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SYSTEMS OF DYADIC CUBES IN A DOUBLING METRIC SPACE

BY

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Abstract. A number of recent results in Euclidean harmonic analysis have exploited several adjacent systems of dyadic cubes, instead of just one fixed system. In this paper, we extend such constructions to general spaces of homogeneous type, making these tools available for analysis on metric spaces. The results include a new (non-random) construction of boundedly many adjacent dyadic systems with useful covering properties, and a streamlined version of the random construction recently devised by H. Martikainen and the first author. We illustrate the usefulness of these constructions with applications to weighted inequalities and the BMO space; further applications will appear in forthcoming work.

1. Introduction. The standard system of dyadic cubes,

$$\mathscr{D} := \{2^{-k}([0,1)^n + m) \colon k \in \mathbb{Z}, m \in \mathbb{Z}^n\},$$

plays an indispensable role in harmonic analysis on the Euclidean space $\mathbb{R}^n$. The fundamental properties of these cubes are that any two of them are either disjoint or one is contained in the other, and that the cubes of a given size partition all space. Also, the cubes are not too far away from balls, which are usually more natural objects from the geometric point of view.

Accordingly, there has been interest in constructing analogues of dyadic cubes in more general metric spaces, to provide tools for analysis in such settings. The first results in this direction, as far as we know, are due to G. David [6, Appendix A]; see also [7, Appendix I]. A more comprehensive construction was provided by M. Christ [4], in the full generality of Coifman–Weiss spaces of homogeneous type [5], and this has become the standard reference on the topic. In addition to the basic geometric properties expected from the cubes, Christ also obtained a certain smallness of the boundary condition (in terms of an underlying doubling measure), which has turned out useful in applications to singular integrals. A more elementary construction, without addressing the boundary control but nevertheless sufficient for many purposes, was provided by E. Sawyer and R. L. Wheeden [26]. Some further variants have been considered by other authors [1, 14].

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Meanwhile, new developments in $\mathbb{R}^n$ have seen the need to consider not just one but several adjacent dyadic systems. Two types of constructions are of particular interest here. On the one hand, there are the *random dyadic systems* due to F. Nazarov, S. Treil and A. Volberg [23, Section 4],

$$\begin{aligned}\mathscr{D}(\omega) &:= \Big\{2^{-k}([0,1)^n + m) + \sum_{j>k} 2^{-j}\omega_j \colon k \in \mathbb{Z},\, m \in \mathbb{Z}^n\Big\},\\ \omega &= (\omega_j)_{j\in\mathbb{Z}} \in \Omega := (\{0,1\}^n)^{\mathbb{Z}},\end{aligned}$$

where Ω is equipped with the natural product probability measure. The randomization provides a powerful way of controlling edge effects, even when the space is equipped with a non-doubling measure, by proving that any given point $x \in \mathbb{R}^n$ has a small probability of ending up close to the boundary of a randomly chosen cube. These random dyadic systems have been instrumental in the development of the non-doubling theory of singular integrals [23, 24] and its applications to analytic capacity [27, 28] as well as sharp one-weight [9, 13] and two-weight [16, 25] inequalities for classical singular integrals. A version of such random cubes in metric spaces, starting from Christ's construction, was recently obtained by the first author and H. Martikainen [12], to study singular integrals in metric spaces with non-doubling measures.

On the other hand, there is a *non-random choice* of just boundedly many dyadic systems, say

$$\mathscr{D}^t := \{2^{-k}([0,1)^n + m + (-1)^k t) \colon k \in \mathbb{Z},\, m \in \mathbb{Z}^n\}, \qquad t \in \{0, 1/3, 2/3\}^n,$$

which have the following useful property: for every ball B, there exists a cube Q in at least one of the $\mathscr{D}^t$ such that $B \subseteq \frac{9}{10}Q$ while $\operatorname{diam}(Q) \le C \operatorname{diam}(B)$. These adjacent systems have been exploited, e.g., in the work of C. Muscalu, T. Tao and C. Thiele [22, 21] on multi-linear operators, and very recently by M. Lacey, E. Sawyer and I. Uriarte-Tuero [15] on two-weight inequalities. We are not aware of the precise original occurrence of this latter set of systems: Lacey et al. [15, Section 2.2] attribute them to J. Garnett and P. Jones; Muscalu et al. [22, Section 5] to M. Christ, at least the observation on $B \subseteq \frac{9}{10}Q$. T. Mei [20] has shown that a similar conclusion can be obtained with just $n+1$ (rather than 3^n) cleverly chosen systems $\mathscr{D}^t$.

The goals of the present paper are two-fold. First, we recall and streamline the construction of the Christ-type dyadic cubes in a metric space, including the recent randomized version $\mathscr{D}(\omega)$ from [12]. Second, we provide a metric space version of the non-random choice of boundedly many dyadic systems $\mathscr{D}^t$, with the property that any B is contained in some $Q \in \bigcup_t \mathscr{D}^t$ with $\operatorname{diam}(Q) \le C \operatorname{diam}(B)$, which is a completely new result. We even combine the two constructions, yielding a random family of adjacent dyadic systems $\mathscr{D}^t(\omega)$.

We have strived for a reasonably comprehensive and transparent presentation, including some results which could be found elsewhere. In the hope of making the paper a useful reference, we have tried to make the statements of our theorems easily applicable as "black boxes", but also paid attention to the details of the proofs. As in $\mathbb{R}^n$, where the open $(2^{-k}((0,1)^n+m))$, half-open $(2^{-k}([0,1)^n+m))$ and closed $(2^{-k}([0,1]^n+m))$ dyadic cubes each serve their different purpose, we also present (as in [12]) a unified construction of three related systems of open, half-open and closed cubes. While this may not be the absolute shortest route to the half-open cubes alone (for which one should probably consult the recent paper of A. Käenmäki, T. Rajala and V. Suomala [14]), we find the properties of the open and closed cubes proven on the way to be useful as well.

The basic idea behind the construction of several adjacent dyadic systems is from [12]: New centre points for the cubes of sidelength δ^k (where $\delta > 0$ is a small parameter playing the role of the constant $1/2$ in the classical Euclidean dyadic system) are chosen among the old centre points of the (one level smaller) cubes of sidelength δ^{k+1}. In the non-probabilistic selection, instead of doing this randomly, we need to do it in a "clever" way. The basic *conflict* to avoid is two new centres getting too close to each other. This is achieved by equipping the points with suitable *labels*, which help us avoid these conflicts. As it turns out, this philosophy can also be used to simplify the original random construction from [12], where originally the conflicts were first allowed among the new centre points, and yet another selection process was needed to remove some of them, thereby yielding the final points. This simplification already proved useful in the consideration of vector-valued singular integrals by the random cubes method by Martikainen [18], and it is expected to be of interest elsewhere. A particular feature of the new selection process is a natural one-to-one correspondence between the old and new cubes, which was not present when some of the new centres were first removed; thus the original cubes may be used as an index set for the new cubes, a technical property which was much exploited in some of the recent Euclidean applications [9, 13].

As an illustration of the use of the new adjacent dyadic systems, we provide easy extensions of two results in Euclidean harmonic analysis to metric spaces X with a doubling measure μ: First, Buckley's theorem [2] on the sharp weighted norm of the Hardy–Littlewood maximal operator,

$$\|Mf\|_{L^p(w)} \leq C\|w\|_{A_p}^{1/(p-1)}\|f\|_{L^p(w)},$$
$$\|w\|_{A_p} := \sup_B \Big(\fint_B w\,d\mu\Big)\Big(\fint_B w^{-1/(p-1)}\,d\mu\Big)^{p-1},$$

where the supremum is over all metric balls in X, and C only depends on X,

μ and $p \in (1, \infty)$. Second, the representation of BMO(μ) as an intersection of finitely many dyadic BMO spaces, extending the Euclidean result in [20].

In extending Buckley's theorem, we follow the Euclidean approach due to Lerner [17]. A noteworthy feature of our argument is the circumvention of the use of the Besicovitch covering theorem, an essentially Euclidean device used in Lerner's original proof, by the trivial covering properties exhibited by the adjacent dyadic systems. We believe that this displays a more general principle of avoiding the Besicovitch theorem and thereby allowing extensions of other Euclidean results to metric spaces.

Note that only these applications, but not the construction of the cubes as such, depends on the existence of a doubling measure μ on X; for the cubes, we only need the weaker geometric doubling property that any ball of radius r can be covered by at most A_1 (a fixed constant) balls of radius $\frac{1}{2}r$. Further applications will be considered in a forthcoming paper by the second author.

2. Definition and construction of a dyadic system

2.1. **Set-up.** Let ρ be a quasi-metric on the space X, i.e., it satisfies the axioms of a metric except for the triangle inequality, which is assumed in the weaker form

$$\rho(x, y) \leq A_0(\rho(x, z) + \rho(z, y))$$

with a constant $A_0 \geq 1$. The quasi-metric space (X, ρ) is assumed to have the following (*geometric*) *doubling property*: There exists a positive integer $A_1 \in \mathbb{N}$ such that for every $x \in X$ and for every $r > 0$, the ball $B(x, r) := \{y \in X : \rho(y, x) < r\}$ can be covered by at most A_1 balls $B(x_i, r/2)$.

Until further notice, no other properties of the quasi-metric space (X, ρ) will be required; in particular, we do not assume any measurability of X. Some of the arguments are valid even without the assumption of geometric doubling.

Set $a_1 := \log_2 A_1$. The following properties are easy to check (cf. [10, Lemmas 2.3 and 2.5]):

(1) Any ball $B(x, r)$ can be covered by at most $A_1\delta^{-a_1}$ balls of radius δr for any $\delta \in (0, 1]$.
(2) Any ball $B(x, r)$ contains at most $A_1\delta^{-a_1}$ centres x_i of pairwise disjoint balls $B(x_i, \delta r)$.
(3) Any disjoint family of balls in X is at most countable.
(4) If $x, y \in X$ have $\rho(x, y) \geq r$, then the balls $B(x, r/(2A_0))$ and $B(y, r/(2A_0))$ are disjoint.

A subset $\Omega \subseteq X$ is *open* if for every $x \in \Omega$ there exist $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq \Omega$. A subset $F \subseteq X$ is *closed* if its complement is open. The usual proof of the fact that $F \subseteq X$ is closed if and only if it contains its limit points, carries over to quasi-metric spaces. However, some balls $B(x,r)$ may fail to be open. (E.g., consider $X = \{-1\} \cup [0,\infty)$ with the usual distance between all other pairs of points except $\rho(-1,0) := 1/2$. Then $B(-1,1) = \{-1,0\}$ does not contain any ball of the form $B(0,\varepsilon)$, and hence cannot be open.)

2.2. THEOREM. *Suppose that constants $0 < c_0 \leq C_0 < \infty$ and $\delta \in (0,1)$ satisfy*

$$12A_0^3 C_0 \delta \leq c_0. \tag{2.3}$$

Given a set of points $\{z_\alpha^k\}_\alpha$, $\alpha \in \mathscr{A}_k$, for every $k \in \mathbb{Z}$, with the properties that

$$\rho(z_\alpha^k, z_\beta^k) \geq c_0\delta^k \quad (\alpha \neq \beta), \qquad \min_\alpha \rho(x, z_\alpha^k) < C_0\delta^k \quad \forall x \in X, \tag{2.4}$$

we can construct families of sets $\tilde{Q}_\alpha^k \subseteq Q_\alpha^k \subseteq \bar{Q}_\alpha^k$ (called open, half-open and closed dyadic cubes) *such that:*

$$\tilde{Q}_\alpha^k \text{ and } \bar{Q}_\alpha^k \text{ are the interior and closure of } Q_\alpha^k; \tag{2.5}$$

$$\text{if } \ell \geq k, \text{ then either } Q_\beta^\ell \subseteq Q_\alpha^k \text{ or } Q_\alpha^k \cap Q_\beta^\ell = \emptyset; \tag{2.6}$$

$$X = \bigcup_\alpha Q_\alpha^k \quad (\text{disjoint union}) \quad \forall k \in \mathbb{Z}; \tag{2.7}$$

$$B(z_\alpha^k, c_1\delta^k) \subseteq Q_\alpha^k \subseteq B(z_\alpha^k, C_1\delta^k) =: B(Q_\alpha^k), \tag{2.8}$$

where $c_1 := (3A_0^2)^{-1}c_0$ and $C_1 := 2A_0C_0$;

$$\text{if } \ell \geq k \text{ and } Q_\beta^\ell \subseteq Q_\alpha^k, \text{ then } B(Q_\beta^\ell) \subseteq B(Q_\alpha^k). \tag{2.9}$$

The open and closed cubes $\tilde{Q}_\alpha^k$ and $\bar{Q}_\alpha^k$ depend only on the points z_β^ℓ for $\ell \geq k$. The half-open cubes Q_α^k depend on z_β^ℓ for $\ell \geq \min(k, k_0)$, where $k_0 \in \mathbb{Z}$ is a preassigned number entering the construction.

To some extent, this combines the benefits of the alternative constructions of Christ and Sawyer–Wheeden: on the one hand, we obtain dyadic cubes on all length scales (rather than from a given level up), as in Christ's construction, and we also obtain an exact partition of the space (rather than up to measure zero), as in Sawyer and Wheeden. The fact that things would be slightly simpler if we started from a fixed finest level of partition, like Sawyer and Wheeden, is reflected by the dependence of the half-open cubes also on the points of the coarser scales once we go past the preassigned threshold level k_0.

The proof consists of several steps.

2.10. LEMMA (Partial order for dyadic points). *Under the assumptions of Theorem 2.2, there is a partial order $\leq$ among the pairs (k,α) such that:*

- *if $\rho(z_\beta^{k+1}, z_\alpha^k) < (2A_0)^{-1}c_0\delta^k$ then $(k+1,\beta) \leq (k,\alpha)$;*

- *if* $(k+1,\beta) \le (k,\alpha)$ *then* $\rho(z_\beta^{k+1}, z_\alpha^k) < C_0\delta^k$*;*
- *for every* $(k+1,\beta)$*, there is exactly one* $(k,\alpha) \ge (k+1,\beta)$*, called its* parent*;*
- *for every* (k,α)*, there are between* 1 *and* M *pairs* $(k+1,\beta) \le (k,\alpha)$*, called its* children*;*
- $(\ell,\beta) \le (k,\alpha)$ *if and only if* $\ell \ge k$ *and there are* $(j+1,\gamma_{j+1}) \le (j,\gamma_j)$ *for all* $j = k, k+1, \dots, \ell-1$*, for some* $\gamma_k = \alpha, \gamma_{k+1}, \dots, \gamma_{\ell-1}, \gamma_\ell = \beta$*; then* (ℓ,β) *and* (k,α) *are called one another's* descendant *and* ancestor*, respectively.*

Proof. Indeed, these properties essentially define $\le$: Given a point $(k+1,\beta)$, check whether there exists α such that $\rho(z_\alpha^k, z_\beta^{k+1}) < (2A_0)^{-1}c_0\delta^k$. If one exists, it is necessarily unique by (2.4) and we decree that $(k+1,\beta) \le (k,\alpha)$. If no such good α exists, choose any α for which $\rho(z_\alpha^k, z_\beta^{k+1}) < C_0\delta^k$ (at least one such α exists by (2.4)) and decree that $(k+1,\beta) \le (k,\alpha)$. (From (2.4) and the geometric doubling property, it follows readily that the index sets $\mathscr{A}_k$, $k \in \mathbb{Z}$, are countable. By assuming that the countable number of indices α are taken from $\mathbb{N}$, we could choose the smallest α, thereby eliminating the arbitrariness of this choice in the construction.) In either case, we decree that $(k+1,\beta)$ is not related to any other (k,γ), and finally we extend $\le$ by transitivity to obtain a partial ordering.

It only remains to check the claim concerning the number of children. If $(k+1,\beta) \le (k,\alpha)$, we have $\rho(z_\alpha^k, z_\beta^{k+1}) < C_0\delta^k$, but also $\rho(z_\beta^{k+1}, z_\gamma^{k+1}) \ge c_0\delta^{k+1}$ for any $\gamma \ne \beta$. Thus the geometric doubling property implies that there can be at most boundedly many, say M, such $(k+1,\beta)$. Conversely, for (k,α), there exists at least one $(k+1,\beta)$ with $\rho(z_\alpha^k, z_\beta^{k+1}) < C_0\delta^{k+1} \le (2A_0)^{-1}c_0\delta^k$, and thus $(k+1,\beta) \le (k,\alpha)$, so there is at least one child. ∎

With the partial order defined, it is possible to formulate the rest of the construction, although proving all the stated properties needs some more work. As a preliminary version, we define, for every $k \in \mathbb{Z}$ and every α,

$$\hat{Q}_\alpha^k := \{z_\beta^\ell \colon (\ell,\beta) \le (k,\alpha)\}.$$

Then

$$\bar{Q}_\alpha^k := \overline{\hat{Q}_\alpha^k},$$

the closure of $\hat{Q}_\alpha^k$, and

$$\tilde{Q}_\alpha^k := \Big(\bigcup_{\beta \ne \alpha} \bar{Q}_\beta^k\Big)^c.$$

It is clear from the definition that these depend only on z_β^ℓ for $\ell \ge k$.

In the following section we prove:

2.11. Proposition (Properties of closed and open dyadic cubes). *Suppose that for every $k \in \mathbb{Z}$ we have a set of points with properties* (2.4) *and constants that satisfy* (2.3) *as in Theorem* 2.2. *Then the cubes $\tilde{Q}^k_\alpha$ and $\bar{Q}^k_\alpha$ satisfy:*

(2.12) *$\tilde{Q}^k_\alpha$ and $\bar{Q}^k_\alpha$ are one another's interior and closure, respectively;*

(2.13) *for $\ell \geq k$, we have $\tilde{Q}^\ell_\beta \subseteq \tilde{Q}^k_\alpha$ if $(\ell,\beta) \leq (k,\alpha)$, and $\tilde{Q}^\ell_\beta \cap \bar{Q}^k_\alpha = \bar{Q}^\ell_\beta \cap \tilde{Q}^k_\alpha = \emptyset$ otherwise;*

$$X = \bigcup_\alpha \bar{Q}^k_\alpha \quad (\textit{possibly with overlap}) \quad \forall k \in \mathbb{Z}; \tag{2.14}$$

$$\text{(i) } B(z^k_\alpha, c_1\delta^k) \subseteq \tilde{Q}^k_\alpha, \qquad \text{(ii) } \bar{Q}^k_\alpha \subseteq B(z^k_\alpha, C_1\delta^k); \tag{2.15}$$

$$\textit{if } (\ell,\beta) \leq (k,\alpha) \textit{ then } B(z^\ell_\beta, C_1\delta^\ell) \subseteq B(z^k_\alpha, C_1\delta^k). \tag{2.16}$$

Moreover,

$$\bar{Q}^k_\alpha = \bigcup_{\beta\colon (\ell,\beta)\leq(k,\alpha)} \bar{Q}^\ell_\beta \quad \forall \ell \geq k. \tag{2.17}$$

Assuming this result, which contains the essence of Theorem 2.2, we can complete the proof of the theorem by the following lemma:

2.18. Lemma (Construction of half-open cubes). *Assuming Proposition* 2.11, *we can construct Q^k_α which satisfy the assertions of Theorem* 2.2.

Proof. Here it is convenient to assume that the pairs (k,α) are parameterized by $\alpha \in \mathbb{N}$ for each $k \in \mathbb{Z}$. For the given threshold level $k = k_0$, we define recursively

$$Q^{k_0}_0 := \bar{Q}^{k_0}_0, \qquad Q^{k_0}_\alpha := \bar{Q}^{k_0}_\alpha \setminus \bigcup_{\beta=0}^{\alpha-1} Q^{k_0}_\beta, \quad \alpha \geq 1.$$

By construction, it is clear that the $Q^{k_0}_\alpha$ are pairwise disjoint and satisfy

$$\begin{gathered} \bar{Q}^{k_0}_\alpha \supseteq Q^{k_0}_\alpha \supseteq \bar{Q}^{k_0}_\alpha \setminus \bigcup_{\beta\neq\alpha} \bar{Q}^{k_0}_\beta \supseteq \tilde{Q}^{k_0}_\alpha \setminus (\tilde{Q}^{k_0}_\alpha)^c = \tilde{Q}^{k_0}_\alpha, \\ \bigcup_{\beta=0}^{\alpha} Q^{k_0}_\beta = \bigcup_{\beta=0}^{\alpha} \bar{Q}^{k_0}_\beta \xrightarrow[\alpha\to\infty]{} X. \end{gathered} \tag{2.19}$$

For $k < k_0$ we define

$$Q^k_\alpha := \bigcup_{\beta\colon (k_0,\beta)\leq(k,\alpha)} Q^{k_0}_\beta.$$

Then clearly $Q^k_\alpha \subseteq \bar{Q}^k_\alpha$, these partition X for a fixed k, and (2.6) holds for all $k \leq \ell \leq k_0$. Finally,

$$(Q^k_\alpha)^c = \bigcup_{\beta\colon (k_0,\beta)\not\leq(k,\alpha)} Q^{k_0}_\beta \subseteq \bigcup_{\gamma\neq\alpha} \bigcup_{\beta\colon (k_0,\beta)\not\leq(k,\gamma)} \bar{Q}^{k_0}_\beta = \bigcup_{\gamma\neq\alpha} \bar{Q}^k_\gamma = (\tilde{Q}^k_\alpha)^c,$$

so that $\tilde{Q}^k_\alpha \subseteq Q^k_\alpha$.

For $k > k_0$, we proceed by induction as follows. Suppose that the cubes Q^ℓ_α, $\ell \leq k-1$, are already defined as required. For every α, consider the finitely many $(k,\beta) \leq (k-1,\alpha)$, and relabel them temporarily with $\beta = 0,1,\ldots$ (up to some finite number). Then define

$$Q^k_0 := Q^{k-1}_\alpha \cap \bar{Q}^k_0, \qquad Q^k_\beta := Q^{k-1}_\alpha \cap \bar{Q}^k_\beta \setminus \bigcup_{\gamma=0}^{\beta-1} Q^k_\gamma, \quad \beta \geq 1.$$

Disjointness is clear, and as in (2.19) we get $\bar{Q}^k_\beta \supseteq Q^k_\beta \supseteq Q^{k-1}_\alpha \cap \tilde{Q}^k_\beta \supseteq \tilde{Q}^{k-1}_\alpha \cap \tilde{Q}^k_\beta = \tilde{Q}^k_\beta$, and

$$\bigcup_{\beta\colon (k,\beta)\leq(k-1,\alpha)} Q^k_\beta = Q^{k-1}_\alpha \cap \bigcup_{\beta\colon (k,\beta)\leq(k-1,\alpha)} \bar{Q}^k_\beta = Q^{k-1}_\alpha.$$

This easily implies (2.6) and (2.7) in all the remaining cases.

Finally, (2.12)⇒(2.5) and (2.15)⇒(2.8) are clear from $\tilde{Q}^k_\alpha \subseteq Q^k_\alpha \subseteq \bar{Q}^k_\alpha$. To see that (2.16)⇒(2.9), observe that if $\ell \geq k$ and $Q^\ell_\beta \subseteq Q^k_\alpha$, then (2.6) and (2.13) imply that $(\ell,\beta) \leq (k,\alpha)$, and thus (2.16) can be used. ∎

2.20. Remark. We work with several adjacent sets of cubes, as stated. Given a system of dyadic points, the preliminary cubes $\hat{Q}^k_\alpha$ determine the open cubes $\tilde{Q}^k_\alpha$ and the closed cubes $\bar{Q}^k_\alpha$ unambiguously but not the half-open cubes Q^k_α as their construction involves selection. If we know the final half-open cubes Q^k_α as point sets, they determine the open cubes $\tilde{Q}^k_\alpha$ and the closed cubes $\bar{Q}^k_\alpha$ unambiguously as these are nothing but the set of interior points and the closure respectively. The family of the preliminary cubes $\hat{Q}^k_\alpha$, however, is not uniquely determined by the point sets Q^k_α. But usually we think that the cubes Q^k_α carry the information of their centre point z^k_α and generation k as well. The ideology here is the very same as when defining a metric ball.

2.21. **Existence of dyadic points.** Consider a maximal collection of points $x^k_\alpha \in X$ satisfying the two inequalities

$$\rho(x^k_\alpha, x^k_\beta) \geq c_0\delta^k \quad (\alpha\neq\beta), \qquad \min_\alpha \rho(x, x^k_\alpha) < C_0\delta^k \quad \forall x \in X \tag{2.22}$$

with constants $c_0 = 1 = C_0$. It follows from the maximality argument that such a point set exists for any given $\delta \in (0,1)$ and every $k \in \mathbb{Z}$. From the first condition and the geometric doubling property it follows that a minimum in the second condition is indeed attained. Note that we may, of course, choose maximal point sets in such a way that given a fixed point $x_0 \in X$, for every

$k \in \mathbb{Z}$, there exists α such that $x_\alpha^k = x_0$. Finally, we may choose δ such that the restriction (2.3) introduced in Theorem 2.2 holds.

3. Verification of the properties. This section contains the proof of Proposition 2.11, which consists of the more technical aspects of Theorem 2.2. Suppose that for every $k \in \mathbb{Z}$ we have a set of points with properties (2.4) and constants that satisfy (2.3) as in Theorem 2.2. Recall that $c_1 := (3A_0^2)^{-1}c_0$ and $C_1 := 2A_0C_0$. We start with simple inclusion properties.

3.1. Lemma. *If $(\ell,\beta) \le (k,\alpha)$, then $\rho(z_\alpha^k, z_\beta^\ell) < C_1\delta^k$.*

Proof. Consider the chain

$$(k,\alpha) = (k,\gamma_0) \ge (k+1,\gamma_1) \ge \cdots \ge (k+(\ell-k),\gamma_{\ell-k}) = (\ell,\beta)$$

with $\rho(z_{\gamma_i}^{k+i}, z_{\gamma_{i+1}}^{k+i+1}) \le C_0\delta^{k+i}$ for all $i \in \{0,\dots,l-k-1\}$. By iterating the triangle inequality,

$$\begin{aligned}\rho(z_\alpha^k, z_\beta^\ell) &\le \sum_{i=0}^{k-\ell-1} A_0^{i+1}\rho(z_{\gamma_i}^{k+i}, z_{\gamma_{i+1}}^{k+i+1})\\ &\le \sum_{i=0}^{k-\ell-1} A_0^{i+1}C_0\delta^{k+i} < \frac{A_0C_0\delta^k}{1-A_0\delta} \le 2A_0C_0\delta^k. \blacksquare\end{aligned}$$

3.2. Lemma (Containing balls; (2.15)(ii) and (2.16)). *We have $\bar{Q}_\alpha^k \subseteq B(z_\alpha^k, C_1\delta^k)$, and also $B(z_\beta^\ell, C_1\delta^k) \subseteq B(z_\alpha^k, C_1\delta^k)$ for all $(\ell,\beta) \le (k,\alpha)$.*

Proof. For the first inclusion, let $x \in \bar{Q}_\alpha^k$; hence it is a limit of some $x_r \in \hat{Q}_\alpha^k$, $r \in \mathbb{Z}_+$. If $x_r = z_\alpha^k$ infinitely often, then also $x = z_\alpha^k$, and there is nothing to prove. Otherwise, infinitely many x_r are of the form z_β^ℓ for some $(\ell,\beta) = (\ell(r),\beta(r)) \le (k+1,\gamma) = (k+1,\gamma(r)) \le (k,\alpha)$, and then

$$\begin{aligned}\rho(z_\alpha^k, x) &\le A_0\rho(z_\alpha^k, z_\gamma^{k+1}) + A_0^2\rho(z_\gamma^{k+1}, z_\beta^\ell) + A_0^2\rho(z_\beta^\ell, x)\\ &\le A_0C_0\delta^k + A_0^2\cdot 2A_0C_0\delta^{k+1} + A_0^2\rho(x_r,x)\\ &< A_0C_0\delta^k + \tfrac{1}{2}A_0C_0\delta^k + \tfrac{1}{2}A_0C_0\delta^k = 2A_0C_0\delta^k\end{aligned} \tag{3.3}$$

for such x_r with $r \ge r_0$, since $\ell \ge k+1$ and $4A_0^2\delta \le 1$.

The inclusion between the balls is clear if $\ell = k$, so let again $(\ell,\beta) \le (k+1,\gamma) \le (k,\alpha)$. Let $x \in B(z_\beta^\ell, C_1\delta^\ell)$. As in (3.3), we deduce (now using $\rho(z_\beta^\ell, x) \le C_1\delta^\ell$ instead of $\rho(z_\beta^\ell, x) = \rho(z_\beta^\ell, x_r)$) that

$$\rho(z_\alpha^k, x) < A_0C_0\delta^k + A_0^2\cdot 2A_0C_0\delta^{k+1} + A_0^2\cdot C_1\delta^\ell \le 2A_0C_0\delta^k. \blacksquare$$

3.4. Lemma. *For any Λ, $\bigcup_{\alpha\in\Lambda}\bar{Q}_\alpha^k$ is the closure of $\bigcup_{\alpha\in\Lambda}\hat{Q}_\alpha^k$; in particular, this union is closed. Hence the open cubes $\tilde{Q}_\alpha^k := (\bigcup_{\gamma\ne\alpha}\bar{Q}_\gamma^k)^c$ are indeed open sets.*

Proof. Let $k \in \mathbb{Z}$ be fixed. It is clear that each $\bar{Q}^k_\alpha$, $\alpha \in \mathit{\Lambda}$, is a subset of the closure of $\bigcup_{\alpha\in\mathit{\Lambda}} \hat{Q}^k_\alpha$. From the geometric doubling property and the inclusion $\hat{Q}^k_\alpha \subseteq B(z^k_\alpha, C_1\delta^k)$, it follows readily that a bounded set can intersect at most finitely many different $\hat{Q}^k_\alpha$. Hence, if a convergent, thus bounded, sequence of points x_r belong to $\bigcup_{\alpha\in\mathit{\Lambda}} \bar{Q}^k_\alpha$, then they belong to some sub-union with a finite $\mathit{\Lambda}_1 \subseteq \mathit{\Lambda}$ in place of $\mathit{\Lambda}$. A union of finitely many closed sets is closed, so also the limit of the sequence (x_r) must belong to the same union. Thus all limit points of $\bigcup_{\alpha\in\mathit{\Lambda}} \hat{Q}^k_\alpha$ belong to $\bigcup_{\alpha\in\mathit{\Lambda}} \bar{Q}^k_\alpha$. ∎

3.5. Lemma (Unions of closed cubes; (2.14) and (2.17)). *For all $k, \ell \in \mathbb{Z}$ with $\ell > k$, we have*

$$X = \bigcup_\alpha \bar{Q}^k_\alpha, \qquad \bar{Q}^k_\alpha = \bigcup_{\beta\colon (\ell,\beta)\le(k,\alpha)} \bar{Q}^\ell_\beta.$$

Proof. The union $\bigcup_\alpha \hat{Q}^k_\alpha$ contains the points z^ℓ_β with $\ell \ge k$ and β arbitrary, which are dense in X by (2.4). Hence the closure of this union is X, but it is also equal to $\bigcup_\alpha \bar{Q}^k_\alpha$ by Lemma 3.4.

We turn to the second identity, first with $\ell = k+1$. It is clear that

$$\hat{Q}^k_\alpha = \{z^k_\alpha\} \cup \bigcup_{\beta\colon (k+1,\beta)\le(k,\alpha)} \hat{Q}^{k+1}_\beta;$$

hence by taking closures with the help of Lemma 3.4,

$$\bar{Q}^k_\alpha = \{z^k_\alpha\} \cup \bigcup_{\beta\colon (k+1,\beta)\le(k,\alpha)} \bar{Q}^{k+1}_\beta.$$

Since the cubes $\bar{Q}^{k+1}_\beta$ cover all X, it is clear that $z^k_\alpha \in \bar{Q}^{k+1}_\beta \subseteq B(z^{k+1}_\beta, C_1\delta^{k+1})$ for some β, and we only need to check that $(k+1,\beta) \le (k,\alpha)$. But this follows, using $4A_0^2C_0\delta \le c_0$, from

$$\rho(z^k_\alpha, z^{k+1}_\beta) < C_1\delta^{k+1} = 2A_0C_0\delta\cdot\delta^k \le \frac{c_0}{2A_0}\delta^k.$$

The case of a general $\ell > k$ follows by an $(\ell - k)$-fold iteration of the identity for $\ell = k+1$. ∎

3.6. Lemma. $\bar{Q}^k_\beta \subseteq B(z^k_\alpha, c_1\delta^k)^c$ *for* $\beta \ne \alpha$.

Proof. By Lemma 3.5, we need to show that $\bar{Q}^{k+1}_\gamma \subseteq B(z^k_\alpha, c_1\delta^k)^c$ for all $(k+1,\gamma) \le (k,\beta)$. If not, then $\bar{Q}^{k+1}_\gamma \subseteq B(z^{k+1}_\gamma, C_1\delta^{k+1})$ and $B(z^k_\alpha, c_1\delta^k)$ have a common point x; whence

$$\begin{aligned}\rho(z^{k+1}_\gamma, z^k_\alpha) &\le A_0\rho(z^{k+1}_\gamma, x) + A_0\rho(x, z^k_\alpha) < A_0C_1\delta^{k+1} + A_0c_1\delta^k \\ &= \left(2A_0^2C_0\delta + \frac{c_0}{3A_0}\right)\delta^k \le \left(\frac{c_0}{6A_0} + \frac{c_0}{3A_0}\right)\delta^k = \frac{c_0}{2A_0}\delta^k,\end{aligned}$$

by $12A_0^3C_0\delta \le c_0$, and this implies that $(k+1,\gamma) \le (k,\alpha)$, a contradiction with $\alpha \ne \beta$. ∎

3.7. LEMMA (Nestedness and contained balls; (2.13) and (2.15)(i)). *If $\ell \geq k$, then $\tilde{Q}^\ell_\beta \subseteq \tilde{Q}^k_\alpha$ for $(\ell,\beta) \leq (k,\alpha)$, or $\tilde{Q}^\ell_\beta \cap \bar{Q}^k_\alpha = \bar{Q}^\ell_\beta \cap \tilde{Q}^k_\alpha = \emptyset$ otherwise. Moreover, $\hat{Q}^k_\alpha \subseteq \tilde{Q}^k_\alpha \subseteq \bar{Q}^k_\alpha$ and $B(z^k_\alpha, c_1\delta^k) \subseteq \tilde{Q}^k_\alpha$.*

Proof. Let first $(\ell,\beta) \leq (k,\alpha)$. Then

$$(\tilde{Q}^k_\alpha)^c = \bigcup_{\gamma\neq\alpha} \bar{Q}^k_\gamma = \bigcup_{\gamma\neq\alpha}\ \bigcup_{\eta\colon(\ell,\eta)\leq(k,\gamma)} \bar{Q}^\ell_\eta = \bigcup_{\eta\colon(\ell,\eta)\not\leq(k,\alpha)} \bar{Q}^\ell_\eta \subseteq \bigcup_{\eta\neq\beta} \bar{Q}^\ell_\eta = (\tilde{Q}^\ell_\beta)^c;$$

hence $\tilde{Q}^\ell_\beta \subseteq \tilde{Q}^k_\alpha$. By Lemma 3.6, we have

$$B(z^\ell_\beta, c_1\delta^\ell)^c \supseteq \bigcup_{\eta\neq\beta} \bar{Q}^\ell_\eta = (\tilde{Q}^\ell_\beta)^c;$$

thus $z^\ell_\beta \in B(z^\ell_\beta, c_1\delta^\ell) \subseteq \tilde{Q}^\ell_\beta \subseteq \tilde{Q}^k_\alpha$. This gives both $\hat{Q}^k_\alpha \subseteq \tilde{Q}^k_\alpha$ and $B(z^k_\alpha, c_1\delta^k) \subseteq \tilde{Q}^k_\alpha$.

To see that $\tilde{Q}^k_\alpha \subseteq \bar{Q}^k_\alpha$, observe from Lemma 3.5 that $X = \bigcup_\alpha \bar{Q}^k_\alpha = \bar{Q}^k_\alpha \cup \bigcup_{\beta\neq\alpha} \bar{Q}^k_\beta$, so that

$$\tilde{Q}^k_\alpha = \Big(\bigcup_{\beta\neq\alpha} \bar{Q}^k_\beta\Big)^c \subseteq \bar{Q}^k_\alpha.$$

Let then $(\ell,\beta) \not\leq (k,\alpha)$, and thus $(\ell,\beta) \leq (k,\gamma)$ for some $\gamma \neq \alpha$. By what was already proven, $\tilde{Q}^\ell_\beta \subseteq \tilde{Q}^k_\gamma \subseteq (\bar{Q}^k_\alpha)^c$, and (taking closures) $\bar{Q}^\ell_\beta \subseteq \bar{Q}^k_\gamma \subseteq \bigcup_{\eta\neq\alpha} \bar{Q}^k_\eta = (\tilde{Q}^k_\alpha)^c$. ∎

3.8. LEMMA (Closure and interior; (2.12)). *The cubes $\bar{Q}^k_\alpha$ and $\tilde{Q}^k_\alpha$ are each other's closure and interior.*

Proof. From $\hat{Q}^k_\alpha \subseteq \tilde{Q}^k_\alpha \subseteq \bar{Q}^k_\alpha$ and the fact that $\bar{Q}^k_\alpha$ is the closure of $\hat{Q}^k_\alpha$, it is clear that it is also the closure of $\tilde{Q}^k_\alpha$.

Concerning the interior, it is clear that the open set $\tilde{Q}^k_\alpha \subseteq \bar{Q}^k_\alpha$ is a subset of the interior of $\bar{Q}^k_\alpha$. For the other direction, observe that $\hat{Q}^k_\beta \subseteq \tilde{Q}^k_\beta \subseteq (\bar{Q}^k_\alpha)^c$ for all $\beta \neq \alpha$; hence

$$\overline{(\bar{Q}^k_\alpha)^c} \supseteq \overline{\bigcup_{\beta\neq\alpha} \hat{Q}^k_\beta} = \bigcup_{\beta\neq\alpha} \bar{Q}^k_\beta = (\tilde{Q}^k_\alpha)^c.$$

Thus the interior of $\bar{Q}^k_\alpha$, which is the complement of $\overline{(\bar{Q}^k_\alpha)^c}$, is a subset of $(\tilde{Q}^k_\alpha)^{cc} = \tilde{Q}^k_\alpha$. ∎

4. Adjacent dyadic systems. In this section we will prove the following theorem.

4.1. THEOREM. *Given a set of reference points $\{x^k_\alpha\}$, $k \in \mathbb{Z}$, $\alpha \in \mathscr{A}_k$, suppose that a constant $\delta \in (0,1)$ satisfies $96A_0^6\delta \leq 1$. Then there exists a finite collection of families $\mathscr{D}^t$, $t = 1,\ldots,K = K(A_0,A_1,\delta) < \infty$, where* *each* $\mathscr{D}^t$ *is a collection of dyadic cubes, associated to dyadic points* $\{z_\alpha^k\}$, $k \in \mathbb{Z}$, $\alpha \in \mathscr{A}_k$, *with the properties* (2.5)–(2.9) *of Theorem* 2.2. *In addition,*

(4.2) *for every ball* $B = B(x, r) \subseteq X$, *there exist* t *and* $Q \in \mathscr{D}^t$ *with* $B \subseteq Q$ *and* $\operatorname{diam}(Q) \leq Cr$.

The constant $C < \infty$ *in* (4.2) *only depends on the quasi-metric constant* A_0 *and parameter* δ.

We will also prove the following variant of Theorem 4.1:

4.3. Proposition. *Given a fixed point* $x_0 \in X$, *there exists a finite collection of families* $\mathscr{D}^t$, $t = 1, \ldots, K = K(A_0, A_1, \delta) < \infty$, *where each* $\mathscr{D}^t$ *is a collection of dyadic cubes with the properties* (2.5)–(2.9) *of Theorem* 2.2, *and the property* (4.2) *of Theorem* 4.1 *is satisfied. In addition, for every* $t = 1, \ldots, K$ *and* k, *there exists* α *such that* $x_0 = z_\alpha^k$, *the centre point of* $Q_\alpha^k \in \mathscr{D}^t$.

4.4. **Reference dyadic points.** Recall from 2.21 that for every $k \in \mathbb{Z}$ there exists a point set $\{x_\alpha^k\}_{\alpha \in \mathscr{A}_k}$ such that

$$\rho(x_\alpha^k, x_\beta^k) \geq \delta^k \quad (\alpha \neq \beta), \qquad \min_\alpha \rho(x, x_\alpha^k) < \delta^k \quad \forall x \in X.$$

We will refer to the set $\{x_\alpha^k\}_{k,\alpha}$ of dyadic points as the set of *reference points.*

Suppose that $\delta \in (0,1)$ satisfies $96A_0^6\delta \leq 1$, and set $c_0 := (4A_0^2)^{-1}$. In particular, $\delta < c_0$ and

$$\frac{1}{2A_0^2} - \delta > \frac{1}{2A_0^2} - c_0 = c_0. \tag{4.5}$$

4.6. Definition. Reference points x_α^k and x_β^k, $\alpha \neq \beta$, of the same generation are *in conflict* if

$$\rho(x_\alpha^k, x_\beta^k) < c_0\delta^{k-1}.$$

4.7. Definition. Reference points x_α^k and x_β^k, $\alpha \neq \beta$, of the same generation are *neighbours* if there is a conflict between their children. More precisely, the points x_α^k and x_β^k, $\alpha \neq \beta$, are neighbours if there exist points $(k+1, \gamma) \leq (k, \alpha)$ and $(k+1, \sigma) \leq (k, \beta)$ such that $\rho(x_\gamma^{k+1}, x_\sigma^{k+1}) < c_0\delta^k$.

Recall from 2.10 that due to the doubling property, a dyadic point can have at most M children (a constant independent of the point). By similar arguments, also the number of neighbours of a dyadic point is bounded from above by a fixed constant, say L.

4.8. **Labelling of the reference points.** Fix $k \in \mathbb{Z}$. We label the reference points x_α^k of generation k as follows: Begin with some index pair (k, α) and label it with 0. Then, for every (k, β), $\beta \neq \alpha$, check whether any of its neighbours (boundedly many) already have a label. If not, label it with 0. Otherwise, pick the *smallest* positive integer not yet in use among the neighbours. As the number of neighbours a point can have is bounded above by L, every point x^k_α gets a *primary label* $\mathrm{label}_1(k,\alpha) := \ell$ not greater than L. Furthermore if (k,α) and (k,β), $\alpha \neq \beta$, have the same label $\ell \in \{0,\dots,L\}$, they are not neighbours.

Next we label the reference points x^{k+1}_γ of the following generation $k+1$ with duplex labels: If $\mathrm{label}_1(k,\alpha) = \ell$, each of its children $(k+1,\beta) \leq (k,\alpha)$ (boundedly many) gets a different *duplex label* $\mathrm{label}_2(k+1,\beta) := (\ell, m), m = m(\beta) \in \{1,\dots,M\}$. Then if $(k+1,\gamma)$ and $(k+1,\sigma)$, $\gamma \neq \sigma$, have the same primary label $\ell \in \{0,\dots,L\}$, they are not in conflict.

We next define new dyadic points z^k_α of generation k by selecting them from the set of reference points of generation $k+1$. We will first allow an almost free selection and then consider a more specific choice.

4.9. Definition (General selection rule). For every $k \in \mathbb{Z}$, pick $\ell = \ell_k \in \{0,\dots,L\}$ which we refer to as the *master label.* For every α, check if $\mathrm{label}_1(k,\alpha) = \ell$. If so, pick any $(k+1,\beta) \leq (k,\alpha)$ and declare that $(k,\alpha) \searrow (k+1,\beta)$ and $(k,\alpha) \not\searrow (k+1,\gamma)$ for every $\gamma \neq \beta$. Also set $z^k_\alpha := x^{k+1}_\beta$.

Otherwise, pick some $(k+1,\beta) \leq (k,\alpha)$ with $\rho(x^{k+1}_\beta, x^k_\alpha) < \delta^{k+1}$ (such a child always exists) and declare that $(k,\alpha) \searrow (k+1,\beta)$ and $(k,\alpha) \not\searrow (k+1,\gamma)$ for every $\gamma \neq \beta$. Also set $z^k_\alpha := x^{k+1}_\beta$. Note that, by construction, every (k,α) is related to some $(k+1,\beta)$ by the relation $\searrow$ but it may or may not be related to any $(k-1,\sigma)$.

The point sets obtained from the reference points by the general selection rule have the distribution property (2.4) of Theorem 2.2:

4.10. Lemma. *Set $c_0 := (4A_0^2)^{-1}$ and $C_0 := 2A_0$. Let $\{z^k_\alpha\}$ be the set of new dyadic points obtained by the general selection rule from the reference points. Then for every $k \in \mathbb{Z}$ we have*

$$\rho(z^k_\alpha, z^k_\beta) \geq c_0\delta^k, \qquad \alpha \neq \beta,$$

and for every $x \in X$ and every $k \in \mathbb{Z}$ we find α such that

$$\rho(x, z^k_\alpha) < C_0\delta^k.$$

Proof. Let us fix $k \in \mathbb{Z}$ and $\ell = \ell_k \in \{0,\dots,L\}$. By the general selection rule, $z^k_\alpha = x^{k+1}_\gamma$ and $z^k_\beta = x^{k+1}_\sigma$ for some reference points x^{k+1}_γ and x^{k+1}_σ. First assume that at least one of the reference points x^k_α and x^k_β has primary label different from the master label ℓ. We may, without loss of generality, assume that $\mathrm{label}_1(k,\alpha) \neq \ell$. This implies $\rho(z^k_\alpha, x^k_\alpha) < \delta^{k+1}$. Since $(k+1,\sigma)$ is not a child of (k,α), we have $\rho(z^k_\beta, x^k_\alpha) \geq (2A_0)^{-1}\delta^k$. Thus,

$$(2A_0)^{-1}\delta^k \leq \rho(z^k_\beta, x^k_\alpha) \leq A_0\rho(x^k_\alpha, z^k_\alpha) + A_0\rho(z^k_\alpha, z^k_\beta) \leq A_0\delta^{k+1} + A_0\rho(z^k_\alpha, z^k_\beta),$$

or equivalently, by (4.5),

$$\rho(z^k_\alpha, z^k_\beta) \geq \frac{1}{A_0}\left(\frac{\delta^k}{2A_0} - A_0\delta^{k+1}\right) = \left(\frac{1}{2A_0^2} - \delta\right)\delta^k > c_0\delta^k.$$

Otherwise, both x^k_α and x^k_β have primary label ℓ. In this case, the points x^k_α and x^k_β are not neighbours and there is no conflict between their children. The first assertion follows.

Fix $x \in X$. There exists α such that $\rho(x, x^k_\alpha) < \delta^k$ where x^k_α is a reference point. As z^k_α is a child of x^k_α, we have $\rho(x^k_\alpha, z^k_\alpha) < \delta^k$. It follows that

$$\rho(x, z^k_\alpha) \leq A_0\rho(x, x^k_\alpha) + A_0\rho(x^k_\alpha, z^k_\alpha) < A_0\delta^k + A_0\delta^k = 2A_0\delta^k. \blacksquare$$

4.11. DEFINITION (Specific selection rule). Fix $(\ell, m) \in \{0, \ldots, L\} \times \{1, \ldots, M\}$. For every index pair (k, α), check whether there exists $(k+1, \beta) \leq (k, \alpha)$ with label pair (ℓ, m). If so, set $z^k_\alpha := x^{k+1}_\beta$. Otherwise, pick some $(k+1, \beta) \leq (k, \alpha)$ with $\rho(x^{k+1}_\beta, x^k_\alpha) < \delta^{k+1}$ and set $z^k_\alpha := x^{k+1}_\beta$.

Note that the specific selection rule is more precise than the general selection rule. Indeed, in case a reference point has primary label the same as the master label ℓ, we do not just choose any child but the one with duplex label (ℓ, m) (if one exists). Thus, it is a special case of the general selection rule. In particular, the point sets obtained from the reference points by the specific selection rule satisfy the distribution properties of Lemma 4.10.

Let φ be a bijection $\{0, \ldots, L\} \times \{1, \ldots, M\} \to \{1, \ldots, K\} \subset \mathbb{N}$, $(\ell, m) \mapsto t$. We identify $t = \varphi(\ell, m)$ with (ℓ, m). Each t gives rise to a set $\{{}^t z^k_\alpha : k \in \mathbb{Z},\, \alpha \in \mathscr{A}_k\}$ of new dyadic points associated with the duplex label $(\ell, m) = t$. Note that, by repeating the specific selection rule for every ordered pair of labels (ℓ, m), Lemma 4.10 and Theorem 2.2 complete the proof of the first part of Theorem 4.1. We denote by $\mathscr{D}^t$ the family of dyadic cubes $Q^k_\alpha = {}^t Q^k_\alpha$ corresponding to the point set $\{{}^t z^k_\alpha \colon k \in \mathbb{Z},\, \alpha \in \mathscr{A}_k\}$, $t = 1, \ldots, K$.

The collection of dyadic families $\mathscr{D}^t$, $t = 1, \ldots, K$, obtained by repeating the specific selection rule with all the choices of (ℓ, m) have the following property:

4.12. LEMMA. *For every ball $B = B(x, r) \subseteq X$ there exist an integer t and a dyadic cube $Q \in \mathscr{D}^t$ such that*

$$B \subseteq Q \quad \textit{and} \quad \operatorname{diam}(Q) \leq Cr,$$

where $C = C(A_0, \delta)$ is a constant independent of x and t.

4.13. REMARK. Note that the proof will show that for r with $\delta^{k+2} < r \leq \delta^{k+1}$, we may assume that the containing cube $Q \in \mathscr{D}^t$ is of generation k. Further, $\rho(x, z^k_\alpha) < \delta^{k+1}$ where z^k_α denotes the centre of Q. Also note that this is the second part of Theorem 4.1.

Proof of Lemma 4.12. Fix $B(x,r) \subseteq X, r > 0$, and pick $k \in \mathbb{Z}$ so that $\delta^{k+2} < r \leq \delta^{k+1}$. There exists a reference point x_β^{k+1} with duplex label, say (ℓ, m), such that

$$\rho(x, x_\beta^{k+1}) < \delta^{k+1}.$$

Let α be the unique index for which $(k+1, \beta) \leq (k, \alpha)$. Then $x_\beta^{k+1} = {}^t z_\alpha^k$, which is a new dyadic point of generation k in the system $\mathscr{D}^t$, $t = (\ell, m)$. We will prove that $B(x,r) \subseteq B({}^t z_\alpha^k, c_1\delta^k)$ where $c_1 = (3A_0^2)^{-1}c_0 = (12A_0^4)^{-1}$. Indeed, suppose $y \in B(x,r)$. Then

$$\rho(y, {}^t z_\alpha^k) \leq A_0\rho(y,x) + A_0\rho(x, {}^t z_\alpha^k) < A_0 r + A_0\delta^{k+1} \leq 2A_0\delta^{k+1} \leq c_1\delta^k,$$

since $24A_0^5\delta \leq 1$. Thus, $y \in Q := {}^t Q_\alpha^k \in \mathscr{D}^t$, $t = (\ell, m)$. For the diameter of Q we get

$$\operatorname{diam}(Q) \leq \operatorname{diam}(B({}^t z_\alpha^k, C_1\delta^k)) \leq 2A_0C_1\delta^k = \frac{2A_0C_1}{\delta^2}\delta^{k+2} \leq Cr,$$

with $C := 2A_0C_1\delta^{-2} = 8A_0^3\delta^{-2}$. ∎

4.14. Definition (Specific selection rule with a distinguished point). Given $x_0 \in X$, recall from 2.21 that the set of reference points can be chosen in such a way that for every $k \in \mathbb{Z}$ there exists α such that $x_\alpha^k = x_0$. Fix $(\ell, m) \in \{0, \dots, L\} \times \{1, \dots, M\}$. For every k, begin with $x_0 = x_\alpha^k$ and set $z_\alpha^k := x_0$. For every $\beta \neq \alpha$, choose the new points z_β^k by the specific selection rule, as defined earlier.

Note that the specific selection rule with a distinguished point is again a special case of the general selection rule. It is not, however, a special case of the specific selection rule. Hence, we need to verify that the dyadic systems obtained by repeating the specific selection rule with a distinguished point with all the choices of (ℓ, m) also satisfy the assertions of Lemma 4.12:

4.15. Lemma. *Given a fixed point $x_0 \in X$, there exist finitely many point sets $\{z_\alpha^k\}_{k,\alpha}$ satisfying the assertions of Lemma* 4.10 *and having the property that for every $k \in \mathbb{Z}$ there exists α such that $z_\alpha^k = x_0$. In addition, the family of dyadic systems defined by these new dyadic points has the property of Lemma* 4.12.

4.16. Remark. Note that the proof will show that for r with $\delta^{k+2} < r \leq \delta^{k+1}$, we may assume that the containing cube $Q \in \mathscr{D}^t$ is of generation $k-1$. Further, $\rho(x, z_\alpha^k) < 2A_0\delta^k$ where z_α^k denotes the centre point of Q. Also note that Lemma 4.15 completes the proof of Proposition 4.3.

Proof of Lemma 4.15. The assertions of Lemma 4.10 are clear, since we are still in the regime of the general selection rule. We consider the assertion of Lemma 4.12. Fix a ball $B = B(x,r)$ in X with $\delta^{k+2} < r \leq \delta^{k+1}$. There exists a reference point x_β^{k+1} such that $\rho(x, x_\beta^{k+1}) < \delta^{k+1}$.

Assume first that $x^k_\alpha = x_0$ for a unique $(k,\alpha) \geq (k+1,\beta)$. Then also $x^{k-1}_\gamma = x_0$ for a unique $(k-1,\gamma) \geq (k,\alpha)$, implying that ${}^t z^{k-1}_\gamma = x_0$ by the specific selection rule with a distinguished point with every $t=(\ell,m)$. Take $y \in B(x,r)$. Then

$$\begin{aligned}\rho(y, {}^t z^{k-1}_\gamma) &= \rho(y,x_0) \leq A_0^2\rho(y,x) + A_0^2\rho(x,x^{k+1}_\beta) + A_0\rho(x^{k+1}_\beta, x_0)\\ &< A_0^2 r + A_0^2\delta^{k+1} + A_0\delta^k\\ &\leq (2A_0^2\delta^2 + A_0\delta)\delta^{k-1} \leq \left(2A_0^2\frac{1}{24^2A_0^{10}} + A_0\frac{1}{24A_0^5}\right)\delta^{k-1}\\ &< \frac{1}{12A_0^4}\delta^{k-1} = c_1\delta^{k-1}\end{aligned}$$

since $24A_0^5\delta \leq 1$ and $c_1 = (3A_0^2)^{-1}c_0 = (12A_0^4)^{-1}$. Thus, $y \in Q := {}^tQ^{k-1}_\gamma$ for any t. For the diameter of Q we have

$$\operatorname{diam}(Q) \leq \operatorname{diam}(B({}^t z^{k-1}_\gamma, C_1\delta^{k-1})) \leq 2A_0C_1\delta^{k-1} = \frac{2A_0C_1}{\delta^3}\delta^{k+2} \leq Cr,$$

with $C := 2A_0C_1\delta^{-3} = 8A_0^3\delta^{-3}$.

If $x^k_\alpha \neq x_0$ for $(k,\alpha) \geq (k+1,\beta)$, then the new dyadic point ${}^t z^k_\alpha$ is chosen among the x^{k+1}_σ with $(k+1,\sigma) \leq (k,\alpha)$ exactly as in the specific selection rule (without a distinguished point). Thus, the reference point x^{k+1}_β is ${}^t z^k_\alpha$ for $t = \text{label}_2(k+1,\beta)$, and the proof is completed by the same argument as in Lemma 4.12. ■

5. Random dyadic systems. In this section we will prove the following theorem, originally from [12]. The present contribution consists of a detailed and streamlined construction of the underlying probability space Ω, the details of which already turned out helpful in an application to singular integrals in [18].

5.1. THEOREM. *Given a set of reference points $\{x^k_\alpha\}$, $k \in \mathbb{Z}$, $\alpha \in \mathscr{A}_k$, suppose that a constant $\delta \in (0,1)$ satisfies $96A_0^6\delta \leq 1$. Then there exists a probability space $(\Omega,\mathbb{P})$ such that every $\omega \in \Omega$ defines a dyadic system $\mathscr{D}(\omega) = \{Q^k_\alpha(\omega)\}_{k,\alpha}$, related to new dyadic points $\{z^k_\alpha(\omega)\}_{k,\alpha}$, with the properties* (2.5)–(2.9) *of Theorem 2.2. Further, $(\Omega,\mathbb{P})$ has the following properties:*

$$\Omega = \prod_{k\in\mathbb{Z}} \Omega_k, \quad \omega = (\omega_k)_{k\in\mathbb{Z}} \in \Omega \text{ with } \omega_k \in \Omega_k \text{ independent}; \tag{5.2}$$

$$z^k_\alpha(\omega) = z^k_\alpha(\omega_k); \tag{5.3}$$

$$\text{if } (k+1,\beta) \leq (k,\alpha), \text{ then } \mathbb{P}(\{\omega \in \Omega\colon z^k_\alpha(\omega) = x^{k+1}_\beta\}) \geq \tau_0 > 0. \tag{5.4}$$

5.5. REMARK. In this section we will construct $(\Omega,\mathbb{P})$ by randomizing the choice of new dyadic points from the reference points with respect to all the possible degrees of freedom. The properties (5.2)–(5.4) can, however, be obtained with much less randomness. We will return to this in Section 6.

We will first state the following theorem which presents the general properties of all the random dyadic systems with the properties (5.2)–(5.4). For the slightly different random systems originally constructed in [12], the property (5.7) below was already established; its consequences stated as (5.8) and (5.9) were observed and applied in [11].

5.6. THEOREM. *Given a set of reference points* $\{x_\alpha^k\}$, $k \in \mathbb{Z}$, $\alpha \in \mathscr{A}_k$, *suppose that a constant* $\delta \in (0,1)$ *satisfies* $144A_0^8\delta \le 1$. *Suppose* $(\Omega, \mathbb{P})$ *is any probability space such that every* $\omega \in \Omega$ *defines a dyadic system* $\mathscr{D}(\omega) = \{Q_\alpha^k(\omega)\}_{k,\alpha}$, *related to new dyadic points* $\{z_\alpha^k(\omega)\}_{k,\alpha}$, *with the properties* (2.5)–(2.9) *of Theorem* 2.2. *Suppose further that* $(\Omega, \mathbb{P})$ *has the properties* (5.2)–(5.4) *of Theorem* 5.1. *Then the following probabilistic statements hold:*

- *For every* $x \in X, \tau > 0$ *and* $k \in \mathbb{Z}$,

$$\mathbb{P}\Big(\Big\{\omega \in \Omega \colon x \in \bigcup_\alpha \partial_{\tau\delta^k} Q_\alpha^k(\omega)\Big\}\Big) \le C_2\tau^\eta \quad \text{for some } C_2, \eta > 0, \tag{5.7}$$

  where

$$\partial_\varepsilon Q := \{x \in \bar{Q} \colon \rho(x, \tilde{Q}^c) \le \varepsilon\}, \qquad \varepsilon > 0.$$

- *For every* $x \in X$,

$$\mathbb{P}\Big(\Big\{\omega \in \Omega \colon x \in \bigcup_{k,\alpha} \partial Q_\alpha^k(\omega)\Big\}\Big) = 0. \tag{5.8}$$

- *Given a positive* σ*-finite measure* μ *on* X,

$$\mu\Big(\bigcup_{k,\alpha} \partial Q_\alpha^k(\omega)\Big) = 0 \quad \text{for a.e. } \omega \in \Omega. \tag{5.9}$$

5.10. **The probability space.** Keeping the fixed set $\{x_\alpha^k\}$, $k \in \mathbb{Z}$, $\alpha \in \mathscr{A}_k$, of reference points, we randomize the construction of new dyadic points from them. This amounts to formalizing the underlying space of all possible choices of new dyadic points allowed by the general selection rule, and then defining a natural probability measure on this space. The underlying probability space Ω will be formed by countable products and unions of finite probability spaces as follows: $\Omega := \prod_{k\in\mathbb{Z}} \Omega_k$ where

$$\begin{aligned}\Omega_k := \bigcup_{\ell_k \in \{0,\ldots,L\}} \Big(\{\ell_k\} \times & \prod_{\substack{\alpha\in\mathscr{A}_k \\ \mathrm{label}_1(k,\alpha)=\ell_k}} \{\gamma \colon (k+1,\gamma) \le (k,\alpha)\} \\ & \times \prod_{\substack{\alpha\in\mathscr{A}_k \\ \mathrm{label}_1(k,\alpha)\ne\ell_k}} \{\gamma \colon \rho(x_\gamma^{k+1}, x_\alpha^k) < \delta^{k+1}\}\Big).\end{aligned}$$

For the finite sets $\{\gamma\colon (k+1,\gamma)\leq(k,\alpha)\}$ and $\{\gamma\colon \rho(x^{k+1}_\gamma,x^k_\alpha)<\delta^{k+1}\}$, we use the σ-algebras consisting of all subsets. The σ-algebra $\mathscr{G}_k$ of the set Ω_k is generated by these sets. We will further consider

$$\mathscr{H}_k := \Big\{\prod_{j<k}\Omega_j\times G_k\times\prod_{j>k}\Omega_j\colon G_k\in\mathscr{G}_k\Big\},\qquad k\in\mathbb{Z}.$$

Then the σ-algebra $\mathscr{H}$ of Ω is generated by the σ-algebras $\mathscr{H}_k$.

The points $\omega\in\Omega$ admit the natural coordinate representation $\omega=(\omega_k)_{k\in\mathbb{Z}}$, where

$$\omega_k=(\ell_k,\omega_{k,\alpha}:\alpha\in\mathscr{A}_k)\in\Omega_k,$$

where $\ell_k\in\{0,\ldots,L\}$ and each $\omega_{k,\alpha}\in\mathscr{A}_{k+1}$ satisfies $(k+1,\omega_{k,\alpha})\leq(k,\alpha)$, as well as $\rho(x^{k+1}_{\omega_{k,\alpha}},x^k_\alpha)<\delta^{k+1}$ if $\mathrm{label}_1(k,\alpha)\neq\ell_k$.

We define a probability $\mathbb{P}$ on Ω by requiring the coordinates ω_k to be independent and distributed as follows: First,

$$\mathbb{P}(\ell_k=\ell)=\frac{1}{L+1}\qquad\forall\ell=0,\ldots,L.$$

Second, given the master label ℓ_k, the subcoordinates $\omega_{k,\alpha}$, $\alpha\in\mathscr{A}_k$, are again independent, with distribution

$$\begin{aligned}&\mathbb{P}(\omega_{k,\alpha}=\beta\mid\ell_k=\ell)\\ &\qquad=\begin{cases}[\#\{\gamma\colon(k+1,\gamma)\leq(k,\alpha)\}]^{-1} & \forall(k+1,\beta)\leq(k,\alpha)\\ & \qquad\text{if } \mathrm{label}_1(k,\alpha)=\ell_k,\\ [\#\{\gamma\colon\rho(x^{k+1}_\gamma,x^k_\alpha)<\delta^{k+1}\}]^{-1} & \forall(k+1,\beta):\rho(x^{k+1}_\beta,x^k_\alpha)<\delta^{k+1}\\ & \qquad\text{if } \mathrm{label}_1(k,\alpha)\neq\ell_k.\end{cases}\end{aligned}$$

Note that there is an obvious one-to-one correspondence between the coordinates $\omega_k\in\Omega_k$ and the admissible choices of the relation $\searrow$ between index pairs on levels k and $k+1$, subject to the general selection rule. This relation in turn uniquely determines the new dyadic points $z^k_\alpha=z^k_\alpha(\omega_k)$ for the given level $k\in\mathbb{Z}$, and thus the choice of $\omega=(\omega_k)_{k\in\mathbb{Z}}$ uniquely determines the new dyadic points $z^k_\alpha=z^k_\alpha(\omega)$ on all levels $k\in\mathbb{Z}$. By a random choice of the new dyadic points, we understand the new dyadic points $z^k_\alpha(\omega)$, where $\omega\in\Omega$ is distributed according to the probability $\mathbb{P}$.

Once the points $z^k_\alpha(\omega)$ are chosen, they uniquely determine the relation $\leq_\omega$ between the index pairs (k,α) (not to be confused with the original relation $\leq$, which is in general not the same); recall that it is possible to make the choice of $\leq_\omega$ in such a way that it only depends on the dyadic points without any arbitrariness. Then the points $z^k_\alpha(\omega)$ and the relation $\leq_\omega$ together determine the new dyadic cubes $Q^k_\alpha(\omega)$ as a function of $\omega\in\Omega$, and their random choice corresponds to the random choice of ω according to the law $\mathbb{P}$.

It is evident, by Lemma 4.10, that for every $\omega \in \Omega$, the dyadic system $\mathscr{D}(\omega)$ has the properties (2.5)–(2.9) of Theorem 2.2.

Note that by construction, for every $(k+1,\beta) \leq (k,\alpha)$,

$$\begin{aligned}\mathbb{P}(\{\omega \in \Omega\colon z_\alpha^k(\omega) = x_\beta^{k+1}\}) \\ \geq [(L+1)\#\{\gamma\colon (k+1,\gamma) \leq (k,\alpha)\}]^{-1} \geq [(L+1)M]^{-1} =: \tau_0 > 0.\end{aligned}$$

This completes the proof of Theorem 5.1.

5.11. **A technical lemma.** Before turning to a more thorough investigation of the random dyadic cubes just defined, we provide a technical lemma, which has nothing to do with the randomness, but is a general property of all dyadic systems. However, we will only make use of this lemma in the randomized context, which is the reason of including it in this section. Roughly speaking, the lemma states that in order to reach the boundary of a cube from its centre, along a direct line of ancestry of dyadic points, one needs to make jumps of non-trivial size at every step. This result goes back to Christ [4], and appeared as part of the proof of his Lemma 17. Christ's lemma concerned the smallness of the boundary region of the dyadic cubes with respect to an underlying doubling measure; the technical intermediate result is valid even without the presence of a measure, and we will apply it to get analogous smallness results for the boundary with respect to the probability $\mathbb{P}$ defined above.

5.12. Lemma. *Suppose that constants $0 < c_0 \leq C_0 < \infty$ and $\delta \in (0,1)$ satisfy $18A_0^5 C_0 \delta \leq c_0$. Let $\{z_\alpha^k\}_{k,\alpha}$ be a set of points as in Theorem 2.2. Given $N \in \mathbb{Z}_+$ and $\tau > 0$, suppose that $12A_0^4 \tau \leq c_0 \delta^N$. Let $x \in \bar{Q}_\alpha^k$ with $\rho(x, (\tilde{Q}_\alpha^k)^c) < \tau\delta^k$. For all chains*

$$(k+N,\sigma) = (k+N,\sigma_{k+N}) \leq \cdots \leq (k+1,\sigma_{k+1}) \leq (k,\sigma_k)$$

such that $x \in \bar{Q}_\sigma^{k+N}$, we have $\rho(z_{\sigma_j}^j, z_{\sigma_i}^i) \geq \varepsilon_1 \delta^j$, $\varepsilon_1 := (12A_0^4)^{-1} c_0$, for all $k \leq j < i \leq k+N$.

Proof. Let (k,α) be fixed and consider $x \in \bar{Q}_\alpha^k$ with $\rho(x, (\tilde{Q}_\alpha^k)^c) < \tau\delta^k$ for some $\tau > 0$. Let (j,σ_j) be the intermediate pairs as in the assertion, and abbreviate $z^j := z_{\sigma_j}^j$ for $k \leq j \leq k+N$. Suppose for contradiction that $\rho(z^j, z^i) < \varepsilon_1 \delta^j$ for some $k \leq j < i \leq k+N$. There are two possibilities: $\sigma_k = \alpha$ or not.

First assume $\sigma_k = \alpha$ (i.e. the chain travels in Q_α^k). Then, as $x \in \bar{Q}_\sigma^{k+N} \subset \bar{Q}_{\sigma_i}^i$, also $x \in B(z^i, C_1\delta^i)$ for $(i,\sigma_i) \geq (k+N,\sigma)$. We also have $B(z^j, c_1\delta^j) \subseteq \tilde{Q}_{\sigma_j}^j \subseteq \tilde{Q}_\alpha^k$, and so it follows that

$$\begin{aligned}c_1\delta^j &\le \rho(z^j,(\tilde{Q}^k_\alpha)^c) \le A_0\rho(x,(\tilde{Q}^k_\alpha)^c) + A_0^2\rho(x,z^i) + A_0^2\rho(z^i,z^j)\\ &< A_0\tau\delta^k + A_0^2C_1\delta^i + A_0^2\varepsilon_1\delta^j \le \frac{1}{4}c_1\delta^{N+k} + \frac{1}{3}c_1\delta^{i-1} + \frac{1}{4}c_1\delta^j \le c_1\delta^j,\end{aligned}$$

since $c_1 := (3A_0^2)^{-1}c_0$, $C_1 := 2A_0C_0$, $4A_0^2\tau \le c_1\delta^N$, $3A_0^2C_1\delta \le c_1$ and $4A_0^2\varepsilon_1 \le c_1$, and this is a contradiction.

If $\sigma_k \ne \alpha$ (and the chain travels outside Q^k_α), we have $x \in \bar{Q}^{k+N}_\sigma \subseteq \bar{Q}^k_{\sigma_k}$ and $\rho(x,(\tilde{Q}^k_{\sigma_k})^c) = 0 < \tau\delta^k$. Thus we are in the identical situation as before but with σ_k in place of α. Hence the same conclusion applies. ∎

5.13. **The proof of Theorem 5.6.** From now on, assume $(\Omega,\mathbb{P})$ is a probability space with the properties (5.2)–(5.4) of Theorem 5.1, and that $144A_0^8\delta \le 1$.

5.14. Definition (Boundary zone of a dyadic cube). For $\varepsilon > 0$, we denote

$$\partial_\varepsilon Q := \{x \in \bar{Q} \colon \rho(x,\tilde{Q}^c) \le \varepsilon\}.$$

We mention that if the space (X,ρ) supports a doubling measure μ, Lemma 5.12 has the following consequence [4, Lemma 17]: For every $\varepsilon > 0$ there exists $\tau \in (0,1]$ such that for every dyadic cube Q^k_α,

$$\mu(\partial_{\tau\delta^k}Q^k_\alpha) < \varepsilon\mu(Q^k_\alpha).$$

Here, we are concerned with the following probabilistic analogue:

5.15. Lemma ((5.7) of Theorem 5.6). *For a given $x \in X$ and $\tau > 0$ and a fixed $k \in \mathbb{Z}$,*

$$\mathbb{P}\Big(\Big\{\omega \in \Omega \colon x \in \bigcup_\alpha \partial_{\tau\delta^k}Q^k_\alpha(\omega)\Big\}\Big) \le C_2\tau^\eta$$

for some constants $C_2, \eta > 0$.

5.16. **Reduction.** We consider the event

$$E = \Big\{\omega \in \Omega \colon x \in \bigcup_\alpha \partial_{\tau\delta^k}Q^k_\alpha(\omega)\Big\}.$$

First note that for every $x \in X$ and $k \in \mathbb{Z}$, there exists a finite set $A = A_k(x)$ of indices such that if $x \in \bar{Q}^k_\alpha(\omega)$ for any $\omega \in \Omega$, then $\alpha \in A_k(x)$. Moreover, $\#A_k(x) \le C < \infty$ where C is independent of x and k. Indeed, if $x \in \bar{Q}^k_\alpha(\omega)$ we have $\rho(x,z^k_\alpha(\omega)) < C_1\delta^k$, and thus

$$\rho(x,x^k_\alpha) \le A_0\rho(x,z^k_\alpha(\omega)) + A_0\rho(z^k_\alpha(\omega),x^k_\alpha) < A_0(C_1+1)\delta^k,$$

since $\rho(z^k_\alpha(\omega),x^k_\alpha) < \delta^k$ by the choice of $z^k_\alpha(\omega)$. By the geometric doubling property, the ball $B(x,A_0(C_1+1)\delta^k)$ can contain at most boundedly many centres x^k_α of the disjoint balls

$$B(x^k_\alpha,(2A_0)^{-1}\delta^k).$$

In particular, if $x \in \bigcup_\alpha \partial_{\tau\delta^k} Q^k_\alpha(\omega)$, then $x \in \bigcup_{\alpha\in A_k(x)} \partial_{\tau\delta^k} Q^k_\alpha(\omega)$ where $\#A_k(x) \le C < \infty$ and C is independent of x and k. Since the closed dyadic cubes of any generation $k+N$ cover X, it follows that

$$\begin{aligned} E &= \Big\{\omega \in \Omega\colon x \in \Big(\bigcup_\alpha \partial_{\tau\delta^k} Q^k_\alpha(\omega)\Big) \cap \Big(\bigcup_\sigma \bar{Q}^{k+N}_\sigma(\omega)\Big)\Big\} \\ &= \Big\{\omega \in \Omega\colon x \in \Big(\bigcup_{\alpha\in A_k(x)} \partial_{\tau\delta^k} Q^k_\alpha(\omega)\Big) \cap \Big(\bigcup_{\sigma\in A_{k+N}(x)} \bar{Q}^{k+N}_\sigma(\omega)\Big)\Big\} \\ &= \Big\{\omega \in \Omega\colon x \in \bigcup_{\substack{\alpha\in A_k(x)\\ \sigma\in A_{k+N}(x)}} (\partial_{\tau\delta^k} Q^k_\alpha(\omega) \cap \bar{Q}^{k+N}_\sigma(\omega))\Big\}. \end{aligned}$$

Here the union is bounded, and we have

$$\mathbb{P}(E) \le \sum_{\substack{\alpha\in A_k(x)\\ \sigma\in A_{k+N}(x)}} \mathbb{P}(\{\omega \in \Omega\colon x \in \partial_{\tau\delta^k} Q^k_\alpha(\omega) \cap \bar{Q}^{k+N}_\sigma(\omega)\}).$$

Thus, in order to prove Lemma 5.15, it suffices to prove that

$$\mathbb{P}(\{\omega \in \Omega\colon x \in \partial_{\tau\delta^k} Q^k_\alpha(\omega) \cap \bar{Q}^{k+N}_\sigma(\omega)\}) \le C_2\tau^\eta$$

for some constants $C_2, \eta > 0$ with fixed $x \in X$, $\tau > 0$, $N \in \mathbb{N}$, $k \in \mathbb{Z}$, α and σ.

We first state the following basic probability lemma, which we have included for the convenience of readers less experienced with conditional expectations; this is essentially the only place where probabilistic reasoning beyond standard measure theory will be needed.

5.17. Lemma. *Let $\{\mathscr{F}_j\}$, $j = 1, \dots, k$, be a finite collection of σ-algebras and suppose that $\mathscr{F}_{j+1} \subseteq \mathscr{F}_j$ and $A_j \in \mathscr{F}_j$ for all j. Then*

$$\mathbb{E}\prod_{j=1}^{k-1} \chi_{A_j} = \mathbb{E}\mathbb{E}_{\mathscr{F}_k}\chi_{A_{k-1}}\mathbb{E}_{\mathscr{F}_{k-1}}\chi_{A_{k-2}} \dots \mathbb{E}_{\mathscr{F}_2}\chi_{A_1}$$

where $\mathbb{E}_{\mathscr{F}_j}[\,\cdot\,] := \mathbb{E}[\,\cdot\,|\mathscr{F}_j]$ denotes the conditional expectation given $\mathscr{F}_j$.

Proof. By the properties of conditional expectation (see for example [29, §9.7],

$$\mathbb{E}\Big(\prod_{j=1}^{k-1}\chi_{A_j}\Big) = \mathbb{E}\Big(\mathbb{E}_{\mathscr{F}_k}\Big[\prod_{j=1}^{k-1}\chi_{A_j}\Big]\Big).$$

First we use the so-called Tower Property: Since $\mathscr{F}_k \subseteq \mathscr{F}_{k-1}$, we have

$$\mathbb{E}_{\mathscr{F}_k}\Big[\prod_{j=1}^{k-1}\chi_{A_j}\Big] = \mathbb{E}_{\mathscr{F}_k}\Big[\mathbb{E}_{\mathscr{F}_{k-1}}\Big[\prod_{j=1}^{k-1}\chi_{A_j}\Big]\Big]. \tag{5.18}$$

Secondly, the fact that $\chi_{A_{k-1}}$ is $\mathscr{F}_{k-1}$-measurable implies that

$$\mathbb{E}_{\mathscr{F}_{k-1}}\Big[\prod_{j=1}^{k-1}\chi_{A_j}\Big] = \chi_{A_{k-1}}\mathbb{E}_{\mathscr{F}_{k-1}}\Big[\prod_{j=1}^{k-2}\chi_{A_j}\Big]. \tag{5.19}$$

We now repeat steps (5.18) and (5.19) $k-1$ times to conclude that

$$\mathbb{E}\Big(\prod_{j=1}^{k-1}\chi_{A_j}\Big) = \mathbb{E}(\mathbb{E}_{\mathscr{F}_k}[\chi_{A_{k-1}}\mathbb{E}_{\mathscr{F}_{k-1}}[\chi_{A_{k-2}}\cdots\chi_{A_2}\mathbb{E}_{\mathscr{F}_2}[\chi_{A_1}]]\ldots]). \blacksquare$$

Proof of Lemma 5.15. Fix $x\in X$ and $k\in\mathbb{Z}$. Given $\tau\in(0,(4A_0^2)^{-1}c_1)$, pick the unique $N\in\mathbb{N}:=\{0,1,\ldots\}$ so that $c_1\delta^{N+1}<4A_0^2\tau\leq c_1\delta^N$. (Since all probability is at most 1, the claim is of course true for all η when $\tau\geq(4A_0^2)^{-1}c_1$, if we take C_2 large enough.) Note that, in particular, $12A_0^4\tau\leq c_0\delta^N$ since $c_1:=(3A_0^2)^{-1}c_0$. Also note that, by the assumption $144A_0^8\delta\leq 1$, we have $18A_0^5C_0\delta\leq c_0$ since $c_0=(4A_0^2)^{-1}$ and $C_0=2A_0$. Thus, the parameter assumptions of Lemma 5.12 hold. By the reduction in 5.16, it suffices to consider the event

$$E_{\alpha,\sigma}:=\{\omega\in\Omega\colon x\in\partial_{\tau\delta^k}Q_\alpha^k(\omega)\cap\bar{Q}_\sigma^{k+N}(\omega)\}$$

for fixed α and σ.

Let $\sigma=:\sigma_{k+N}$. For every $j=k,k+1,\ldots,k+N-1$, let us denote

$$A_j:=\{\omega\in\Omega\colon \rho(z_{\sigma_j}^j(\omega),z_{\sigma_{j+1}}^{j+1}(\omega))\geq\varepsilon_1\delta^j \quad \text{for } (j,\sigma_j)\geq_\omega(j+1,\sigma_{j+1})\geq_\omega(k+N,\sigma)\}$$

where $\varepsilon_1:=(2A_0)^{-2}c_1$ is the constant from Lemma 5.12. Note that the set A_j only depends on the choice of points of levels from $k+N$ to j and, by (5.3), the choice of these points only depends on Ω_j for $j=k,k+1,\ldots,k+N-1$. By Lemma 5.12, we have in particular

$$E_{\alpha,\sigma}\subseteq\bigcap_{j=k}^{k+N-1}A_j.$$

Let us denote by

$$\mathscr{F}_j:=\sigma(\mathscr{H}_i\colon i\geq j)$$

the σ-algebra generated by the class of subsets of Ω with the points of level $i\geq j$ fixed. Note that

$$A_j\in\mathscr{F}_j \text{ for } j=1,\ldots,N \quad\text{and}\quad \mathscr{F}_{j+1}\subseteq\mathscr{F}_j \text{ for } j=k,\ldots,k+N-1.$$

By Lemma 5.17, we have

$$\begin{aligned}(5.20)\qquad \mathbb{P}(E_{\alpha,\sigma}) &= \mathbb{E}(\chi_{E_{\alpha,\sigma}}) \le \mathbb{E}\Big(\prod_{j=k}^{k+N-1} \chi_{A_j}\Big) \\ &= \mathbb{E}\mathbb{E}_{\mathscr{F}_{k+N}}\chi_{A_{k+N-1}}\mathbb{E}_{\mathscr{F}_{k+N-1}}\chi_{A_{k+N-2}}\dots\mathbb{E}_{\mathscr{F}_{k+1}}\chi_{A_k}.\end{aligned}$$

We first calculate

$$\mathbb{E}_{\mathscr{F}_{k+1}}[\chi_{A_k}] = \mathbb{P}(A_k|\mathscr{F}_{k+1}).$$

Note that for a given index pair $(k+1, \sigma_{k+1})$, there always exists a reference point x_γ^{k+1} such that $\rho(x_\gamma^{k+1}, z_{\sigma_{k+1}}^{k+1}(\omega)) < \delta^{k+1} < \varepsilon_1\delta^k$. On the other hand, by (5.4), there is a positive probability τ_0 that $x_\gamma^{k+1} = z_\gamma^k(\omega)$. Thus, for a given index pair $(k+1, \sigma_{k+1})$, there is a positive probability that the pair (k, σ_k) for which $(k, \sigma_k) \ge_\omega (k+1, \sigma_{k+1})$ satisfies $\rho(z_{\sigma_k}^k, z_{\sigma_{k+1}}^{k+1}) < \varepsilon_1\delta^k$. Since the negation of the event A_k with given $\mathscr{F}_{k+1}$ is that for some index pair $(k+1, \sigma_{k+1})$, the parent is within distance $\varepsilon_1\delta^k$, we conclude that

$$(5.21)\qquad \mathbb{E}_{\mathscr{F}_{k+1}}\chi_{A_k} = \mathbb{P}(A_k|\mathscr{F}_{k+1}) \le 1 - \tau_0, \quad \tau_0 > 0.$$

Further, by the above, monotonicity and linearity we have

$$\begin{aligned}(5.22)\qquad \mathbb{E}_{\mathscr{F}_{k+2}}[\chi_{A_{k+1}}\mathbb{E}_{\mathscr{F}_{k+1}}\chi_{A_k}] &\le \mathbb{E}_{\mathscr{F}_{k+2}}[\chi_{A_{k+1}}(1-\tau_0)] \\ &= (1-\tau_0)\mathbb{E}_{\mathscr{F}_{k+2}}[\chi_{A_{k+1}}]\end{aligned}$$

since $1 - \tau_0$ is a constant. We now proceed backwards and travel from the end of the chain in (5.20) repeating the steps in (5.21) and (5.22) $N-1$ times. Each time the term $\mathbb{E}_{\mathscr{F}_{k+i}}\chi_{A_i}$, $i = 1, \dots, N$, is estimated from above by the constant $1 - \tau_0 \in (0,1)$, which can then be relocated by (5.22). We obtain

$$\mathbb{P}(E_{\alpha,\sigma}) \le (1-\tau_0)^N < C_2\tau^\eta$$

with $C_2 := 4A_0^2(c_1\delta)^{-1}$ and $\eta := \log(1-\tau_0)/\log\delta > 0$. ∎

5.23. Corollary ((5.8) of Theorem 5.6). *For $x \in X$,*

$$\mathbb{P}\Big(\Big\{\omega \in \Omega\colon x \in \bigcup_{\alpha,k} \partial Q_\alpha^k(\omega)\Big\}\Big) = 0.$$

Proof. Recall from (2.5) that the cubes $\tilde{Q}_\alpha^k$ and $\bar{Q}_\alpha^k$ are the interior and closure of Q_α^k, respectively. Thus,

$$\partial Q_\alpha^k = \bar{Q}_\alpha^k \setminus \tilde{Q}_\alpha^k = \bar{Q}_\alpha^k \cap (\tilde{Q}_\alpha^k)^c \subseteq \{x \in \bar{Q}_\alpha^k\colon \rho(x, (\tilde{Q}_\alpha^k)^c) = 0\}.$$

It follows that, for any $\tau > 0$,

$$\partial Q_\alpha^k \subseteq \{x \in \bar{Q}_\alpha^k\colon \rho(x, (\tilde{Q}_\alpha^k)^c) \le \tau\delta^k\} = \partial_{\tau\delta^k} Q_\alpha^k.$$

Thus,

$$\Big\{\omega \in \Omega\colon x \in \bigcup_\alpha \partial Q_\alpha^k(\omega)\Big\} \subseteq \Big\{\omega \in \Omega\colon x \in \bigcup_\alpha \partial_{\tau\delta^k} Q_\alpha^k(\omega)\Big\},$$

and consequently, by Lemma 5.15,

$$\mathbb{P}\Big(\Big\{\omega\in\Omega\colon x\in\bigcup_\alpha\partial Q^k_\alpha(\omega)\Big\}\Big)\le\mathbb{P}\Big(\Big\{\omega\in\Omega\colon x\in\bigcup_\alpha\partial_{\tau\delta^k}Q^k_\alpha(\omega)\Big\}\Big)\le C_2\tau^\eta.$$

Thus, letting $\tau\to 0$ we obtain

$$\mathbb{P}\Big(\Big\{\omega\in\Omega\colon x\in\bigcup_\alpha\partial Q^k_\alpha(\omega)\Big\}\Big)=0.$$

Finally,

$$\mathbb{P}\Big(\Big\{\omega\in\Omega\colon x\in\bigcup_{\alpha,k}\partial Q^k_\alpha(\omega)\Big\}\Big)\le\sum_k\mathbb{P}\Big(\Big\{\omega\in\Omega\colon x\in\bigcup_\alpha\partial Q^k_\alpha(\omega)\Big\}\Big)=0. \blacksquare$$

5.24. Lemma ((5.9) of Theorem 5.6). *Assume that μ is a positive σ-finite measure on X. Then*

$$\mu\Big(\bigcup_{\alpha,k}\partial Q^k_\alpha(\omega)\Big)=0\quad\text{for a.e. }\omega\in\Omega. \tag{5.25}$$

In particular, given μ we may choose $\omega\in\Omega$ such that (5.25) *holds.*

Proof. For a fixed $\omega\in\Omega$, denote

$$B_\omega:=\bigcup_{\alpha,k}\partial Q^k_\alpha(\omega),$$

and for a fixed $x\in X$, denote

$$B^x:=\{\omega\in\Omega\colon x\in B_\omega\}.$$

By Fubini's Theorem,

$$\begin{aligned}\int_\Omega\mu(B_\omega)\,d\mathbb{P}(\omega)&=\int_\Omega\int_X\chi_{B_\omega}(x)\,d\mu(x)\,d\mathbb{P}(\omega)\\&=\int_X\int_\Omega\chi_{B^x}(\omega)\,d\mathbb{P}(\omega)\,d\mu(x)=\int_X\mathbb{P}(B^x)\,d\mu(x)=0\end{aligned}$$

since $\mathbb{P}(B^x)=0$ by Corollary 5.23. The assertion follows. ∎

6. Random adjacent dyadic systems. In this section we will prove the following theorem.

6.1. Theorem. *Given a set $\{x^k_\alpha\}_{k,\alpha}$ of reference points, suppose that a constant $\delta\in(0,1)$ satisfies $144A_0^8\delta\le 1$. Then there exists a probability space $(\Omega,\mathbb{P})$ such that every $\omega\in\Omega$ defines a family $(\mathscr{D}^t(\omega))_{t=1}^K$ of dyadic systems where each $\mathscr{D}^t(\omega)=\{{}^tQ^k_\alpha(\omega)\}_{k,\alpha}$, related to new dyadic points $\{{}^tz^k_\alpha(\omega)\}_{k,\alpha}$, has the properties* (2.5)–(2.9) *of Theorem* 2.2. *Further,*

(6.2) *for every $\omega\in\Omega$, $(\mathscr{D}^t(\omega))_{t=1}^K$ has the property of Lemma* 4.12*;*
(6.3) *for every $t\in\{1,\dots,K\}$, $(\mathscr{D}^t(\omega))_{\omega\in\Omega}$ has the properties* (5.2)–(5.4) *of Theorem* 5.1.

Notice that this statement makes no reference to randomness but the proof does, and it is not clear how to prove something like this without the help of randomization. In more classical set-ups, similar conclusions could be reached with the help of strongly Euclidean devices like rotations (cf. [19, Theorem 2 and its proof]).

One immediate application of such a construction is the following.

6.4. COROLLARY. *Given a set $\{x^k_\alpha\}_{k,\alpha}$ of reference points, suppose that a constant $\delta \in (0,1)$ satisfies $144A_0^8\delta \le 1$. Let μ be a positive σ-finite measure on X. Then the finite collection of adjacent dyadic systems $\mathscr{D}^t$, $t = 1, \ldots, K$, as in Theorem* 4.1 *may be chosen to have the additional property that*

$$\mu(\partial Q) = 0 \qquad \forall Q \in \bigcup_{t=1}^{K} \mathscr{D}^t.$$

Proof. Let $\mathscr{D}^t(\omega)$ be the random adjacent systems guaranteed by Theorem 6.1. By (6.3) and (5.9) of Theorem 5.6,

$$\text{for all } t \in \{1, \ldots, K\}, \text{ for a.e. } \omega \in \Omega, \quad \mu\Big(\bigcup_{Q \in \mathscr{D}^t(\omega)} \partial Q\Big) = 0.$$

Since there are only finitely many choices of t, we can reverse the order of "for all t" and "for a.e. $\omega \in \Omega$" above, and then it suffices to choose any $\omega \in \Omega$ outside the event of probability zero implicit in the "a.e.", and take $\mathscr{D}^t := \mathscr{D}^t(\omega)$ for this $\omega \in \Omega$. ∎

6.5. **The probability space.** We define a probability space Ω by setting

$$\Omega := \prod_{k\in\mathbb{Z}} \Omega_k, \qquad \Omega_k := \{1, \ldots, K\}.$$

The points $\omega \in \Omega$ admit the natural coordinate representation $\omega = (\omega_k)_{k\in\mathbb{Z}}$ where $\omega_k \in \{1, \ldots, K\}$.

We define a probability $\mathbb{P}$ on Ω by requiring the coordinates ω_k to be independent and distributed with equal probabilities

$$\mathbb{P}(\omega_k = T) = 1/K \qquad \forall T = 1, \ldots, K.$$

Given $\omega_k = T_k$, define a permutation of $(1, \ldots, K)$ by

$$\pi_k(t) := t + T_k \ (\mathrm{mod}\ K), \quad t = 1, \ldots, K.$$

Given T_k and $t \in \{1, \ldots, K\}$, they together define an ordered pair $\pi_k(t) = (\ell_k(t), m_k(t))$ via the bijection φ introduced in 4.11. We define the new dyadic points $\{{}^t z^k_\alpha\}_\alpha$ of generation k as follows. For every (k, α), check whether there exists $(k+1, \beta) \le (k, \alpha)$ such that $\mathrm{label}_2(k+1, \beta) = \pi_k(t)$. If so, set ${}^t z^k_\alpha := x^{k+1}_\beta$. Otherwise, pick any $(k+1, \beta) \le (k, \alpha)$ with $\rho(x^{k+1}_\beta, x^k_\alpha) <$

δ^{k+1} and set ${}^t z^k_\alpha := x^{k+1}_\beta$. (We could, for example, choose the nearest child of x^k_α and eliminate the arbitrariness of this choice.)

The choice of $\omega = (\omega_k)$ then uniquely determines the permutation π_k on every level k, which in turn determines a family of new dyadic points ${}^t z^k_\alpha(\omega) = {}^t z^k_\alpha(\omega_k) = z^k_\alpha(\pi_k(t))$ for each $t = 1, \ldots, K$.

Once the points ${}^t z^k_\alpha(\omega_k)$, $t = 1, \ldots, K$, are chosen, they uniquely determine the relation $\leq_{\omega,t}$ between the index pairs (k, α). Then, for every $t = 1, \ldots, K$, the points ${}^t z^k_\alpha(\omega_k)$ and the relation $\leq_{\omega,t}$ together determine the new dyadic cubes ${}^t Q^k_\alpha(\omega_k)$, and their random choice corresponds to the random choice of ω according to the law $\mathbb{P}$. Note that the choice of the new dyadic points ${}^t z^k_\alpha(\omega) = z^k_\alpha(\pi_k(t))$ coincides with the specific selection rule defined in 4.11.

It is evident, by Lemma 4.10 in view of 4.11, that for every $\omega \in \Omega$ and every $t = 1, \ldots, K$, the dyadic system $\mathscr{D}^t(\omega)$ has the properties (2.5)–(2.9) of Theorem 2.2. We may complete the proof of Theorem 6.1 by the following lemma:

6.6. Lemma. *For every $\omega \in \Omega$, the family $(\mathscr{D}^t(\omega))_{t=1}^K$ has the property of Lemma* 4.12. *For every $t = 1, \ldots, K$, $(\mathscr{D}^t(\omega))_{\omega\in\Omega}$ has the properties* (5.2)–(5.4) *of Theorem* 5.1.

Proof. Suppose $\omega \in \Omega$, $\omega = (\omega_k)_{k\in\mathbb{Z}}$, and let π_k be the permutation defined by $T_k := \omega_k$. Going back to the proof of Lemma 4.12, it suffices to prove that for every $(k+1, \beta)$, we have $x^{k+1}_\beta = {}^t z^k_\alpha$ for $(k, \alpha) \geq (k+1, \beta)$ and some $t = 1, \ldots, K$. But this is clear since $\mathrm{label}_2(k+1, \beta) =: t = \pi_k(t - T_k)$ where $t - T_k$ is defined modulo K. By construction, ${}^t z^k_\alpha = x^{k+1}_\beta$, and the first assertion follows.

With fixed $t \in \{1, \ldots, K\}$ and $(k+1, \beta) \leq (k, \alpha)$, there is a positive probability $\tau_0 = 1/K$ that $\pi_k(t) = s$ for $s = \mathrm{label}_2(k+1, \beta)$. Thus,

$$\mathbb{P}(\{\omega \in \Omega \colon {}^t z^k_\alpha(\omega) = x^{k+1}_\beta\}) = \mathbb{P}(\{\omega \in \Omega \colon \pi_k(t) = \mathrm{label}_2(k+1, \beta)\}) = \tau_0 > 0,$$

and hence (5.4) of Theorem 5.1 holds. The other properties follow directly from the construction of $(\Omega, \mathbb{P})$. The second assertion follows. ∎

6.7. Remark. By the construction of the probability space $(\Omega, \mathbb{P})$, while the random choice of new dyadic points is independent on different levels, on a given level k it only depends on the choice of ω_k. Thus, the choice of points $\{{}^t z^k_\alpha\}_\alpha$ is not independent. By slightly changing the construction of Ω_k, we may obtain independence also in the choice of non-neighbouring points on the same level: We define a probability space Ω by setting

$$\Omega := \prod_{k\in\mathbb{Z}} \Omega_k, \qquad \Omega_k := \{1, \ldots, K\} \times \prod_{\alpha\in\mathscr{A}_k} \{1, \ldots, M_{k,\alpha}\},$$

where $M_{k,\alpha} := \#\{\gamma\colon (k+1,\gamma) \leq (k,\alpha)\}$, the number of children of the reference point (k,α).

The points $\omega \in \Omega$ again admit the natural coordinate representation $\omega = (\omega_k)_{k\in\mathbb{Z}}$. Moreover, $\omega_k = (T_k, m_{k,\alpha}\colon \alpha \in \mathscr{A}_k) \in \Omega_k$ where $T_k \in \{1,\dots,K\}$ and $m_{k,\alpha} \in \{1,\dots,M_{k,\alpha}\}$.

We define a probability $\mathbb{P}$ on Ω by requiring the coordinates ω_k to be independent and distributed as follows. First,

$$\mathbb{P}(T_k = T) = 1/K \quad \forall T = 1,\dots,K.$$

Second, the subcoordinates $m_{k,\alpha}$ are again independent with distribution

$$\mathbb{P}(m_{k,\alpha} = m) = 1/M_{k,\alpha} \quad \forall m = 1,\dots,M_{k,\alpha}.$$

Given $\omega_k = (T_k, m_{k,\alpha}\colon \alpha \in \mathscr{A}_k)$, we define the new dyadic points as follows. First, T_k defines a cyclic permutation π_k of $(1,\dots,K)$ as before. Then, for every $t = 1,\dots,K$ and (k,α), check whether there exists $(k+1,\beta) \leq (k,\alpha)$ such that

$$\mathrm{label}_2(k+1,\beta) = (\mathrm{pr}_1(\pi_k(t)), \mathrm{pr}_2(\pi_k(t)) + m_{k,\alpha} \pmod{M_{k,\alpha}}).$$

If so, set ${}^t z^k_\alpha := x^{k+1}_\beta$. Otherwise, pick any $(k+1,\beta) \leq (k,\alpha)$ with $\rho(x^{k+1}_\beta, x^k_\alpha) < \delta^{k+1}$ and set ${}^t z^k_\alpha := x^{k+1}_\beta$. (We could again choose, for example, the nearest child of x^k_α and eliminate the arbitrariness of this choice.)

7. Applications

7.1. **Set-up.** Let (X,ρ) be a quasi-metric space and suppose that μ is a positive Borel measure on X satisfying the doubling condition

$$\mu(2B) \leq C\mu(B) \quad \text{for all balls } B. \tag{7.2}$$

Note that then $0 < \mu(B) < \infty$ for all balls B. Let us state the following well-known lemma.

7.3. Lemma. *For every $x \in X$ and $0 < r \leq R$ we have*

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leq C_\mu \left(\frac{R}{r}\right)^{c_\mu}$$

where $c_\mu = \log_2 C_\mu$ and C_μ is the smallest constant satisfying (7.2).

There is an immediate sequel for dyadic cubes:

7.4. Corollary. *There exists a constant $C \geq 1$ such that for every dyadic cube Q we have $\mu(B_Q) \leq C\mu(Q)$, where B_Q is the containing ball of Q as in* (2.8). *Conversely, given a ball $B := B(x,r)$, there exists a dyadic system $\mathscr{D}^t$ and a dyadic cube $Q_B \in \mathscr{D}^t$ such that $B \subseteq Q_B$ and $\mu(Q_B) \leq C\mu(B)$.*

Proof. Given a dyadic cube Q and a ball $B(x,r)$, consider the balls $B(x_\alpha^k, c_1\delta^k) \subseteq Q_\alpha^k \subseteq B(x_\alpha^k, C_1\delta^k) =: B_Q$ from Theorem 2.2 and the cube Q_B from Lemma 4.12 with $B(x,r) \subseteq Q_B \subseteq B(x, Cr)$. The assertion follows readily from Lemma 7.3. ∎

7.5. **Maximal operators.** Let ω be a weight on X, i.e. $\omega \geq 0$ and $\omega \in L^1_{\mathrm{loc}}(X,\mu)$. Given a measurable set E, denote $\omega(E) := \int_E \omega \, d\mu$. Define the *weighted Hardy–Littlewood maximal operator* M_ω by

$$M_\omega f(x) = \sup_{B \ni x} \frac{1}{\omega(B)} \int_B |f| \, \omega \, d\mu, \quad f \in L^1_{\mathrm{loc}}(X,\mu),\, x \in X, \tag{7.6}$$

where the supremum is over all balls B containing x. We drop the subscript ω in M_ω if $\omega \equiv 1$.

Given a weight ω and $p > 1$, set $\sigma = \omega^{-1/(p-1)}$. We say that ω satisfies the A_p-*condition* and denote $\omega \in A_p$ if

$$\|\omega\|_{A_p} := \sup_B \frac{\omega(B)\sigma(B)^{p-1}}{\mu(B)^p} < \infty.$$

Note that $\omega \in A_p$ if and only if $\sigma \in A_{p'}$ where $1/p + 1/p' = 1$.

By the fundamental result of B. Muckenhoupt, the classical Hardy–Littlewood maximal operator M is bounded on L^p_ω, $1 < p < \infty$, if and only if $\omega \in A_p$. In this section, we will provide a quantitative formulation of this well-known result on the metric space (X,μ).

Let $\mathscr{D}^t$ denote any fixed dyadic grid of cubes Q_α^k, $k \in \mathbb{Z}$, $\alpha \in \mathscr{A}_k$, constructed in Section 4. We define the *weighted dyadic maximal operator* $M_\omega^{\mathscr{D}^t}$ by

$$M_\omega^{\mathscr{D}^t} f(x) = \sup_{Q \ni x} \frac{1}{\omega(Q)} \int_Q |f| \omega \, d\mu, \quad f \in L^1_{\mathrm{loc}}(X,\mu),\, x \in X, \tag{7.7}$$

where the supremum is over all dyadic cubes $Q \in \mathscr{D}^t$ containing x.

By a well-known fact (see for example [29, Theorem 14.11]), the dyadic maximal operators $M_\omega^{\mathscr{D}^t}$ are bounded on L^p_ω, $1 < p < \infty$, uniformly in all weights ω: For $1 < p < \infty$,

$$\|M_\omega^{\mathscr{D}^t} f\|_{L^p_\omega} \leq p' \|f\|_{L^p_\omega} \tag{7.8}$$

for all $f \in L^p_\omega$.

We further consider the closely related *sharp maximal operator* $M^\#$ defined by

$$M^\# f(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu, \quad f \in L^1_{\mathrm{loc}}(X,\mu),\, x \in X,$$

where the supremum is over all balls containing x, and we have used the notation

$$f_E := \frac{1}{\mu(E)} \int_E |f|\, d\mu$$

for the integral average of f over a bounded measurable set E with $\mu(E) > 0$. Further define the *dyadic sharp maximal operator* $M^{\#}_{\mathscr{D}^t}$ by

$$M^{\#}_{\mathscr{D}^t} f(x) := \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f - f_Q|\, d\mu, \quad f \in L^1_{\mathrm{loc}}(X,\mu),\ x \in X,$$

where the supremum is over all dyadic cubes $Q \in \mathscr{D}^t$ containing x.

We will first show the equivalence between the classical operators M and $M^{\#}$ and their dyadic counterparts:

7.9. Proposition. *Let $f \in L^1_{\mathrm{loc}}(X,\mu)$. We have the pointwise estimates*

$$M^{\mathscr{D}^t} f(x) \le CMf(x) \quad \textit{and} \quad Mf(x) \le C \sum_{t=1}^{K} M^{\mathscr{D}^t} f(x), \tag{7.10}$$

$$M^{\#}_{\mathscr{D}^t} f(x) \le CM^{\#} f(x) \quad \textit{and} \quad M^{\#} f(x) \le C \sum_{t=1}^{K} M^{\#}_{\mathscr{D}^t} f(x), \tag{7.11}$$

where both of the first inequalities hold for every $t = 1, \dots, K$ and the constant $C \ge 1$ is independent of f.

Proof. Fix t and assume $x \in Q$, $Q \in \mathscr{D}^t$. Let B be the containing ball of Q as in (2.8). Then $\mu(B) \le C\mu(Q)$ by Lemma 7.4, and therefore

$$\begin{aligned} \frac{1}{\mu(Q)} \int_Q |f - f_Q|\, d\mu &\le \frac{1}{\mu(Q)} \int_Q |f - f_B|\, d\mu + |f_B - f_Q| \\ &\le \frac{2}{\mu(Q)} \int_Q |f - f_B|\, d\mu \le \frac{2C}{\mu(B)} \int_B |f - f_B|\, d\mu \\ &\le 2CM^{\#} f(x). \end{aligned}$$

The first inequality in (7.11) follows by taking the supremum over all dyadic cubes in $\mathscr{D}^t$ containing x. Also the first inequality in (7.10) follows by putting $f_B = f_Q = 0$ in the above and making the obvious simplifications.

For the reverse inequalities, consider a ball $B \ni x$ and let $Q = Q(t)$ be the dyadic cube as in Lemma 4.12 with $B \subseteq Q$ and $\mu(Q) \le C\mu(B)$. By repeating the argument above with the roles of B and Q interchanged we may conclude with

$$\frac{1}{\mu(B)} \int_B |f - f_B|\, d\mu \le 2CM^{\#}_{\mathscr{D}^t} f(x) \le 2C \sum_{t=1}^{K} M^{\#}_{\mathscr{D}^t} f(x).$$

The second inequality in (7.11) follows again by taking the supremum over all balls containing x. Also the second inequality in (7.10) follows as earlier. ∎

7.12. **The sharp weighted norm of the Hardy–Littlewood maximal operator.** To illustrate the use of the new adjacent dyadic systems constructed in Section 4, we will provide an easy extension of Buckley's theorem [2] on the sharp dependence of $\|M\|_{L^p_\omega}$ on $\|\omega\|_{A_p}$ in Muckenhoupt's theorem for the Hardy–Littlewood maximal function to metric spaces X with a doubling measure μ:

7.13. Proposition. *Let $1 < p < \infty$. Then*

$$\|Mf\|_{L^p_w} \leq C\|w\|_{A_p}^{1/(p-1)}\|f\|_{L^p_w},$$

where the constant C depends only on X, μ and p.

Proof. By Proposition 7.9, it suffices to prove the analogous estimate for the dyadic maximal operator $M^{\mathscr{D}^t}$. We follow the Euclidean approach due to Lerner [17] with the deviation that we utilize the universal maximal function estimate (7.8) instead of the corresponding result for the centred maximal operator which, in the Euclidean case, is a well-known consequence of the Besicovitch covering theorem, a powerful classical tool generally unavailable in abstract metric spaces. We repeat the details for the reader's convenience.

Suppose $0 \leq \omega \in L^1_{\text{loc}}(X,\mu)$. Denote $\sigma = \omega^{-1/(p-1)}$ and

$$A_p(Q) := \frac{\omega(Q)\sigma(Q)^{p-1}}{\mu(Q)^p}.$$

Fix t and suppose $x \in Q$, $Q \in \mathscr{D}^t$. We have

$$\begin{aligned}
\frac{1}{\mu(Q)}\int_Q |f|\,d\mu &= A_p(Q)^{1/(p-1)}\left[\frac{\mu(Q)}{\omega(Q)}\left(\frac{1}{\sigma(Q)}\int_Q |f|\,d\mu\right)^{p-1}\right]^{1/(p-1)} \\
&\leq \|\omega\|_{A_p}^{1/(p-1)}\left[\frac{1}{\omega(Q)}\int_Q \left(\frac{1}{\sigma(Q)}\int_Q |f\sigma^{-1}|\,\sigma d\mu\right)^{p-1} d\mu\right]^{1/(p-1)} \\
&\leq \|\omega\|_{A_p}^{1/(p-1)}\left[\frac{1}{\omega(Q)}\int_Q \left(M_\sigma^{\mathscr{D}^t}(f\sigma^{-1})(x)\right)^{p-1} d\mu(x)\right]^{1/(p-1)} \\
&\leq \|\omega\|_{A_p}^{1/(p-1)}\left[\frac{1}{\omega(Q)}\int_Q \left([M_\sigma^{\mathscr{D}^t}(f\sigma^{-1})]^{p-1}\omega^{-1}\right)\omega\,d\mu\right]^{1/(p-1)},
\end{aligned}$$

and hence

$$M^{\mathscr{D}^t}f(x) \leq \|\omega\|_{A_p}^{1/(p-1)} M_\omega^{\mathscr{D}^t}\left([M_\sigma^{\mathscr{D}^t}(f\sigma^{-1})]^{p-1}\omega^{-1}\right)(x)^{1/(p-1)}.$$

Therefore, since $p/(p-1) = p'$ and $\sigma = \omega^{-1/(p-1)}$, the above estimate together with the universal maximal function estimate (7.8) implies that

$$\begin{aligned}\|M^{\mathscr{D}^t} f\|_{L^p_\omega} &\le \|\omega\|_{A_p}^{1/(p-1)} \big\|M_\omega^{\mathscr{D}^t}\big([M_\sigma^{\mathscr{D}^t}(f\sigma^{-1})]^{p-1}\omega^{-1}\big)\big\|_{L^{p'}_\omega}^{1/(p-1)} \\ &\le \|\omega\|_{A_p}^{1/(p-1)} p^{1/(p-1)} \|[M_\sigma^{\mathscr{D}^t}(f\sigma^{-1})]^{p-1}\omega^{-1}\|_{L^{p'}_\omega}^{1/(p-1)} \\ &= \|\omega\|_{A_p}^{1/(p-1)} p^{1/(p-1)} \|M_\sigma^{\mathscr{D}^t}(f\sigma^{-1})\|_{L^p_\sigma} \\ &\le \|\omega\|_{A_p}^{1/(p-1)} p^{1/(p-1)} p' \|f\sigma^{-1}\|_{L^p_\sigma} = \|\omega\|_{A_p}^{1/(p-1)} p^{1/(p-1)} p' \|f\|_{L^p_\omega}. \ \blacksquare\end{aligned}$$

7.14. **Functions of bounded mean oscillation.** Recall that *the classical* BMO(μ) *space* is the set of equivalence classes of functions $f \in L^1_{\mathrm{loc}}(X,\mu)$, modulo additive constants, such that the L^1-averages

$$\frac{1}{\mu(B)}\int_B |f - f_B|\,d\mu, \qquad f_B := \frac{1}{\mu(B)}\int_B |f|\,d\mu,$$

are bounded (uniformly in B). The non-negative real number

$$\|f\|_{\mathrm{BMO}} := \sup_B \frac{1}{\mu(B)}\int_B |f - f_B|\,d\mu < \infty, \tag{7.15}$$

where the supremum is over all balls, is then called the BMO-norm of f.

For every $t = 1, \dots, K$, we define the *dyadic* BMO(μ) *space* $\mathrm{BMO}_{\mathscr{D}^t}$ as the set of equivalence classes of functions $f \in L^1_{\mathrm{loc}}(X,\mu)$, modulo additive constants, such that

$$\|f\|_{\mathrm{BMO}_{\mathscr{D}^t}} := \sup_{Q\in\mathscr{D}^t} \frac{1}{\mu(Q)}\int_Q |f - f_Q|\,d\mu < \infty, \qquad f_Q := \frac{1}{\mu(Q)}\int_Q |f|\,d\mu,$$

where the supremum is over all dyadic cubes $Q \in \mathscr{D}^t$. The quantity $\|f\|_{\mathrm{BMO}_{\mathscr{D}^t}}$ is then the dyadic BMO-norm of f.

The relationship between the two kinds of BMO spaces has been studied in the Euclidean setting in [8] and [20].

As a second illustration of the use of the new adjacent dyadic systems, we provide a representation of BMO(μ) as an intersection of finitely many dyadic BMO(μ) spaces. This extends the Euclidean result, which was explicitly stated by T. Mei [20], but already implicit in some earlier work (cf. [20, Remark 6]). A related result in metric spaces was also proven by Caruso and Fanciullo [3].

7.16. Proposition. *Suppose (X,ρ) is a quasi-metric space and μ is a positive Borel measure on X with the doubling property* (7.2)*. There exist constants $C > 0$ and $C' > 0$ depending only on X and μ such that for every $f \in L^1_{\mathrm{loc}}(X,\mu)$,*

$$C\|f\|_{\mathrm{BMO}_{\mathscr{D}^t}} \le \|f\|_{\mathrm{BMO}} \le C' \sum_{t=1}^{K} \|f\|_{\mathrm{BMO}_{\mathscr{D}^t}}$$

where the first estimate holds for every $t = 1, \dots, K$. *Thus,*

$$\mathrm{BMO}(\mu) = \bigcap_{t=1}^{K} \mathrm{BMO}_{\mathscr{D}^t}(\mu)$$

with equivalent norms.

Proof. This is an immediate corollary of Proposition 7.9. ∎

7.17. Remark. It is well-known that both the classical and dyadic $\mathrm{BMO}(\mu)$ spaces satisfy the John–Nirenberg inequality. The proof for the dyadic version is slightly easier. One may represent the space $\mathrm{BMO}^p(\mu)$ as an intersection of finitely many dyadic spaces $\mathrm{BMO}^p_{\mathscr{D}^t}$, $p > 1$, as stated in Proposition 7.16. With this representation, one may derive the John–Nirenberg inequality and the exponential integrability of $\mathrm{BMO}(\mu)$ functions from their dyadic counterparts, thereby avoiding some technicalities in the proof.

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