C O L L O Q U I U M M A T H E M A T I C U M

VOL. 103 2005 NO. 2

CHARACTERIZATION OF LOCAL DIMENSION FUNCTIONS OF SUBSETS OF $\mathbb{R}^d$

BY

L. OLSEN (St. Andrews)

Abstract. For a subset $E \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, the local Hausdorff dimension function of E at x is defined by

$$\dim_{\mathsf{H},\mathsf{loc}}(x,E) = \lim_{r \searrow 0} \dim_{\mathsf{H}}(E \cap B(x,r))$$

where $\dim_{\mathsf{H}}$ denotes the Hausdorff dimension. We give a complete characterization of the set of functions that are local Hausdorff dimension functions. In fact, we prove a significantly more general result, namely, we give a complete characterization of those functions that are local dimension functions of an arbitrary regular dimension index.

1. Introduction and statement of results. For a subset $E \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we define the local Hausdorff dimension function of E at x by

$$\dim_{\mathsf{H},\mathsf{loc}}(x,E) = \lim_{r \searrow 0} \dim_{\mathsf{H}}(E \cap B(x,r))$$

where $\dim_{\mathsf{H}}$ denotes the Hausdorff dimension. The reader is referred to [Fa2] for the definition of the Hausdorff dimension. The local Hausdorff dimension function of a set has recently found several applications in fractal geometry and information theory (cf. [JS, Ru]). In [Ol] we proved that any continuous function is the local Hausdorff dimension function of some set, i.e. if $f : \mathbb{R}^d \to [0,d]$ is continuous, then there exists a set $E \subseteq \mathbb{R}^d$ such that $f(x) = \dim_{\mathsf{H},\mathsf{loc}}(x,E)$ for all $x \in \mathbb{R}^d$. In [Ol] we also showed that there are discontinuous functions which are local Hausdorff dimension functions, and discontinuous functions which are not. This suggests the following natural problem:

> Find a characterization of those functions that are local Hausdorff dimension functions.

In Theorem 1 we will give a complete characterization of such functions. In fact, we will address a significantly more general problem, namely:

2000 *Mathematics Subject Classification*: Primary 28A80.

Key words and phrases: Hausdorff dimension, packing dimension, dimension index, local Hausdorff dimension, local packing dimension.

> Find a characterization of those functions that are local dimension functions of an arbitrary regular dimension index.

In Theorem 4 we provide a complete solution to this problem.

We need to introduce the notion of punctured upper semicontinuity. Recall that the upper limit of a function $f : \mathbb{R}^d \to \mathbb{R}$ as y tends to x is defined by

$$\limsup_{y \to x} f(y) = \inf_{r>0} \sup_{|x-y|<r} f(y).$$

The *punctured upper limit* of f as y tends to x is defined by

$$\limsup_{y \to x}{}_{\mathsf{p}} f(y) = \inf_{r>0} \sup_{0<|x-y|<r} f(y).$$

Also, recall that a function f is called upper semicontinuous at a point x if $\limsup_{y\to x} f(y) \le f(x)$. However, since clearly $f(x) \le \limsup_{y\to x} f(y)$, we see that f is upper semicontinuous at x if

$$\limsup_{y \to x} f(y) = f(x).$$

In analogy with this result, we define punctured upper semicontinuity as follows.

Definition. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called *punctured upper semicontinuous at x* if

$$\limsup_{y \to x}{}_{\mathsf{p}} f(y) = f(x). \tag{1.1}$$

A function $f : \mathbb{R}^d \to \mathbb{R}$ is called *punctured upper semicontinuous* if it is punctured upper semicontinuous at all x.

It is easily seen that if f is continuous at x, then f is punctured upper semicontinuous at x, and that if f is punctured upper semicontinuous at x, then f is upper semicontinuous at x. There exist punctured upper semicontinuous functions which are discontinuous (for example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \ge 0$), and upper semicontinuous functions which are not punctured upper semicontinuous (for example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for $x \ne 0$ and $f(0) = 1$).

We can now give a complete characterization of those functions that are local Hausdorff dimension functions.

Theorem 1. *Let $f : \mathbb{R}^d \to [0, \infty)$ be an arbitrary function. Then the following two statements are equivalent.*

(1) *There exists a set $E \subseteq \mathbb{R}^d$ such that*

$$f(x) = \dim_{\mathsf{H,loc}}(x, E) \quad \textit{for all } x \in \mathbb{R}^d.$$

(2) *The function f satisfies the following two conditions:*

(i) *f is punctured upper semicontinuous.*
(ii) *For all $0 \le t < \sup_{x\in\mathbb{R}^d} f(x)$, we have*

$$t < \dim_{\mathsf{H},\mathsf{loc}}(x, \{t < f\}) \quad \textit{for all } x \in \{t < f\}.$$

EXAMPLE. Let $0 < s < 1$. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for $x \le 0$ and $f(x) = s$ for $0 < x$ clearly satisfies condition (2)(ii) in Theorem 1 but is not punctured upper semicontinuous. It therefore follows that f is not the local Hausdorff dimension function of any set $E \subseteq \mathbb{R}$.

EXAMPLE. Let $C \subseteq \mathbb{R}$ denote the usual Cantor set and suppose that $\dim_{\mathsf{H}}(C) = \log 2/\log 3 < s \le 1$. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for $x \notin C$ and $f(x) = s$ for $x \in C$ is easily seen to be punctured upper semicontinuous but it does not satisfy condition (2)(ii) in Theorem 1. Indeed, if $\dim_{\mathsf{H}}(C) = \log 2/\log 3 \le t < s$, then $\{t < f\} = C$, so $\dim_{\mathsf{H},\mathsf{loc}}(x, \{t < f\}) = \dim_{\mathsf{H},\mathsf{loc}}(x, C) \le \dim_{\mathsf{H}}(C) \le t$ for all $x \in \{t < f\} = C$. Therefore f is not the local Hausdorff dimension function of any set $E \subseteq \mathbb{R}$.

The following result was also obtained in [Ol] using different methods.

COROLLARY 2. *If $f : \mathbb{R}^d \to [0,\infty)$ is a continuous function with $f(x) \le d$ for all $x \in \mathbb{R}^d$, then there exists a set $E \subseteq \mathbb{R}^d$ such that*

$$f(x) = \dim_{\mathsf{H},\mathsf{loc}}(x, E) \quad \textit{for all } x \in \mathbb{R}^d.$$

Proof. This follows from the fact that a continuous function f with $f(x) \le d$ for all x clearly satisfies the conditions of Theorem 1. ■

In fact, we prove a significantly more general result characterizing local dimension functions of arbitrary regular dimension indices. Below we define a regular dimension index.

DEFINITION. A function $\dim : \{E \mid E \subseteq \mathbb{R}^d\} \to [0,\infty)$ is called a *dimension index* if it satisfies the following three conditions:

(1) If $E \subseteq F \subseteq \mathbb{R}^d$, then $\dim(E) \le \dim(F)$.
(2) If $E_1, E_2, \ldots \subseteq \mathbb{R}^d$, then

$$\dim\Big(\bigcup_n E_n\Big) = \sup_n \dim(E_n).$$

(3) If $x \in \mathbb{R}^d$, then $\dim(\{x\}) = 0$.

A dimension index dim is called *regular* if it, in addition, satisfies the following condition:

(4) If $t \ge 0$ and $E \subseteq \mathbb{R}^d$ is a Borel set satisfying $t < \dim(E)$, then there exists a compact set $C \subseteq E$ such that $\dim(C) = t$.

Properties of general dimension indices have been studied by, for example, Cutler [Cu] and Tricot [Tr]. It is clear that the Hausdorff dimension

$\dim_{\mathsf{H}}$ and the packing dimension $\dim_{\mathsf{P}}$ are dimension indices, and Proposition 3 shows that they are also regular. The reader is referred to [Fa2] for the definition of the packing dimension.

PROPOSITION 3. *The Hausdorff dimension* $\dim_{\mathsf{H}}$ *and the packing dimension* $\dim_{\mathsf{P}}$ *are regular dimension indices.*

Proof. To prove the regularity, we need the following two (deep) results. For $t \ge 0$, we let $\mathcal{H}^t$ denote the t-dimensional Hausdorff measure and we let $\mathcal{P}^t$ denote the t-dimensional packing measure.

(1) If $t \ge 0$ and E is a Suslin subset of $\mathbb{R}^d$ such that $\mathcal{H}^t(E) = \infty$, then there exists a compact set $C \subseteq E$ such that $0 < \mathcal{H}^t(C) < \infty$.
(2) If $t \ge 0$ and E is a Suslin subset of $\mathbb{R}^d$ such that $\mathcal{P}^t(E) = \infty$, then there exists a compact set $C \subseteq E$ such that $0 < \mathcal{P}^t(C) < \infty$.

Result (1) follows from [Fa1, Theorem 5.5] and result (2) is proved in [JP]. It follows immediately from (1) and (2) that $\dim_{\mathsf{H}}$ and $\dim_{\mathsf{P}}$ are regular dimension indices. ■

For an arbitrary dimension index dim and a subset $E \subseteq \mathbb{R}^d$ we define the *local dimension of* E *at* $x \in \mathbb{R}^d$ by

$$\dim_{\mathsf{loc}}(x, E) = \lim_{r \searrow 0} \dim(E \cap B(x, r)).$$

We can now state our main result.

THEOREM 4. *Let* $f : \mathbb{R}^d \to [0, \infty)$ *be an arbitrary function and let* dim *be a regular dimension index.* (*In particular*, *this condition is satisfied if* dim *equals the Hausdorff dimension* $\dim_{\mathsf{H}}$ *or the packing dimension* $\dim_{\mathsf{P}}$.) *Then the following three statements are equivalent*:

(1) *There exists a set* $E \subseteq \mathbb{R}^d$ *such that*

$$f(x) = \dim_{\mathsf{loc}}(x, E) \quad \text{for all } x \in \mathbb{R}^d.$$

(2) *There exists an* $\mathcal{F}_\sigma$ *set* $E \subseteq \mathbb{R}^d$ *such that*

$$f(x) = \dim_{\mathsf{loc}}(x, E) \quad \text{for all } x \in \mathbb{R}^d.$$

(3) *The function* f *satisfies the following two conditions*:

(i) f *is punctured upper semicontinuous.*
(ii) *For all* $0 \le t < \sup_{x \in \mathbb{R}^d} f(x)$, *we have*

$$t < \dim_{\mathsf{loc}}(x, \{t < f\}) \quad \text{for all } x \in \{t < f\}.$$

Observe that Theorem 1 follows immediately from Theorem 4. We also note that in order to prove Theorem 4 it clearly suffices to show that (3) implies (2) (which is done in Section 2), and that (1) implies (3) (Section 3). Finally, the proof of the following consequence of Theorem 4 is similar to that of Corollary 2 and is therefore omitted.

COROLLARY 5. *Let* dim *be a regular dimension index such that*

$$\dim(G) = \dim(\mathbb{R}^d)$$

for all open non-empty subsets $G \subseteq \mathbb{R}^d$. (*In particular*, *this condition is satisfied if* dim *equals* $\dim_{\mathsf{H}}$ *or* $\dim_{\mathsf{P}}$.) *If* $f : \mathbb{R}^d \to [0,\infty)$ *is a continuous function with* $f(x) \le \dim(\mathbb{R}^d)$ *for all* $x \in \mathbb{R}^d$, *then there exists a set* $E \subseteq \mathbb{R}^d$ *such that*

$$f(x) = \dim_{\mathsf{loc}}(x, E) \quad \text{for all } x \in \mathbb{R}^d.$$

REMARK. It follows from Theorem 4 that if M is an arbitrary subset of $\mathbb{R}^d$, then there exists an $\mathcal{F}_\sigma$ subset E of $\mathbb{R}^d$ whose local Hausdorff dimension function coincides with that of M, i.e.

$$\dim_{\mathsf{H,loc}}(x, E) = \dim_{\mathsf{H,loc}}(x, M) \quad \text{for all } x \in \mathbb{R}^d.$$

This result is the best possible and cannot be improved. More precisely, if M is an arbitrary subset of $\mathbb{R}^d$, then it is in general not possible to choose a closed subset F of $\mathbb{R}^d$ such that

$$\dim_{\mathsf{H,loc}}(x, F) = \dim_{\mathsf{H,loc}}(x, M) \quad \text{for all } x \in \mathbb{R}^d.$$

Indeed, let $f : \mathbb{R}^d \to \mathbb{R}$ be any continuous function such that $0 < f(x) \le d$ for all $x \in \mathbb{R}^d$ and $f(x_0) < d$ for some x_0. It follows from Theorem 4 that there exists a set M such that $f(x) = \dim_{\mathsf{H,loc}}(x, M)$ for all $x \in \mathbb{R}^d$. We now claim that there is no closed set F such that $\dim_{\mathsf{H,loc}}(x, F) = f(x)$ for all $x \in \mathbb{R}^d$. To see this observe that if F is closed and $\dim_{\mathsf{H,loc}}(x, F) > 0$, then $x \in F$. Hence, if F is closed and $\dim_{\mathsf{H,loc}}(x, F) = f(x) > 0$ for all $x \in \mathbb{R}^d$, then $F = \mathbb{R}^d$, whence $\dim_{\mathsf{H,loc}}(x, F) = d$ for all x, contradicting the fact that $\dim_{\mathsf{H,loc}}(x_0, F) = f(x_0) < d$.

2. Proof of Theorem 4: (3) implies (2). First we introduce some notation. For a function $f : \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{R}^d$ and $r > 0$ write

$$\begin{aligned} B_{\mathsf{p}}(x, r) &= \{y \in \mathbb{R}^d \mid 0 < |x - y| < r\}, \\ M_{\mathsf{p}}(f; x, r) &= \sup_{0<|x-y|<r} f(y), \end{aligned} \tag{2.1}$$

i.e. $B_{\mathsf{p}}(x, r)$ is the punctured ball centered at x and with radius equal to r and $M_{\mathsf{p}}(f; x, r)$ is the supremum of f over $B_{\mathsf{p}}(x, r)$.

Proof that (3) *implies* (2) *in Theorem* 4. Let $0 \le t < \sup_{x\in\mathbb{R}^d} f(x)$. For $x \in \{t < f\}$ and $r > 0$ we have

$$t < \dim_{\mathsf{loc}}(x, \{t < f\}) \le \dim(B(x, r) \cap \{t < f\}). \tag{2.2}$$

Also, since f is punctured upper semicontinuous, and so in particular upper semicontinuous, $B(x, r) \cap \{t < f\}$ is Borel. It therefore follows from (2.2) and the fact dim is regular that there exists a compact set $E_t(x, r)$ satisfying

$$E_t(x, r) \subseteq B(x, r) \cap \{t < f\}, \quad \dim(E_t(x, r)) = t.$$

Next choose a countable dense subset U_t of $\{t < f\}$ and define

$$E = \bigcup_{\substack{0\le t<\sup_{y\in\mathbb{R}^d} f(y)\\ t\in\mathbb{Q}_+}} \ \bigcup_{\substack{r\in\mathbb{Q}_+\\ x\in U_t}} E_t(x,r).$$

The set E is clearly $\mathcal{F}_\sigma$. We will now prove that f is the local dimension function of E.

Claim 1. *For all $x\in\mathbb{R}^d$, we have*

$$\dim_{\mathsf{loc}}(x,E) \le f(x).$$

Proof. Fix $x\in\mathbb{R}^d$ and $r>0$. Then

$$\begin{aligned}(2.3)\qquad E\cap B(x,r) &\subseteq (E\cap B_{\mathsf{p}}(x,r))\cup\{x\}\\ &= \bigcup_{\substack{0\le t<\sup_{y\in\mathbb{R}^d} f(y)\\ t\in\mathbb{Q}_+}} \Big(\bigcup_{\substack{s\in\mathbb{Q}_+\\ x\in U_t}} (E_t(x,s)\cap B_{\mathsf{p}}(x,r))\Big)\cup\{x\}.\end{aligned}$$

Next observe that since $E_t(x,s)\subseteq\{t<f\}$, we have

$$(2.4)\quad E_t(x,s)\cap B_{\mathsf{p}}(x,r)\subseteq\{t<f\}\cap B_{\mathsf{p}}(x,r)=\emptyset \quad \text{for } M_{\mathsf{p}}(f;x,r)\le t.$$

Combining (2.3) and (2.4) yields

$$\begin{aligned}E\cap B(x,r) &\subseteq \bigcup_{\substack{0\le t<M_{\mathsf{p}}(f;x,r)\\ t\in\mathbb{Q}_+}} \Big(\bigcup_{\substack{s\in\mathbb{Q}_+\\ x\in U_t}} (E_t(x,s)\cap B_{\mathsf{p}}(x,r))\Big)\cup\{x\}\\ &\subseteq \bigcup_{\substack{0\le t<M_{\mathsf{p}}(f;x,r)\\ t\in\mathbb{Q}_+}} \Big(\bigcup_{\substack{s\in\mathbb{Q}_+\\ x\in U_t}} E_t(x,s)\Big)\cup\{x\}.\end{aligned}$$

It follows that

$$\begin{aligned}\dim(E\cap B(x,r)) &\le \max\Big(\sup_{\substack{0\le t<M_{\mathsf{p}}(f;x,r)\\ t\in\mathbb{Q}_+}}\ \sup_{\substack{s\in\mathbb{Q}_+\\ x\in U_t}} \dim(E_t(x,s)),\dim(\{x\})\Big)\\ &= \sup_{\substack{0\le t<M_{\mathsf{p}}(f;x,r)\\ t\in\mathbb{Q}_+}}\ \sup_{\substack{s\in\mathbb{Q}_+\\ x\in U_t}} \dim(E_t(x,s))\\ &= \sup_{\substack{0\le t<M_{\mathsf{p}}(f;x,r)\\ t\in\mathbb{Q}_+}}\ \sup_{\substack{s\in\mathbb{Q}_+\\ x\in U_t}} t = M_{\mathsf{p}}(f;x,r)\end{aligned}$$

for all $r>0$. Finally, using the fact that f is punctured upper semicontinuous at x, we infer that

$$\dim_{\mathsf{loc}}(x,E)=\lim_{r\searrow 0}\dim(E\cap B(x,r))\le\lim_{r\searrow 0} M_{\mathsf{p}}(f;x,r)=f(x). \ \blacksquare$$

Claim 2. *For all $x\in\mathbb{R}^d$, we have*

$$f(x)\le\dim_{\mathsf{loc}}(x,E).$$

Proof. Fix $x \in \mathbb{R}^d$ and $r > 0$. Next, let $\varepsilon > 0$. We can choose $s \in \mathbb{Q}_+$ with $\frac{1}{8}r \le s \le \frac{1}{4}r$ and $y \in B_{\mathsf{p}}(x, s)$ such that $M_{\mathsf{p}}(f; x, s) - \varepsilon < f(y)$. Finally, choose $t \in \mathbb{Q}_+$ with $M_{\mathsf{p}}(f; x, s) - 2\varepsilon \le t \le M_{\mathsf{p}}(f; x, s) - \varepsilon$. It follows that $y \in \{t < f\}$ and we can thus find $u \in U_t$ with $|u - y| < s$. It is now clear that

$$E_t(u, s) \subseteq E,$$

and that $E_t(u, s) \subseteq B(u, s) \subseteq B(x, r)$, whence

$$E_t(u, s) \cap B(x, r) = E_t(u, s).$$

We therefore conclude that

$$\begin{aligned}\dim(E \cap B(x, r)) &\ge \dim(E_t(u, s) \cap B(x, r)) = \dim(E_t(u, s)) = t \\ &\ge M_{\mathsf{p}}(f; x, s) - 2\varepsilon \ge M_{\mathsf{p}}\big(f; x, \tfrac{1}{8}r\big) - 2\varepsilon.\end{aligned}$$

Since f is punctured upper semicontinuous at x, we infer that

$$\dim_{\mathsf{loc}}(x, E) = \lim_{r \searrow 0} \dim(E \cap B(x, r)) \ge \lim_{r \searrow 0} M_{\mathsf{p}}\big(f; x, \tfrac{1}{8}r\big) - 2\varepsilon = f(x) - 2\varepsilon.$$

Finally, letting $\varepsilon \searrow 0$, shows that $\dim_{\mathsf{loc}}(x, E) \ge f(x)$. ∎

3. Proof of Theorem 4: (1) implies (3). We begin with a small lemma.

LEMMA 6. *Let $M \subseteq \mathbb{R}^d$. Then*

$$\dim(M) \le \sup_{x \in M} \dim_{\mathsf{loc}}(x, M).$$

Proof. Let $\varepsilon > 0$. For each $x \in M$ we can choose a positive number $r_x > 0$ such that

$$\dim(M \cap B(x, r_x)) \le \dim_{\mathsf{loc}}(x, M) + \varepsilon.$$

The family $(B(x, r_x))_{x \in M}$ forms an open cover of M, and it therefore follows from Lindelöf's theorem (cf. [BBT, p. 7, Exercise 1:1.14]) that there exists a countable subset $U \subseteq M$ such that the family $(B(x, r_x))_{x \in U}$ covers M. This implies that

$$\begin{aligned}\dim(M) &= \dim\Big(\bigcup_{x \in U} (M \cap B(x, r_x))\Big) = \sup_{x \in U} \dim(M \cap B(x, r_x)) \\ &\le \sup_{x \in U} \dim_{\mathsf{loc}}(x, M) + \varepsilon \le \sup_{x \in M} \dim_{\mathsf{loc}}(x, M) + \varepsilon.\end{aligned}$$

Letting $\varepsilon \searrow 0$ gives the desired result. ∎

Proof that (1) *implies* (3)(i) *in Theorem* 4. We prove this by proving the two claims below.

CLAIM 1. *For all $x \in \mathbb{R}^d$, we have*

$$\limsup_{y \to x}{}_{\mathsf{p}} f(y) \le f(x).$$

Proof. Fix $x \in \mathbb{R}^d$ and $r > 0$. For all $y \in B_{\mathsf{p}}(x,r)$ and $0 < s < r - |x-y|$ we see that $B(y,s) \subseteq B(x,r)$, whence $\dim(E \cap B(x,r)) \geq \dim(E \cap B(y,s))$. Letting $s \searrow 0$ now gives

$$\dim(E \cap B(x,r)) \geq \dim_{\mathsf{loc}}(y,E) = f(y).$$

Since $y \in B_{\mathsf{p}}(x,r)$ was arbitrary, this implies that

$$\dim(E \cap B(x,r)) \geq \sup_{0<|x-y|<r} f(y).$$

Finally, letting $r \searrow 0$ shows that

$$f(x) = \dim_{\mathsf{loc}}(x,E) \geq \limsup_{y \to x}{}_{\mathsf{p}} f(y). \ \blacksquare$$

CLAIM 2. *For all $x \in \mathbb{R}^d$, we have*

$$f(x) \leq \limsup_{y \to x}{}_{\mathsf{p}} f(y).$$

Proof. Fix $x \in \mathbb{R}^d$ and $r > 0$. Let $\varepsilon > 0$. For each $y \in B_{\mathsf{p}}(x,r)$ we can find $r_y > 0$ such that $B(y,r_y) \subseteq B(x,r)$ and

$$\dim(E \cap B(y,r_y)) \leq \dim_{\mathsf{loc}}(y,E) + \varepsilon = f(y) + \varepsilon.$$

The family $(B(y,r_y))_{y \in B_{\mathsf{p}}(x,r)}$ forms an open cover of $B_{\mathsf{p}}(x,r)$, so by Lindelöf's theorem there exists a countable subset $U \subseteq B_{\mathsf{p}}(x,r)$ such that $(B(y,r_y))_{y\in U}$ covers $B_{\mathsf{p}}(x,r)$. Hence,

$$\begin{aligned}
\dim(E \cap B(x,r)) &\leq \dim((E \cap B_{\mathsf{p}}(x,r)) \cup \{x\}) \\
&= \max(\dim(E \cap B_{\mathsf{p}}(x,r)), \dim(\{x\})) \\
&= \dim(E \cap B_{\mathsf{p}}(x,r)) \\
&\leq \dim\Big(\bigcup_{y\in U} (E \cap B(y,r_y))\Big) \\
&= \sup_{y\in U} \dim(E \cap B(y,r_y)) \\
&\leq \sup_{y\in U} f(y) + \varepsilon \leq \sup_{0<|x-y|<r} f(y) + \varepsilon.
\end{aligned}$$

Next, letting $\varepsilon \searrow 0$ shows that $\dim(E \cap B(x,r)) \leq \sup_{0<|x-y|<r} f(y)$. Finally, letting $r \searrow 0$ yields

$$f(x) = \dim_{\mathsf{loc}}(x,E) \leq \limsup_{y \to x}{}_{\mathsf{p}} f(y). \ \blacksquare$$

Proof that (1) *implies* (3)(ii) *in Theorem* 4. Let $t < \sup_{v \in \mathbb{R}^d} f(v)$ and $x \in \{t < f\}$. Also, let $r > 0$. Next, observe that

$$\begin{aligned}
(3.1) \qquad & \dim(E \cap B(x,r)) \\
&= \max(\dim((E \setminus \{f \leq t\}) \cap B(x,r)), \dim((E \cap \{f \leq t\}) \cap B(x,r))).
\end{aligned}$$

Using Lemma 6 we see that

$$\begin{aligned}\dim((E\cap\{f\le t\})\cap B(x,r)) &\le \dim(E\cap\{f\le t\})\\ &\le \sup_{y\in E\cap\{f\le t\}} \dim_{\mathsf{loc}}(y,E\cap\{f\le t\})\\ &\le \sup_{y\in\{f\le t\}} \dim_{\mathsf{loc}}(y,E) \le \sup_{y\in\{f\le t\}} f(y)\le t.\end{aligned}$$

We also have

$$\dim(E\cap B(x,r)) \ge \dim_{\mathsf{loc}}(x,E) = f(x) > t. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) shows that

$$\dim(E\cap B(x,r)) = \dim((E\setminus\{f\le t\})\cap B(x,r))$$

for all $r>0$. This clearly implies that $\dim_{\mathsf{loc}}(x,E)=\dim_{\mathsf{loc}}(x,E\setminus\{f\le t\})$, whence

$$t<f(x)=\dim_{\mathsf{loc}}(x,E)=\dim_{\mathsf{loc}}(x,E\setminus\{f\le t\})\le\dim_{\mathsf{loc}}(x,\{t<f\}).$$

This completes the proof. ■

Acknowledgements. I thank an anonymous referee for providing the remark following Corollary 5.

REFERENCES

[BBT] A. Bruckner, J. Bruckner and B. Thomson, *Real Analysis*, Prentice-Hall, 1997.

[Cu] C. D. Cutler, *Measure disintegrations with respect to σ-stable monotone indices and the pointwise representation of packing dimension*, Measure Theory (Oberwolfach, 1990), Rend. Circ. Mat. Palermo (2) Suppl. 28 (1992), 319–339.

[Fa1] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge Tracts in Math. 85, Cambridge Univ. Press, 1986.

[Fa2] —, *Fractal Geometry. Mathematical Foundations and Applications*, Wiley, 1990.

[JP] H. Joyce and D. Preiss, *On the existence of subsets of finite positive packing measure*, Mathematika 42 (1995), 15–24.

[JS] H. Jürgensen and L. Staiger, *Local Hausdorff dimension*, Acta Inform. 32 (1995), 491–507.

[Ol] L. Olsen, *Applications of divergence points to local dimension functions of subsets of* $\mathbb{R}^d$, Proc. Edinburgh Math. Soc. 48 (2005), 213–218.

[Ru] T. Rushing, *Hausdorff dimension of wild fractals*, Trans. Amer. Math. Soc. 334 (1992), 597–613.

[Tr] C. Tricot, *Rarefaction indices*, Mathematika 27 (1980), 46–57.

Department of Mathematics
University of St. Andrews
St. Andrews, Fife KY16 9SS, Scotland
E-mail: lo@st-and.ac.uk

Received 26 November 2004;
revised 7 March 2005 (4533)