \frac{\mu(A\cap[1,n[)}{F_\mu(n)}.$$

It is intended that $\underline{d}$ and $\overline{d}$ (resp. $\underline{\partial}$ and $\overline{\partial}$) are the symbols for the lower and upper arithmetic (resp. logarithmic) densities.

(2.8) REMARK. In the particular case of a measure μ concentrated on a subset $\mathbb{H}$ of $\mathbb{N}^*$, the associated (lower and upper) densities are called *conditional densities* in [3].

Let A be a subset of $\mathbb{N}^*$, neither finite nor cofinite, so that it can be uniquely represented as

$$A = \mathbb{N}^* \cap \bigcup_{n\geq 1} [p_n, q_n[, \tag{2.9}$$

where $(p_n)_n$ and $(q_n)_n$ are two sequences of integers, with $1 \leq p_n < q_n < p_{n+1}$. We are now ready to state our main results.

(2.10) THEOREM. *Let $\mathbb{H}$ be a regular set with exponent λ, and A a subset of $\mathbb{N}^*$ of the form* (2.9)*. Assume moreover that the sequences* (p_n) *and* (q_n) *of* (2.9) *satisfy the relation*

$$q_n \sim \sigma p_n \tag{2.11}$$

(*as $n \to \infty$*) *for a suitable number $\sigma > 1$. Let β be a fixed real number, with $0 \leq \beta < \lambda$, and consider the measure defined as*

$$\nu = \sum_{n\in\mathbb{H}} \frac{1}{n^\beta}\,\varepsilon_n.$$

Then the lower (*resp. upper*) *ν-density of A* (*i.e. $\underline{\delta}_\nu(A)$* (*resp. $\overline{\delta}_\nu(A)$*)) *is equal to the lower* (*resp. upper*) *arithmetic density of A. In other words,*

$$\underline{\delta}_\nu(A) = \underline{d}(A), \qquad \overline{\delta}_\nu(A) = \overline{d}(A).$$

(2.12) THEOREM. *Let $\mathbb{H}$ be a regular set with exponent λ and assume that the counting function H of $\mathbb{H}$ is of the form*

$$H(x) = x^\lambda K(\log x), \tag{2.13}$$

where K is slowly varying and such that there exists a non-increasing function M with

$$K(x) \sim M(x) \quad \text{as } x \to \infty.$$

Consider the measure defined as

$$\nu = \sum_{n \in \mathbb{H}} \frac{1}{n^\lambda} \varepsilon_n.$$

Let A be a subset of $\mathbb{N}^*$ satisfying (2.11) *and assume that A has logarithmic density $\partial(A)$ (resp. ν-density $\delta_\nu(A)$). Then A has ν-density $\delta_\nu(A)$ (resp. logarithmic density $\partial(A)$) and*

$$\partial(A) = \delta_\nu(A).$$

(2.14) REMARK.

(i) Let A_q be the subset of integers whose first digit is q. Then

$$A_q = \mathbb{N}^* \cap \bigcup_{n \geq 1} [q \cdot 10^n, (q+1) \cdot 10^n[,$$

hence A_q safisfies (2.11) with $\sigma = 1 + 1/q$.

(ii) For A_q we have $\partial(A_q) = \log(1+1/q)$; hence the above Theorem (2.12) extends Theorem (4.1) of [9] to "any" regular set $\mathbb{H}$ (in Theorem (4.1) of [9] only the set $\mathbb{P}$ of primes is considered; from our result it follows that any regular set obeys Benford's law (in the sense of the ν-density)).

In order to apply the above result to practical situations, we give in Proposition (6.8) a condition which ensures that a slowly varying function $x \mapsto L(x)$ can be put into the form $x \mapsto K(\log x)$ (with K slowly varying).

The proofs of the above Theorems (2.10) and (2.12) result by combining part 1 (Sect. 3) and part 2 (Sect. 6) below.

The rest of the paper is organized as follows: Sections 3 and 6 give the statements of part 1 and part 2 respectively; Section 5 contains the proofs for part 1, Section 8 contains the proofs for part 2. Some preliminary results are given in Sections 4 and 7. In Section 9 we present some applications, while in the Appendix we construct a counterexample for Remark (2.7).

3. Part 1: results on conditional densities. In connection with conditional densities we are going to prove the following results:

(3.1) THEOREM. *Let $\mathbb{H}$ be an infinite regular subset of $\mathbb{N}^*$, with counting function H and regularity exponent λ. Put as usual $L(x) = H(x)x^{-\lambda}$. Let*

β be a fixed real number, with $0 \le \beta < \lambda$, and consider the two measures defined as

$$\mu = \sum_{n\in\mathbb{N}^*} \frac{H(n)}{n^{\beta+1}}\,\varepsilon_n = \sum_{n\in\mathbb{N}^*} \frac{L(n)}{n^{1-(\lambda-\beta)}}\,\varepsilon_n, \qquad \nu = \sum_{n\in\mathbb{H}} \frac{1}{n^{\beta}}\,\varepsilon_n.$$

Then:

(i) $\displaystyle\int_1^\infty \frac{L(t)}{t^{1-(\lambda-\beta)}}\,dt = \infty$.

(ii) $\displaystyle\sum_{n\in\mathbb{N}^*} \frac{L(n)}{n^{1-(\lambda-\beta)}} = \infty$.

(iii) $\displaystyle\sum_{n\in\mathbb{H}} \frac{1}{n^{\beta}} = \infty$.

(iv) *Moreover, let A be a subset of $\mathbb{N}^*$ of the form* (2.9)*, and assume that the sequences (p_n) and (q_n) of* (2.9) *satisfy* (2.11)*. Then*

$$\underline{\delta}_\mu(A) = \underline{\delta}_\nu(A), \qquad \bar{\delta}_\mu(A) = \bar{\delta}_\nu(A).$$

(3.2) Remark.

(i) For $\beta = 0$, the above theorem says that the statement holds for any regular set $\mathbb{H}$ (i.e. for any $\lambda \le 1$).

(ii) The first application in [6] is a particular case of Theorem (3.1), obtained for $\beta = 0$, $\lambda = 1$.

Theorem (3.1), though rather general, does not say anything for $\beta = \lambda$. It turns out that in order to manage this case, more restrictive assumptions are needed. We have in fact the following result:

(3.3) Theorem. *Let $\mathbb{H}$ be as in Theorem* (3.1)*. Assume in addition that there exists a positive decreasing function M, defined on $[1,\infty)$, such that*

$$M(x) \sim L(x) \quad \textit{as } x \to \infty, \tag{3.4}$$

$$\int_1^\infty \frac{M(t)}{t}\,dt = \infty. \tag{3.5}$$

Then statements (i)–(iv) *of Theorem* (3.1) *hold true for $\beta = \lambda$.*

(3.6) Remark. The second application in [6] is obtained from Theorem (3.3) for $\beta = \lambda = 1$, $M(t) = 1/\log t$.

In Section 4 we give some preliminary results; Section 5 contains the proofs of both Theorems (3.1) and (3.3).

4. Preliminary results. In connection with μ-densities, the following result (see [2, Th. 8.2]) holds:

(4.1) PROPOSITION. *Let A be a subset of $\mathbb{N}^*$, neither finite nor cofinite, of the form* (2.9). *Then*

$$\underline{\delta}_\mu(A) = \liminf_{n\to\infty} \frac{\sum_{k=1}^n (F_\mu(q_k) - F_\mu(p_k))}{F_\mu(p_{n+1})}, \tag{4.2}$$

$$\overline{\delta}_\mu(A) = \limsup_{n\to\infty} \frac{\sum_{k=1}^n (F_\mu(q_k) - F_\mu(p_k))}{F_\mu(q_n)}. \tag{4.3}$$

Formulas (4.2) and (4.3) easily yield the following comparison result (see [6, Th. 1.1]), which we shall use subsequently:

(4.4) THEOREM. *Let μ, ν be two measures on $\mathbb{N}^*$, both having infinite total mass. Let A be a subset of $\mathbb{N}^*$ of the form* (2.9); *assume that*

(a) $F_\mu(p_n) \sim \alpha F_\nu(p_n)$,
(b) $F_\mu(q_n) - F_\mu(p_n) \sim \alpha(F_\nu(q_n) - F_\nu(p_n))$ *as* $n \to \infty$

for a constant $\alpha > 0$. Then

$$\underline{\delta}_\mu(A) = \underline{\delta}_\nu(A), \qquad \overline{\delta}_\mu(A) = \overline{\delta}_\nu(A).$$

We now give some lemmas concerning slowly varying functions. The first one is proved in [1, p. 282].

(4.5) LEMMA. *A function L varies slowly as $x \to \infty$ iff it can be put into the form*

$$L(x) = \psi(x) \exp\Big(\int_1^x \frac{\phi(t)}{t}\, dt\Big),$$

where

$$\lim_{x\to\infty} \psi(x) = c \quad \textit{with } 0 < c < \infty, \qquad \lim_{x\to\infty} \phi(x) = 0. \tag{4.6}$$

Lemma (4.5) yields easily

(4.7) LEMMA. *Let L be a slowly varying function. For every fixed $\delta > 0$, there exists x_0 such that*

$$x^{-\delta} < L(x) < x^{\delta} \quad \textit{for } x > x_0.$$

The next lemma relates the behaviour of $L(x)$ to the behaviour of the truncated moments $\int_1^x t^p L(t)\, dt$.

(4.8) LEMMA. *Let L be a slowly varying function. Then*:

(i) *for every $p \ge -1$ we have*

$$\lim_{x\to\infty} \frac{x^{p+1} L(x)}{\int_1^x t^p L(t)\, dt} = p + 1;$$

(ii) *for every* $p > -1$ *we have*

$$\lim_{x\to\infty} \frac{x^{p+1}L(x)}{\sum_{k=1}^{\lfloor x\rfloor} k^p L(k)} = p+1;$$

hence

(iii) *for every* $p > -1$ *we have*

$$\int_1^x t^p L(t)\,dt \sim \sum_{k=1}^{\lfloor x\rfloor} k^p L(k).$$

Proof. Part (i) is proved in [1, Th. 1, p. 281, (b)]. We prove part (ii). Fix ε, $0 < \varepsilon < c$. By Lemma (4.5), we can find an integer n_0 such that, for $k > n_0$, $t > n_0$, we have both

$$c - \varepsilon \le \psi(k) \le c + \varepsilon, \qquad -\varepsilon \le \phi(t) \le \varepsilon,$$

where ψ, ϕ, c are as in Lemma (4.5). For $\lfloor x\rfloor \ge n_0 + 1$ we can write

$$\frac{\sum_{k=1}^{\lfloor x\rfloor} k^p L(k)}{x^{p+1}L(x)} = \frac{\sum_{k=1}^{n_0} k^p L(k)}{x^{p+1}L(x)} + \frac{\sum_{k=n_0+1}^{\lfloor x\rfloor} k^p L(k)}{x^{p+1}L(x)}. \tag{4.9}$$

Fix δ with $0 < \delta < p+1$. By Lemma (4.7) for x large enough we have

$$x^{p+1}L(x) > x^{p+1-\delta};$$

hence the first term of the sum in (4.9) tends to 0 as $x \to \infty$.

By Lemma (4.5) the second term can be written as

$$\frac{\sum_{k=n_0+1}^{\lfloor x\rfloor} k^p (\psi(k)/\psi(x)) \exp(\int_x^k (\phi(t)/t)\,dt)}{x^{p+1}}.$$

The fraction $\psi(k)/\psi(x)$ is between $(c-\varepsilon)/(c+\varepsilon)$ and $(c+\varepsilon)/(c-\varepsilon)$. As to the remaining term, since $\lim_{x\to\infty}\phi(x) = 0$, for every $\varepsilon > 0$ and x large enough we have

$$\frac{\sum_{k=n_0+1}^{\lfloor x\rfloor} k^{p+\varepsilon}}{x^{p+1+\varepsilon}} \le \frac{\sum_{k=n_0+1}^{\lfloor x\rfloor} k^p \exp(\int_x^k (\phi(t)/t)\,dt)}{x^{p+1}} \le \frac{\sum_{k=n_0+1}^{\lfloor x\rfloor} k^{p-\varepsilon}}{x^{p+1-\varepsilon}}.$$

Choose $\varepsilon < p+1$. Then, by the equivalence

$$\sum_{k=n_0+1}^{\lfloor x\rfloor} k^r \sim \frac{x^{r+1}}{r+1}$$

(as $x \to \infty$, $r > -1$), the left-hand side of the above inequality tends to $(1+p+\varepsilon)^{-1}$, while the right-hand side tends to $(1+p-\varepsilon)^{-1}$, proving the statement since ε is arbitrary. ∎

(4.10) Lemma. *Let L be a slowly varying function, a, b two fixed numbers with $0 < a \le b$, and E the set defined as*

$$E = \{(t,x) \in \mathbb{R}^2 : t > 0,\ x > 0,\ a \le t/x \le b\}.$$

Then

$$\lim_{\substack{t,x\to\infty \\ (t,x)\in E}} \frac{L(t)}{L(x)} = 1.$$

Proof. Without loss of generality, we can assume that $a = 1$, so that $t \ge x$. By Lemma (4.5) we can write

$$\frac{L(t)}{L(x)} = \frac{\psi(t)}{\psi(x)} \exp\biggl(\int_x^t \frac{\phi(u)}{u}\,du\biggr). \tag{4.11}$$

Fix $\varepsilon > 0$ and let r_0 be large enough in order that the following relations hold for $t \ge u \ge x > r_0$:

$$c-\varepsilon \le \psi(t) \le c+\varepsilon, \qquad c-\varepsilon \le \psi(x) \le c+\varepsilon, \qquad -\varepsilon \le \phi(u) \le \varepsilon.$$

By (4.11) we get (for $(t,x) \in E$, $t \ge x > r_0$)

$$\begin{aligned}\frac{c-\varepsilon}{c+\varepsilon}\,\frac{1}{b^\varepsilon} \le \frac{c-\varepsilon}{c+\varepsilon}\exp\biggl(-\varepsilon\int_x^t \frac{1}{u}\,du\biggr) &\le \frac{L(t)}{L(x)} \\ &\le \frac{c+\varepsilon}{c-\varepsilon}\exp\biggl(\varepsilon\int_x^t \frac{1}{u}\,du\biggr) \le \frac{c+\varepsilon}{c-\varepsilon}\,b^\varepsilon.\end{aligned}$$

Hence we can conclude the proof by going to the limit in t and x, since ε is arbitrary. ■

(4.12) Lemma. *Let L be a slowly varying function, $m \ne 1$ a fixed number. Then, for every $p > -1$:*

(i)
$$\lim_{\substack{x,y\to\infty \\ y/x\to m}} \frac{x^{p+1}L(x)}{\int_x^y t^p L(t)\,dt} = \frac{p+1}{m^{p+1}-1};$$

(ii)
$$\lim_{\substack{x,y\to\infty \\ y/x\to m}} \frac{x^{p+1}L(x)}{\sum_{k=\lfloor x\rfloor}^{y} k^p L(k)} = \frac{p+1}{m^{p+1}-1};$$

hence

(iii)
$$\lim_{\substack{x,y\to\infty \\ y/x\to m}} \frac{\int_x^y t^p L(t)\,dt}{\sum_{k=\lfloor x\rfloor}^{y} k^p L(k)} = 1.$$

Proof. Without loss of generality we can assume $m > 1$. The ratio y/x is ultimately bounded from above by a constant C, so that for $x \le t \le y$ we get

$$1 \le \frac{t}{x} \le \frac{y}{x} \le C;$$

hence (i) follows by applying Lemmas (4.8) and (4.10).

The same technique as in (i) and the relation

$$\lim_{\substack{x,y\to\infty \\ y/x\to m}} \frac{\int_x^y t^p\,dt}{\sum_{k=\lfloor x\rfloor}^{y} k^p} = 1$$

give statement (ii). ■

We state the last lemma (whose proof is similar to the previous ones):

(4.13) Lemma. *Let L be slowly varying, $m \neq 1$. Then*

$$\lim_{\substack{x,y\to\infty \\ y/x\to m}} \frac{\int_x^y (L(t)/t)\,dt}{L(x)} = \lim_{\substack{x,y\to\infty \\ y/x\to m}} \frac{\int_x^y (L(t)/t)\,dt}{L(y)} = \log m.$$

5. Proofs of Theorems (3.1) and (3.3). We begin with the proof of Theorem (3.1). For every $n \in \mathbb{N}^*$ put

$$F_\mu(n) = \mu([1,n[) = \sum_{k=1}^{n-1} \frac{L(k)}{k^{1-(\lambda-\beta)}}, \qquad F_\nu(n) = \nu([1,n[) = \sum_{\substack{k=1 \\ k\in\mathbb{H}}}^{n-1} \frac{1}{k^\beta}.$$

It is easy to see that

$$\int_1^\infty \frac{L(t)}{t^{1-(\lambda-\beta)}}\,dt = \infty \tag{5.1}$$

(simply apply Lemma (4.7) with $\delta < \lambda - \beta$).

The relation (5.1) also yields

$$\sum_{n\in\mathbb{N}^*} \frac{L(n)}{n^{1-(\lambda-\beta)}} = \infty,$$

by Lemma (4.8)(iii).

Statement (iii) of Theorem (3.1) will follow if we prove that

$$F_\nu(n) \sim \lambda F_\mu(n) \tag{5.2}$$

as $n \to \infty$. By the Abel summation formula, we have

$$F_\nu(n) = \sum_{k=1}^{n-1} \frac{H(k)-H(k-1)}{k^\beta} = \frac{H(n-1)}{(n-1)^\beta} + \beta \int_1^{n-1} \frac{H(t)}{t^{\beta+1}}\,dt$$

$$= (n-1)^{\lambda-\beta} L(n-1) + \beta \int_1^{n-1} \frac{L(t)}{t^{1-(\lambda-\beta)}}\,dt$$

$$\sim (\lambda-\beta)\int_1^{n-1}\frac{L(t)}{t^{1-(\lambda-\beta)}}\,dt+\beta\int_1^{n-1}\frac{L(t)}{t^{1-(\lambda-\beta)}}\,dt$$
$$=\lambda\int_1^{n-1}\frac{L(t)}{t^{1-(\lambda-\beta)}}\,dt\sim\lambda F_\mu(n),$$

where the first equivalence follows from Lemma (4.8)(i) and the second one from (4.8)(iii), with $p=\lambda-\beta-1>-1$.

Let now A be a subset of $\mathbb{N}^*$ satisfying (2.11). Since (5.2) yields $F_\nu(p_n)\sim\lambda F_\mu(p_n)$, the last statement of Theorem (3.1) will follow from Theorem (4.4) if we prove that

$$F_\nu(q_n)-F_\nu(p_n)\sim\lambda(F_\mu(q_n)-F_\mu(p_n)). \tag{5.3}$$

Again by integration by parts, we have

$$\begin{aligned}&F_\nu(q_n)-F_\nu(p_n)\\&=(q_n-1)^{\lambda-\beta}L(q_n-1)-(p_n-1)^{\lambda-\beta}L(p_n-1)+\beta\int_{p_n-1}^{q_n-1}\frac{L(t)}{t^{1-(\lambda-\beta)}}\,dt.\end{aligned}\tag{5.4}$$

From Lemma (4.12)(iii) (with $p=\lambda-\beta-1$) we get

$$\int_{p_n-1}^{q_n-1}\frac{L(t)}{t^{1-(\lambda-\beta)}}\,dt\sim\sum_{k=p_n}^{q_n-1}\frac{L(k)}{k^{1-(\lambda-\beta)}}.\tag{5.5}$$

Moreover

$$\begin{aligned}&(q_n-1)^{\lambda-\beta}L(q_n-1)-(p_n-1)^{\lambda-\beta}L(p_n-1)\\&\qquad\sim(\sigma^{\lambda-\beta}-1)p_n^{\lambda-\beta}L(p_n)\sim(\lambda-\beta)\sum_{k=p_n}^{q_n-1}\frac{L(k)}{k^{1-(\lambda-\beta)}},\end{aligned}\tag{5.6}$$

where the first equivalence follows from Lemma (4.10) and the second one from Lemma (4.12)(ii) (with $m=\sigma$).

Relations (5.4)–(5.6) now yield (5.3) easily, and this concludes the proof of Theorem (3.1). ∎

We now pass to the proof of Theorem (3.3). Relation (3.4) easily yields

$$\int_1^x\frac{M(t)}{t}\,dt\sim\int_1^x\frac{L(t)}{t}\,dt,\tag{5.7}$$

hence, by (3.5), we get

$$\int_1^\infty\frac{L(t)}{t}\,dt=\infty.$$

Put again

$$F_\mu(n) = \mu([1,n[) = \sum_{k=1}^{n-1} \frac{L(k)}{k},$$

$$F_\nu(n) = \nu([1,n[) = \sum_{\substack{k=1 \\ k\in\mathbb{H}}}^{n-1} \frac{1}{k^\lambda} = L(n-1) + \lambda \int_1^{n-1} \frac{L(t)}{t}\, dt$$

(integration by parts). Since L is slowly varying, Lemma (4.8)(i) (with $p=-1$) yields

$$\lim_{n\to\infty} \frac{L(n-1)}{\int_1^{n-1} \frac{L(t)}{t}\, dt} = 0. \tag{5.8}$$

Since M is decreasing, by using assumptions (3.4), (3.5) and Cesàro's theorem we get

$$\int_1^{n-1} \frac{M(t)}{t}\, dt \sim \sum_{k=1}^{n-1} \frac{M(k)}{k} \sim \sum_{k=1}^{n-1} \frac{L(k)}{k}. \tag{5.9}$$

Relations (5.7)–(5.9) allow us to conclude that

$$F_\nu(n) \sim \lambda \int_1^{n-1} \frac{L(t)}{t}\, dt \sim \lambda F_\mu(n). \tag{5.10}$$

The above relation yields the first two statements of Theorem (3.3). We now pass to the last one.

By arguing as in the proof of (3.1), by (5.10) it will be enough to prove that, for $A = \mathbb{N}^* \cap \bigcup_n [p_n, q_n[$ satisfying (2.11), we have

$$F_\nu(q_n) - F_\nu(p_n) \sim \lambda(F_\mu(q_n) - F_\mu(p_n)). \tag{5.11}$$

From the equivalence $L \sim M$ we easily get

$$F_\mu(q_n) - F_\mu(p_n) \sim \sum_{k=p_n}^{q_n-1} \frac{M(k)}{k} \tag{5.12}$$

as $n \to \infty$. Now

$$F_\nu(q_n) - F_\nu(p_n) = L(q_n-1) - L(p_n-1) + \lambda \int_{p_n-1}^{q_n-1} \frac{L(t)}{t}\, dt.$$

By (5.12), the equivalence (5.11) will be proved if we show that

$$\int_{p_n-1}^{q_n-1} \frac{L(t)}{t}\, dt \sim \int_{p_n-1}^{q_n-1} \frac{M(t)}{t}\, dt \sim \sum_{k=p_n}^{q_n-1} \frac{M(k)}{k} \tag{5.13}$$

and

$$\lim_{n\to\infty} \frac{L(q_n-1)-L(p_n-1)}{\int_{p_n-1}^{q_n-1} \frac{M(t)}{t}\,dt} = 0. \tag{5.14}$$

Since M is decreasing, for $x \geq 1$ we have

$$M(2\lfloor x\rfloor) \leq M(x) \leq M(\lfloor x\rfloor).$$

Since M is slowly varying, we see from the above relation that $M(x) \sim M(\lfloor x\rfloor)$ as $x \to \infty$, hence

$$\int\limits_{p_n-1}^{q_n-1} \frac{M(t)}{t} dt \sim \int\limits_{p_n-1}^{q_n-1} \frac{M(\lfloor t\rfloor)}{\lfloor t\rfloor} dt = \sum_{k=p_n-1}^{q_n-1} \frac{M(k)}{k} \sim \sum_{k=p_n}^{q_n-1} \frac{M(k)}{k},$$

where the last equivalence holds true since $M(n)/n \to 0$ as $n \to \infty$. This gives the second relation in (5.13). The first one is again easily implied by the equivalence $L \sim M$.

We now pass to the proof of (5.14). Fix $\varepsilon > 0$. For n large enough we have

$$\begin{aligned} M(q_n-1) - M(p_n-1) - 2\varepsilon M(q_n-1) &\leq L(q_n-1) - L(p_n-1) \\ &\leq M(q_n-1) - M(p_n-1) + 2\varepsilon M(p_n-1). \end{aligned}$$

The equality (5.14) now follows from Lemma (4.13), since ε is arbitrary. ∎

6. Part 2: a theorem of comparison. Preliminaries and main result. We begin by giving a definition. Let μ, ν be two measures on $\mathbb{N}^*$, and consider the associated asymptotic densities.

(6.1) Definition. We shall say that the ν-density is an *extension* of the μ-density if, for every positive bounded function f, the relation

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mu(k) f(k)}{\mu([1,n])} = l$$

yields the analogous relation for ν:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \nu(k) f(k)}{\nu([1,n])} = l.$$

We shall say that the μ-density and the ν-density are *equivalent* if the converse also holds.

In [5] and [2, pp. 268–271] the following theorem is proved:

(6.2) Theorem. *Let μ be a positive measure on $\mathbb{N}^*$, having infinite total mass. For $n \in \mathbb{N}^*$ put*

$$G(n) = \mu([1,n]).$$

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $h(x) \uparrow \infty$ as $x \to \infty$; denote by ν the positive measure on $\mathbb{N}^*$ defined by

$$\nu([1,n]) = h(G(n)), \qquad n \in \mathbb{N}^*.$$

Assume that, for every increasing sequence $(x_n)_n$ of positive numbers such that $x_n \sim G(n)$ (with $x_{n+1} = x_n \Leftrightarrow G(n+1) = G(n)$), one has

$$h(x_n) \sim h(G(n)), \qquad \frac{h(x_{n+1}) - h(x_n)}{x_{n+1} - x_n} \sim \frac{h(G(n+1)) - h(G(n))}{G(n+1) - G(n)}.$$

Then the ν-density is an extension of the μ-density.

The above theorem has the following obvious

(6.3) Corollary. *Let μ and h be as in Theorems* (6.2) *and* (6.3). *Assume in addition that, for every increasing sequence $(y_n)_n$ of positive numbers such that $y_n \sim h(G(n))$ (with $y_{n+1} = y_n \Leftrightarrow h(G(n+1)) = h(G(n))$), one has*

$$h^{-1}(y_n) \sim G(n), \qquad \frac{h^{-1}(y_{n+1}) - h^{-1}(y_n)}{y_{n+1} - y_n} \sim \frac{G(n+1) - G(n)}{h(G(n+1)) - h(G(n))}.$$

Then the μ-density and the ν-density are equivalent.

A particular case of the above situation is obtained by taking a measure μ such that $G(n+1) \sim G(n)$ as $n \to \infty$ (where, as usual, we define $G(n) = \mu([1,n])$) and, for $p > -1$ fixed, $h(x) = x^{p+1}$. Then the measure ν is given by

$$\nu(n) = G^{p+1}(n+1) - G^{p+1}(n) \sim (p+1)(G(n+1) - G(n))G^p(n). \tag{6.4}$$

By a known result on densities (see for instance [2, Th. 3.2, p. 258], we get the following

(6.5) Corollary. *Let μ and G be as above, and, for $p > -1$ fixed, let ν be defined by*

$$\nu(n) = (G(n+1) - G(n))G^p(n).$$

Then the μ-density and the ν-density are equivalent.

In this section we are concerned with the following extension of the above corollary:

(6.6) Theorem. *Let μ and G be as above. Assume moreover that there exists a sequence $(r_n)_n$ of integers such that*

$$\lim_{n\to\infty} \frac{G(r_n)}{G(n)} = 0. \tag{6.7}$$

Let M be a regularly varying function with exponent λ, and put

$$B(n) = M(G(n)), \qquad n \in \mathbb{N}^*.$$

For $p > -1-\lambda$ fixed, consider the measure ν defined by

$$\nu(n) = (G(n+1) - G(n))G^p(n)B(n).$$

Then:

(i) *ν has infinite total mass, i.e.*

$$\sum_n (G(n+1) - G(n))G^p(n)B(n) = \infty;$$

(ii) *the μ-density and the ν-density are equivalent.*

In the particular case of the logarithmic density (i.e. $G(n) \sim \log n$) we have the following result (which enables us to use the above theorem in practical situations):

(6.8) Proposition. *Let L be slowly varying, and let ϕ be the function of Lemma* (4.5). *Assume that ϕ is positive and*

$$\lim_{t\to\infty} \phi(t)\log t = \lambda, \qquad \lambda \in \mathbb{R}^+.$$

Then there exists M regularly varying with exponent λ such that

$$L(n) = M(\log n).$$

The proof is an easy consequence of Lemma (4.5) and is omitted.

Despite its being evident, we stress the following particular case, since it concerns logarithmic density:

(6.9) Corollary. *Let L be slowly varying, and let ϕ be the function of Lemma* (4.5). *Assume that ϕ is positive and*

$$\lim_{t\to\infty} \phi(t)\log t = \lambda, \qquad \lambda \in \mathbb{R}^+.$$

Then, for every $p > -1-\lambda$, the density defined by

$$\nu(n) = \frac{1}{n}(\log n)^p L(n)$$

is equivalent to the logarithmic density.

7. Preliminary results. We begin by stating and proving some additional results concerning slowly varying functions.

Recall the characterization of slowly varying functions given in Lemma (4.5). By using that lemma, it is easy to prove that

(7.1) Lemma. *Let L be a slowly varying function defined on* $[1,\infty)$, *and $p > -1$ a fixed number. Then*

$$n^{p+1}L(n) \sim \sum_{k=1}^{n} k^p L(k) \sim \Big(\sum_{k=1}^{n} k^p\Big)L(n).$$

The proof of Lemma (7.1) is quite similar to that of Lemma (4.8)(ii).

Lemma (4.5) also yields

(7.2) Lemma. *Assume* $G(x) = x$, *and let* L, p *be as in Lemma* (7.1); *assume that* $(r_n)_n$ *is a sequence of integers such that* (6.7) *holds* (*for* $G(x) = x$). *Then*:

(i) $\displaystyle\lim_{n\to\infty} \frac{\sum_{k=1}^{r_n} k^p L(k)}{\sum_{k=1}^{n} k^p L(k)} = 0;$

(ii) *there exists* $c > 0$ *such that*, *for every* $\varepsilon > 0$ *and* n *large enough* (*depending on* ε), *we have*

$$\frac{c-\varepsilon}{c+\varepsilon}\left(\frac{k}{n}\right)^{\varepsilon} \le \frac{L(k)}{L(n)} \le \frac{c+\varepsilon}{c-\varepsilon}\left(\frac{n}{k}\right)^{\varepsilon}$$

for every k *with* $r_n \le k \le n$.

Proof. (i) Fix ε with $0 < \varepsilon < p+1$. By Lemma (4.5), there exists n_0 such that, for $n_0 \le k \le r_n$, we have

$$\frac{L(k)}{L(n)} \le 2\exp\left(\varepsilon\int_k^n \frac{1}{u}\,du\right) = 2\left(\frac{n}{k}\right)^{\varepsilon};$$

hence, by Lemma (4.5) we get

$$\begin{aligned} 0 &\le \frac{\sum_{k=n_0}^{r_n} k^p L(k)}{\sum_{k=1}^{n} k^p L(k)} \sim \frac{\sum_{k=n_0}^{r_n} k^p L(k)/L(n)}{\sum_{k=1}^{n} k^p} \\ &\le 2\,\frac{\sum_{k=n_0}^{r_n} k^{p-\varepsilon}}{(\sum_{k=1}^{n} k^p)n^{-\varepsilon}} \sim \text{const}\cdot\left(\frac{r_n}{n}\right)^{p+1-\varepsilon} \to 0. \end{aligned}$$

Therefore (i) will follow if we prove that for $p > -1$ we have

$$\lim_{n\to\infty}\sum_{k=1}^{n} k^p L(k) = \infty. \tag{7.3}$$

Now, by a well known formula (see [8]), the Dirichlet series $\sum_n n^p L(n)$ has abscissa of convergence given by the formula

$$\limsup_{n\to\infty}\frac{\log(\sum_{k=1}^{n} L(k))}{\log n}$$

and, by Lemmas (4.5) and (4.7), we have

$$\frac{\log(\sum_{k=1}^{n} L(k))}{\log n} \sim \frac{\log nL(n)}{\log n} = 1 + \frac{\log L(n)}{\log n} \sim 1.$$

(ii) follows again from the characterization of slowly varying functions (Lemma (4.5)).

8. Proof of Theorem (6.6). We prove Theorem (6.6) for the case $G(x) = x$ (i.e. $\mu(n) = 1$ for each $n \in \mathbb{N}^*$) and $\lambda = 0$. (The case of a generic

G needs only the relation $G(n+1) \sim G(n)$ and the existence of a sequence $(r_n)_n$ satisfying (6.7)).

(i) has already been proved (see the proof of (3.1)).

(ii) Let f be a bounded positive function defined on $\mathbb{N}^*$. Without loss of generality we can assume that $0 \le f \le 1$. Let $(r_n)_n$ be a sequence of integers such that (6.7) holds. Then we have

$$\frac{\sum_{k=1}^{n} k^p L(k) f(k)}{\sum_{k=1}^{n} k^p L(k)} = \frac{\sum_{k=1}^{r_n} k^p L(k) f(k)}{\sum_{k=1}^{n} k^p L(k)} + \frac{\sum_{k=r_n+1}^{n} k^p L(k) f(k)}{\sum_{k=1}^{n} k^p L(k)}.$$

The first term on the right hand side is positive and bounded by

$$\frac{\sum_{k=1}^{r_n} k^p L(k)}{\sum_{k=1}^{n} k^p L(k)},$$

which goes to 0 as $n \to \infty$ by Lemma (7.2)(i). As to the second term, by Lemma (7.1) it is equivalent to

$$\frac{\sum_{k=r_n+1}^{n} k^p (L(k)/L(n)) f(k)}{\sum_{k=1}^{n} k^p} = A_n,$$

and, by Lemma (7.2)(ii), for every ε with $0 < \varepsilon < p+1$ and sufficiently large n, we have

$$\text{(8.1)} \qquad \frac{c-\varepsilon}{c+\varepsilon} \cdot \frac{\sum_{k=r_n+1}^{n} f(k) k^{p+\varepsilon}}{(\sum_{k=1}^{n} k^p) n^{\varepsilon}} \le A_n \le \frac{c+\varepsilon}{c-\varepsilon} \cdot \frac{\sum_{k=r_n+1}^{n} f(k) k^{p-\varepsilon}}{(\sum_{k=1}^{n} k^p) n^{-\varepsilon}}.$$

By the relation

$$\Big(\sum_{k=1}^{n} k^p\Big) n^{\beta} \sim \sum_{k=1}^{n} k^{p+\beta}$$

(valid for every β with $p + \beta > -1$) and by Lemma (7.2)(i) (applied for $L \equiv 1$), the theorem is proved (go to the limit as $n \to \infty$ in (8.1) and conclude by the arbitrariness of ε). ∎

9. Applications

(9.1) Example. Let $r \ge 1$ be a fixed integer and

$$\mathbb{H} = \{n^r : n \in \mathbb{N}^*\}.$$

It is easy to see that $H(x) = \lfloor x^{1/r} \rfloor$, hence

$$L(x) = \frac{H(x)}{x^{1/r}} \sim 1.$$

This means that $\mathbb{H}$ is regular with exponent $\lambda = 1/r$, so the conclusion of Theorem (2.10) holds.

Observe that in this case also Theorem (2.12) applies, since $L(x) \sim 1$ (a non-increasing function).

(9.2) Example. Let $\mathbb{H}$ be the set of all powers. Once again both Theorems (2.10) and (2.12) apply because of the following

(9.3) Lemma. $H(x) \sim \sqrt{x}$.

Proof. We have

$$\mathbb{H} = \bigcup_{k \geq 2} \{1^k, 2^k, 3^k, \ldots\}.$$

First, for all $x \geq 1$, we have $H(x) \geq \lfloor \sqrt{x} \rfloor$ because $\mathbb{H}$ contains the squares. Let us prove that, for all $x \geq 4$,

$$H(x) \leq \sqrt{x} + \sqrt[3]{x}\, \frac{\log x}{\log 2}. \tag{9.4}$$

Let 2^{k_0} be the greatest power of 2 not exceeding x: $2^{k_0} \leq x < 2^{k_0} + 1$. This yields

$$2 \leq k_0 = \left\lfloor \frac{\log x}{\log 2} \right\rfloor \leq \frac{\log x}{\log 2}.$$

For each k, $2 \leq k \leq k_0$, there are $\lfloor \sqrt[k]{x} \rfloor$ kth powers that are less than or equal to x. Consequently,

$$H(x) \leq \lfloor \sqrt{x} \rfloor + \lfloor \sqrt[3]{x} \rfloor + \lfloor \sqrt[4]{x} \rfloor + \cdots + \lfloor \sqrt[k_0]{x} \rfloor \leq \sqrt{x} + (k_0 - 2)\sqrt[3]{x},$$

and (9.4) follows. ■

(9.5) Remark. Observe that the sets in Example (9.1) with $r > 1$ and in Example (9.2) have zero arithmetic density.

(9.6) Example. Let $r > 1$ be a fixed integer and

$$\mathbb{H} = \{rn : n \in \mathbb{N}^*\}.$$

It is easy to see that, for $h = 0, 1, \ldots, r-1$, we have

$$H(x) = \frac{\lfloor x - h \rfloor}{r}, \qquad rn + h \leq x < rn + h + 1.$$

Hence

$$\frac{H(x)}{x} \sim \frac{1}{r},$$

and both Theorems (2.10) and (2.12) apply.

(9.7) Remark. Observe that in Example (9.6), $\mathbb{H}$ has arithmetic density equal to $1/r$ (in particular, strictly positive).

Appendix. We are going to construct the function H and the set $\mathbb{H}$ of Remark (2.7). We start by building the function M announced in (2.7) by means of the characterization (4.5). Take $\psi \equiv 1$ and define $\phi : [1, \infty[\to \mathbb{R}^+$ as

$$\phi(y) = \frac{(-1)^{n+1}}{2n+1} \quad \text{for } 2^{n^2} \leq y < 2^{(n+1)^2}.$$

We observe that $\lim_{y\to\infty}\phi(y)=0$. Moreover

$$\int_1^{2^{n^2}} \frac{\phi(y)}{y}\,dy = \sum_{k=0}^{n-1} \int_{2^{k^2}}^{2^{(k+1)^2}} \frac{\phi(y)}{y}\,dy = \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{2k+1} \int_{2^{k^2}}^{2^{(k+1)^2}} \frac{1}{y}\,dy$$
$$= \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{2k+1} \log 2^{(k+1)^2-k^2} = \log 2 \sum_{k=0}^{n-1} (-1)^{k+1};$$

hence, for $2^{n^2} \le x < 2^{(n+1)^2}$, we have

$$M(x) = \exp\left(\int_1^x \frac{\phi(y)}{y}\,dy\right)$$
$$= \exp\left(\log 2\left(\sum_{k=0}^{n-1}(-1)^{k+1} + (-1)^{n+1}\frac{\log_2 x - n^2}{2n+1}\right)\right).$$

It is now easy to prove that M is bounded by 1 but has no limit as $x\to\infty$. In fact, we have

$$M(2^{n^2}) = \begin{cases} 1/2 & \text{for } n = 2r+1, \\ 1 & \text{for } n = 2r. \end{cases}$$

Put now

$$\widetilde{H}(x) = \int_1^x M(t)\,dt \ (\sim xM(x)).$$

By [7, Lemma 4, p. 182] it is possible to construct a set $\mathbb{H}$ with counting function H such that $H(x)\sim\widetilde{H}(x)$. Hence

$$L(x) \equiv \frac{H(x)}{x} \ (\sim M(x))$$

has no limit.

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Received 23 February 2004 (4431)