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FINITE GROUPS OF OTP PROJECTIVE REPRESENTATION TYPE

BY

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Abstract. Let K be a field of characteristic $p > 0$, K^* the multiplicative group of K and $G = G_p \times B$ a finite group, where G_p is a p-group and B is a p'-group. Denote by $K^\lambda G$ a twisted group algebra of G over K with a 2-cocycle $\lambda \in Z^2(G, K^*)$. We give necessary and sufficient conditions for G to be of OTP projective K-representation type, in the sense that there exists a cocycle $\lambda \in Z^2(G, K^*)$ such that every indecomposable $K^\lambda G$-module is isomorphic to the outer tensor product $V \# W$ of an indecomposable $K^\lambda G_p$-module V and a simple $K^\lambda B$-module W. We also exhibit finite groups $G = G_p \times B$ such that, for any $\lambda \in Z^2(G, K^*)$, every indecomposable $K^\lambda G$-module satisfies this condition.

0. Introduction. Let K be a field of characteristic $p > 0$ and $G = G_p \times B$ a finite group, where G_p is a Sylow p-subgroup and $|G_p| > 1$, $|B| > 1$. Given $\mu \in Z^2(G_p, K^*)$ and $\nu \in Z^2(B, K^*)$, the map $\mu \times \nu\colon G \times G \to K^*$ defined by

$$(\mu \times \nu)_{x_1 b_1, x_2 b_2} = \mu_{x_1, x_2} \cdot \nu_{b_1, b_2},$$

for all $x_1, x_2 \in G_p$, $b_1, b_2 \in B$, belongs to $Z^2(G, K^*)$. Every cocycle $\lambda \in Z^2(G, K^*)$ is cohomologous to $\mu \times \nu$, where μ is the restriction of λ to $G_p \times G_p$ and ν is the restriction of λ to $B \times B$.

From now on, we suppose that each cocycle $\lambda \in Z^2(G, K^*)$ under consideration satisfies the condition $\lambda = \mu \times \nu$, and all $K^\lambda G$-modules are assumed to be left and finite-dimensional (as vector spaces over K).

Let $\lambda = \mu \times \nu \in Z^2(G, K^*)$ and $\{u_g : g \in G\}$ be a canonical K-basis of $K^\lambda G$. Then $\{u_h : h \in G_p\}$ is a canonical K-basis of $K^\mu G_p$ and $\{u_b : b \in B\}$ is a canonical K-basis of $K^\nu B$. Moreover, if $g = hb$, where $g \in G$, $h \in G_p$, $b \in B$, then $u_g = u_h u_b = u_b u_h$. It follows that $K^\lambda G \cong K^\mu G_p \otimes_K K^\nu B$.

Given a $K^\mu G_p$-module V and a $K^\nu B$-module W, we denote by $V \# W$ the $K^\lambda G$-module whose underlying vector space is $V \otimes_K W$ with the $K^\lambda G$-module structure given by

$$u_{hb}(v \otimes w) = u_h v \otimes u_b w,$$

for all $h \in G_p$, $b \in B$, $v \in V$, $w \in W$, and extended to $K^\lambda G$ and $V \otimes_K W$

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by K-linearity. The module $V \# W$ is called the *outer tensor product* of V and W (see [21, p. 122]).

We recall from [7, p. 10] the following definitions.

(a) The algebra $K^\lambda G$ is defined to be of *OTP representation type* if every indecomposable $K^\lambda G$-module is isomorphic to the outer tensor product $V \# W$, where V is an indecomposable $K^\mu G_p$-module and W is a simple $K^\nu B$-module.

(b) A group $G = G_p \times B$ is defined to be of *OTP projective K-representation type* if there exists a cocycle $\lambda \in Z^2(G, K^*)$ such that the algebra $K^\lambda G$ is of OTP representation type.

(c) A group $G = G_p \times B$ is said to be of *purely OTP projective K-representation type* if $K^\lambda G$ is of OTP representation type for any $\lambda \in Z^2(G, K^*)$.

In [13] Brauer and Feit proved that if K is algebraically closed, then the group algebra KG is of OTP representation type. Blau [10] and Gudyvok [17, 18] have independently shown that if K is an arbitrary field, then KG is of OTP representation type if and only if G_p is cyclic or K is a splitting field for B. Gudyvok [19, 20] also investigated a similar problem for group rings SG, where S is a complete discrete valuation ring. In [3, 6], the results of Blau and Gudyvok are generalized to the twisted group rings $S^\lambda G$, where $G = G_p \times B$, $S = K$ or S is a complete discrete valuation ring of characteristic $p > 0$. Let $S = K[[X]]$ be the ring of formal power series in the indeterminate X with coefficients in the field K. In [7], necessary and sufficient conditions on G and K are given for G to be of OTP projective S-representation type and of purely OTP projective S-representation type.

In the present work we determine finite groups $G = G_p \times B$ of OTP projective K-representation type and of purely OTP projective K-representation type.

Denote by l_B the product of all pairwise distinct prime divisors of $|B|$. Unless stated otherwise, we assume that if G_p is non-abelian, then $[K(\varepsilon)\colon K]$ is not divisible by p, where ε is a primitive l_Bth root of 1. This condition is satisfied if K contains a primitive qth root of 1 for every prime q dividing $|B|$ such that the characteristic p divides $q-1$. For simplicity of presentation, we set

$$i(K) = \begin{cases} t & \text{if } [K : K^p] = p^t, \\ \infty & \text{if } [K : K^p] = \infty. \end{cases}$$

Let s be the number of invariants of the abelian group G_p/G'_p, and D the subgroup of G_p such that $G'_p \subset D$ and $D/G'_p = \mathrm{soc}(G_p/G'_p)$. Suppose that if $p \neq 2$, $s = i(K) + 1$, G'_p is cyclic and D is a non-abelian group of exponent p, then $|D : Z(D)| = p^2$, where $Z(D)$ is the center of D. We prove in Theorem 3.1 that the group $G = G_p \times B$ is of OTP projective

K-representation type if and only if one of the following three conditions is satisfied:

(i) $s \leq i(K)$ and G'_p is cyclic;
(ii) $s = i(K) + 1$, G'_p is cyclic and there exists a cyclic subgroup T of G_p such that $G'_p \subset T$ and G_p/T has $i(K)$ invariants;
(iii) K is a splitting field for $K^\nu B$ for some $\nu \in Z^2(B, K^*)$.

We also prove in Proposition 3.6 that if $G = G_p \times B$ is abelian, then G is of OTP projective K-representation type if and only if one of the following conditions is satisfied:

(i) $s \leq i(K) + 1$;
(ii) B has a subgroup H such that B/H is of symmetric type and K contains a primitive mth root of 1, where $m = \max\{\exp(B/H), \exp H\}$.

Now suppose that K is an arbitrary field of characteristic p. We establish in Proposition 3.11 that if every prime divisor of $|B'|$ is also a divisor of $|B : B'|$, then $G = G_p \times B$ is of purely OTP projective K-representation type if and only if either G_p is cyclic, or $K = K^q$ and K contains a primitive qth root of 1, for each prime q dividing $|B|$.

In the general case, a finite group $G = G_p \times B$ is of purely OTP projective K-representation type if and only if either G_p is cyclic, or there exists a finite central group extension $1 \to A \to \widehat{B} \to B \to 1$ such that any projective K-representation of B lifts projectively to an ordinary K-representation of $\widehat{B}$ and K is a splitting field for $\widehat{B}$ (Theorem 3.12).

Let $t(K^*)$ denote the torsion subgroup of the multiplicative group K^* of K. Assume that either $t(K^*) = t(K^*)^q$ for every prime q dividing $|B'|$, or every prime divisor of $|B'|$ is also a divisor of $|B : B'|$. Then G is of purely OTP projective K-representation type if and only if either G_p is cyclic, or there exists a covering group $\widehat{B}$ of B over K such that K is a splitting field for $\widehat{B}$ (Proposition 3.13).

1. Preliminaries. Throughout the paper, we use the standard group representation theory notation and terminology introduced in the monographs by Alperin [1], Benson [9], Curtis and Reiner [14], and Karpilovsky [21, 22]. The books by Karpilovsky give a systematic account of the projective representation theory. For classical problems and solutions of group representation theory, we refer to [1, 9, 14] and to the old and nice papers [11, 12]. A background of the representation theory of finite-dimensional algebras can be found in the monographs by Assem, Simson and Skowroński [2], Drozd and Kirichenko [16], Simson [23], and Simson and Skowroński [24], where among other things the representation types (finite, tame, wild) of finite groups and algebras are discussed.

In particular, we use the following notation: $p \geq 2$ is a prime; K is a field of characteristic p, $K^q = \{\alpha^q : \alpha \in K\}$; K^* is the multiplicative group of K; $t(K^*)$ is the torsion subgroup of K^*; $\mathrm{o}(\xi)$ is the order of $\xi \in t(K^*)$; $G = G_p \times B$ is a finite group, where G_p is a p-group, B is a p'-group, $|G_p| > 1$ and $|B| > 1$; H' is the commutant of a group H, $Z(H)$ is the center of H, e is the identity element of H, $|h|$ is the order of $h \in H$ and $\exp H$ is the exponent of H; $\operatorname{soc} A$ is the socle of an abelian group A. Let l_B be the product of all pairwise distinct prime divisors of $|B|$. Unless stated otherwise, we assume that if G_p is non-abelian, then $[K(\varepsilon) : K]$ is not divisible by p, where ε is a primitive l_Bth root of 1. It is not difficult to see that $[K(\varepsilon) : K]$ is not divisible by p if and only if $[K(\xi) : K]$ is not divisible by p, where ξ is a primitive $(\exp B)$th root of 1. Given $\lambda \in Z^2(H, K^*)$, $K^\lambda H$ denotes the twisted group algebra of a group H over K with a 2-cocycle λ, and $\operatorname{rad} K^\lambda H$ the radical of $K^\lambda H$. A K-basis $\{u_h : h \in H\}$ of $K^\lambda H$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in H$ is called *canonical* (corresponding to λ). If D is a subgroup of a group H, the restriction of $\lambda \in Z^2(H, K^*)$ to $D \times D$ is also denoted by λ. In this case, $K^\lambda D$ is a subalgebra of $K^\lambda H$.

Throughout this paper we assume that all cocycle groups are defined with respect to the trivial action of the underlying group on K^*. By Theorem 4.7 in [21, p. 40], the embedding $t(K^*) \to K^*$ induces an injective homomorphism

$$H^2(B, t(K^*)) \to H^2(B, K^*).$$

We shall identify $H^2(B, t(K^*))$ with the subgroup of $H^2(B, K^*)$ which consists of all cohomology classes containing cocycles of finite order.

Given $\mu \in Z^2(G_p, K^*)$, the kernel $\mathrm{Ker}(\mu)$ of μ is the union of all cyclic subgroups $\langle g \rangle$ of G_p such that the restriction of μ to $\langle g \rangle \times \langle g \rangle$ is a coboundary. We recall from [4, p. 196] that $G_p' \subset \mathrm{Ker}(\mu)$, $\mathrm{Ker}(\mu)$ is a normal subgroup of G_p and the restriction of μ to $\mathrm{Ker}(\mu) \times \mathrm{Ker}(\mu)$ is a coboundary.

Let M be a finite group, N a normal subgroup of M and $T = M/N$. Given $\mu \in Z^2(T, K^*)$, denote by $\inf(\mu)$ (see [21, p. 14]) the element of $Z^2(M, K^*)$ defined by

$$\inf(\mu)_{a,b} = \mu_{aN,bN} \quad \text{for all } a, b \in M.$$

We have $\inf(\mu)_{x,y} = 1$ for all $x, y \in N$. Therefore

$$K^{\inf(\mu)} N = KN.$$

Let $\lambda = \inf(\mu)$, $\{v_{aN} : a \in M\}$ be a canonical K-basis of $K^\mu T$ corresponding to μ, and $\{u_a : a \in M\}$ a canonical K-basis of $K^\lambda M$ corresponding to λ. The formula

$$f\Big(\sum_{a \in M} \alpha_a u_a\Big) = \sum_{a \in M} \alpha_a v_{aN}$$

defines a K-algebra epimorphism $f : K^\lambda M \to K^\mu T$ with the kernel $U :=$

$K^\lambda M \cdot I(N)$, where $I(N)$ is the augmentation ideal of the group algebra KN (see [21, p. 88]). Hence $K^\lambda M/U \cong K^\mu T$. We recall that

$$I(N) = \bigoplus_{x \in N \setminus \{e\}} K(u_x - u_e).$$

Assume that N and M are groups. An *extension* of N by M is a short exact sequence of groups

$$E : 1 \xrightarrow{\varphi} N \to \widehat{M} \to M \to 1.$$

If $\varphi(N)$ is contained in the center of $\widehat{M}$, then E is called a *central extension.* If N and M are finite groups, then E is a *finite extension.*

Let V be a finite-dimensional vector space over K, $\mathrm{GL}(V)$ the group of all automorphisms of V, 1_V the identity automorphism of V, M a finite group, and let

$$1 \to N \to \widehat{M} \xrightarrow{\psi} M \to 1$$

be a finite central group extension. Denote by $\pi : \mathrm{GL}(V) \to GL(V)/K^*1_V$ the canonical group epimorphism. Assume that Γ is an ordinary K-representation of $\widehat{M}$ in V with $\Gamma(x) \in K^*1_V$ for any $x \in N$. There exists a projective K-representation Δ of M in V such that the diagram

$$\begin{array}{ccccc} \widehat{M} & \xrightarrow{\Gamma} & \mathrm{GL}(V) & \xrightarrow{\pi} & \mathrm{GL}(V)/K^*1_V \\ \psi \downarrow & & & & \downarrow \mathrm{id} \\ M & \xrightarrow{\Delta} & \mathrm{GL}(V) & \xrightarrow{\pi} & \mathrm{GL}(V)/K^*1_V \end{array}$$

is commutative. We say that Δ *lifts projectively* to the ordinary K-representation Γ of $\widehat{M}$. If $|N| = |H^2(M, K^*)|$ and any projective K-representation of M lifts projectively to an ordinary K-representation of $\widehat{M}$, then $\widehat{M}$ is called a *covering group* of M over K [21, p. 138].

We recall that, for any cocycle $\lambda \in Z^2(G_p, K^*)$, the quotient algebra $K^\lambda G_p/\mathrm{rad}\, K^\lambda G_p$ is K-isomorphic to a field that is a finite purely inseparable field extension of K [21, p. 74]. We call $K^\lambda G_p$ *uniserial* if the left regular and the right regular $K^\lambda G_p$-modules have a unique composition series. It should be noted that some authors use the terminology "uniserial algebra" to mean principal ideal algebras [16, p. 171] and serial algebras (see [15, p. 505] and [16, p. 175]) that are Nakayama algebras [2, p. 168]. By the Morita theorem in [15, p. 507], the algebra $K^\lambda G_p$ is uniserial if and only if $\mathrm{rad}\, K^\lambda G_p = K^\lambda G_p \cdot v = v \cdot K^\lambda G_p$ for some $v \in K^\lambda G_p$. By [16, p. 170], the algebra $K^\lambda G_p$ is uniserial if and only if $\mathrm{rad}\, K^\lambda G_p$ is a principal left (equivalently, right) ideal of $K^\lambda G_p$.

We say that an algebra $K^\lambda G_p$ satisfies the *Q-condition* if there exists a K-algebra epimorphism $K^\lambda G_p \to K^\mu T$, where T is a p-group and T contains an abelian subgroup A such that $K^\mu A$ is not a uniserial algebra.

The following four facts are proved in [6].

Lemma 1.1. *Let K be an arbitrary field of characteristic p, $G = G_p \times B$, $\mu \in Z^2(G_p, K^*)$, $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. If $K^\mu G_p$ is a uniserial algebra or K is a splitting field for $K^\nu B$, then $K^\lambda G$ is of OTP representation type.*

Lemma 1.2. *Let $G = G_p \times B$, $\mu \in Z^2(G_p, K^*)$, $\nu \in Z^2(B, K^*)$, $\lambda = \mu \times \nu$ and assume that $K^\mu G_p$ satisfies the Q-condition. The algebra $K^\lambda G$ is of OTP representation type if and only if K is a splitting field for $K^\nu B$.*

Theorem 1.3. *Let $G = G_p \times B$, $\mu \in Z^2(G_p, K^*)$, $\nu \in Z^2(B, K^*)$, $\lambda = \mu \times \nu$ and $d = \dim_K(K^\mu G_p / \operatorname{rad} K^\mu G_p)$. Denote by D the subgroup of G_p such that $G'_p \subset D$ and $D/G'_p = \operatorname{soc}(G_p/G'_p)$. Assume that if $K^\mu G_p$ is not uniserial, $pd = |G_p : G'_p|$ and $|G'_p| = p$, then $\operatorname{Ker}(\mu) \neq G'_p$ or $|D : Z(D)| \in \{1, p^2\}$. The K-algebra $K^\lambda G$ is of OTP representation type if and only if either $K^\mu G_p$ is uniserial, or K is a splitting field for $K^\nu B$.*

Proposition 1.4. *Let K be an arbitrary field of characteristic p, $G = G_p \times B$, $\nu \in Z^2(B, K^*)$ and $K^\lambda G = KG_p \otimes_K K^\nu B$. The K-algebra $K^\lambda G$ is of OTP representation type if and only if either G_p is cyclic, or K is a splitting field for $K^\nu B$.*

2. On splitting fields for twisted group algebras. We say that an abelian group is of *symmetric type* if it can be decomposed into a direct product of two isomorphic subgroups.

Let G be an abelian group, F an arbitrary field, $\lambda \in Z^2(G, F^*)$, $\{u_g : g \in G\}$ a canonical F-basis of $F^\lambda G$ corresponding to λ, Z the center of $F^\lambda G$ and $H = \{h \in G : u_h \in Z\}$. Then H is a subgroup of G and $Z = F^\lambda H$. Obviously

$$H = \{h \in G : \lambda_{h,g} = \lambda_{g,h} \text{ for any } g \in G\}.$$

We call H the λ-*center* of G.

Proposition 2.1. *Let G be abelian, $\lambda \in Z^2(G, F^*)$, H the λ-center of G, $\overline{G} = G/H$ and $\bar{x} = xH$ for any $x \in G$. Assume that $G \neq H$.*

(i) *The algebra $F^\lambda G$ may be viewed as a twisted group ring $Z^{\bar{\lambda}}\overline{G}$ of $\overline{G}$ over the ring $Z = F^\lambda H$. Moreover*

$$\bar{\lambda}_{\bar{x},\bar{y}} \cdot \bar{\lambda}^{-1}_{\bar{y},\bar{x}} \in t(F^*) \quad \textit{for all } x, y \in G.$$

(ii) *There exists a direct product decomposition $\overline{G} = \overline{C}_1 \times \cdots \times \overline{C}_s$ such that $\overline{C}_i = \langle \bar{a}_i \rangle \times \langle \bar{b}_i \rangle$ is a q_i-group of type $(q_i^{n_i}, q_i^{n_i})$,*

$$Z^{\bar{\lambda}}\overline{G} \cong Z^{\bar{\lambda}}\overline{C}_1 \otimes_Z \cdots \otimes_Z Z^{\bar{\lambda}}\overline{C}_s$$

and

(2.1)
$$Z^{\bar{\lambda}}\overline{C}_i = \bigoplus_{j,k=0}^{q_i^{n_i}-1} Z v_{\bar{a}_i}^j v_{\bar{b}_i}^k$$

with
$$v_{\bar{a}_i}^{q_i^{n_i}} = \alpha_i v_{\bar{e}}, \quad v_{\bar{b}_i}^{q_i^{n_i}} = \beta_i v_{\bar{e}}, \quad v_{\bar{a}_i} v_{\bar{b}_i} = \varepsilon_i v_{\bar{b}_i} v_{\bar{a}_i},$$
where $\alpha_i, \beta_i \in Z$, $\varepsilon_i \in t(F^*)$ *and* $o(\varepsilon_i) = q_i^{n_i}$ *for every* $i \in \{1, \dots, s\}$.

(iii) $\overline{G}$ *is a group of symmetric type and* F *contains a primitive* m*th root of* 1, *where* $m = \exp \overline{G}$.

Proof. Let $\{g_1, \dots, g_r\}$ be a cross section of H in G and $g_1 = e$. Then
$$F^\lambda G = Z u_{g_1} \oplus \cdots \oplus Z u_{g_r}.$$
Put $v_{\bar{g}_i} = u_{g_i}$ for every $i \in \{1, \dots, r\}$. The algebra $F^\lambda G$ may be viewed as a twisted group ring $Z^{\bar{\lambda}}\overline{G}$ of the group $\overline{G}$ over the ring Z with a canonical Z-basis $v_{\bar{g}_1}, \dots, v_{\bar{g}_r}$. For any $x, y \in G$ we have $v_{\bar{x}} v_{\bar{y}} = \xi v_{\bar{y}} v_{\bar{x}}$, where $\xi \in t(F^*)$. The ring Z is the center of $Z^{\bar{\lambda}}\overline{G}$. We also have
$$Z^{\bar{\lambda}}\overline{G} \cong Z^{\bar{\lambda}}\overline{G}_{q_1} \otimes_Z \cdots \otimes_Z Z^{\bar{\lambda}}\overline{G}_{q_k},$$
where $\overline{G}_{q_i}$ is the Sylow q_i-subgroup of $\overline{G}$ for each $i \in \{1, \dots, k\}$.

Let q be a prime and $\overline{G}_q = \langle \bar{x}_1 \rangle \times \cdots \times \langle \bar{x}_t \rangle$ be a Sylow q-subgroup of $\overline{G}$. Assume that $|\bar{x}_j| = q^{m_j}$ and $m_1 \geq \cdots \geq m_t$. The set
$$\{v_{\bar{x}_1}^{k_1} \dots v_{\bar{x}_t}^{k_t} : k_i = 0, 1, \dots, q^{m_i} - 1 \text{ for every } i \in \{1, \dots, t\}\}$$
is a Z-basis of the algebra $Z^{\bar{\lambda}}\overline{G}_q$. We have
$$v_{\bar{x}_1} v_{\bar{x}_j} = \xi_j v_{\bar{x}_j} v_{\bar{x}_1}$$
for any $j \in \{2, \dots, t\}$, where $\xi_j \in F^*$ and $o(\xi_j) \leq q^{m_j}$. If $m_1 > m_2$, then $v_{\bar{x}_1}^{q^{m_2}} \neq v_{\bar{e}}$ and $v_{\bar{x}_1}^{q^{m_2}}$ belongs to the center of $Z^{\bar{\lambda}}\overline{G}$. Hence there exists an $\bar{x}_{j_0}$ such that $|\bar{x}_{j_0}| = q^{m_1}$ and $o(\xi_{j_0}) = q^{m_1}$. Let $j_0 = 2$ and $\xi = \xi_2$. We have
$$v_{\bar{x}_1} v_{\bar{x}_2} = \xi v_{\bar{x}_2} v_{\bar{x}_1}, \quad v_{\bar{x}_i} v_{\bar{x}_j} = \xi^{\gamma_{ij}} v_{\bar{x}_j} v_{\bar{x}_i}$$
for all i, j, where $0 \leq \gamma_{ij} < q^{m_1}$ and $o(\xi^{\gamma_{ij}}) \leq \max\{|\bar{x}_i|, |\bar{x}_j|\}$ for all $i, j \in \{1, \dots, t\}$.

Let $\bar{y}_1 = \bar{x}_1$, $\bar{y}_2 = \bar{x}_2$, $\bar{y}_3 = \bar{x}_1^{\alpha_{31}} \bar{x}_2^{\alpha_{32}} \bar{x}_3, \dots, \bar{y}_t = \bar{x}_1^{\alpha_{t1}} \bar{x}_2^{\alpha_{t2}} \bar{x}_t$ and
$$w_{\bar{y}_1} = v_{\bar{x}_1}, \quad w_{\bar{y}_2} = v_{\bar{x}_2}, \quad w_{\bar{y}_3} = v_{\bar{x}_1}^{\alpha_{31}} v_{\bar{x}_2}^{\alpha_{32}} v_{\bar{x}_3}, \quad \dots, \quad w_{\bar{y}_t} = v_{\bar{x}_1}^{\alpha_{t1}} v_{\bar{x}_2}^{\alpha_{t2}} v_{\bar{x}_t},$$
where
$$\alpha_{j1} = \gamma_{2j}, \quad \alpha_{j2} = q^{m_1} - \gamma_{1j}$$
for every $j \in \{3, \dots, t\}$. Then
$$w_{\bar{y}_1} w_{\bar{y}_j} = w_{\bar{y}_j} w_{\bar{y}_1}, \quad w_{\bar{y}_2} w_{\bar{y}_j} = w_{\bar{y}_j} w_{\bar{y}_2}$$
for every $j \in \{3, \dots, t\}$, and $\overline{G}_q = \langle \bar{y}_1 \rangle \times \cdots \times \langle \bar{y}_t \rangle$. Therefore
$$Z^{\bar{\lambda}}\overline{G}_q \cong Z^{\bar{\lambda}}\overline{G}_q^{(1)} \otimes_Z Z^{\bar{\lambda}}\overline{G}_q^{(2)},$$

where $\overline{G}_q^{(1)} = \langle \bar{y}_1 \rangle \times \langle \bar{y}_2 \rangle$, $\overline{G}_q^{(2)} = \langle \bar{y}_3 \rangle \times \cdots \times \langle \bar{y}_t \rangle$ and $Z^{\bar{\lambda}}\overline{G}_q^{(2)}$ is Z-central. By induction on t, we conclude that

$$Z^{\bar{\lambda}}\overline{G}_q \cong Z^{\bar{\lambda}}\overline{D}_1 \otimes_Z \cdots \otimes_Z Z^{\bar{\lambda}}\overline{D}_{s_q},$$

where $\overline{D}_j$ is a q-group of type (q^{k_j}, q^{k_j}) and $Z^{\bar{\lambda}}\overline{D}_j$ is a central Z-algebra of the form (2.1), for any $j \in \{1, \dots, s_q\}$.

The group $\overline{G}_q$ is of symmetric type. Hence $\overline{G}$ is a group of symmetric type. The field F contains a primitive m_qth root of 1, where $m_q = \exp \overline{G}_q$. It follows that F contains a primitive mth root of 1, where $m = \exp \overline{G}$. ∎

We note that Proposition 2.1 is a generalization of Theorem 2.12 in [22, p. 375]. From Proposition 2.1 one can also deduce Corollary 1.12 in [22, p. 368].

PROPOSITION 2.2. *Let B be an abelian p'-group, $\lambda \in Z^2(B, K^*)$, H the λ-center of B and $\overline{B} = B/H$. Assume that K is a splitting field for $K^\lambda B$.*

- (i) *The field K contains a primitive* $(\exp H)$*th root of* 1*, and there exists $\mu \in Z^2(\overline{B}, K^*)$ such that λ is cohomologous to* $\inf(\mu)$.
- (ii) *The algebra $K^\lambda B$ is K-algebra isomorphic to $K^{\mu_1}\overline{B} \times \cdots \times K^{\mu_l}\overline{B}$, where $l = |H|$, $\mu_1 = \mu$ and $K^{\mu_i}\overline{B}$ is K-algebra isomorphic to $\mathbb{M}_n(K)$, $n^2 = |\overline{B}|$, for every $i \in \{1, \dots, l\}$.*

Proof. (i) K is a splitting field for $Z = K^\lambda H$. It follows that the restriction of λ to $H \times H$ is a coboundary and K contains a primitive $(\exp H)$th root of 1. The algebra $K^\lambda H$ is isomorphic to KH. We may assume that $K^\lambda H = KH$. Denote by $I(H)$ the augmentation ideal of KH. By Lemma 5.5 in [21, p. 91], $K^\lambda B / K^\lambda B \cdot I(H) \cong K^\mu \overline{B}$ for some $\mu \in Z^2(\overline{B}, K^*)$ such that λ is cohomologous to $\inf(\mu)$.

(ii) Let $l = |H|$, $e_1, \dots, e_l$ be a complete system of primitive pairwise orthogonal idempotents of Z and $u_h e_1 = e_1$ for any $h \in H$. Then Ze_i is K-algebra isomorphic to K and, by Proposition 2.1,

$$K^\lambda B e_i \cong (Ze_i)^{\sigma_i}\overline{B} \cong K^{\mu_i}\overline{B}$$

for every $i \in \{1, \dots, l\}$. Moreover, $K^{\mu_1}\overline{B} \cong K^\mu \overline{B}$, $K^{\mu_i}\overline{B}$ is a central K-algebra and K is a splitting field for $K^{\mu_i}\overline{B}$ for each i. Hence $K^{\mu_i}\overline{B}$ is K-algebra isomorphic to $\mathbb{M}_n(K)$, $n^2 = |\overline{B}|$, for every $i \in \{1, \dots, l\}$. ∎

LEMMA 2.3. *Let B be an abelian p'-group of symmetric type. Assume that the field K contains a primitive* $(\exp B)$*th root of* 1*. Then there exists a cocycle $\mu \in Z^2(B, t(K^*))$ such that $K^\mu B \cong \mathbb{M}_n(K)$, where $n^2 = |B|$.*

Proof. We may suppose that B is an abelian q-group of type (q^r, q^r), where $q \neq p$. Let ξ be a primitive q^rth root of 1, F a finite subfield of K

which contains ξ, $B = \langle x \rangle \times \langle y \rangle$ and

$$F^{\mu}B = \bigoplus_{i,j=0}^{q^r-1} Fu_x^i u_y^j, \qquad u_x^{q^r} = u_e, \qquad u_y^{q^r} = u_e, \qquad u_x u_y = \xi u_y u_x.$$

The F-algebra $F^{\mu}B$ is central. Since a finite division algebra is a field, $F^{\mu}B$ is F-algebra isomorphic to $\mathbb{M}_n(F)$, where $n = q^r$. It follows that the K-algebra $K^{\mu}B := K \otimes_F F^{\mu}B$ is K-isomorphic to $\mathbb{M}_n(K)$. ∎

PROPOSITION 2.4. *Assume that B is an abelian p'-group and H is a subgroup of B such that $\overline{B} := B/H$ is of symmetric type and K contains a primitive mth root of* 1*, where $m = \max\{\exp \overline{B}, \exp H\}$. Let $\mu \in Z^2(\overline{B}, K^*)$ and $\lambda = \inf(\mu)$.*

(i) *If $K^{\mu}\overline{B}$ is a central K-algebra then $K^{\lambda}B$ can be decomposed into a direct product of central twisted group algebras of $\overline{B}$ over K.*
(ii) *If $\mu \in Z^2(\overline{B}, t(K^*))$ and $K^{\mu}\overline{B}$ is a central K-algebra, then K is a splitting field for the algebra $K^{\lambda}B$.*
(iii) *Let K contain a primitive* $(\exp B)$*th root of* 1*. If $K^{\mu}\overline{B}$ is K-algebra isomorphic to $\mathbb{M}_n(K)$, where $n^2 = |\overline{B}|$, then $K^{\lambda}B$ is K-algebra isomorphic to the direct product of l copies of $\mathbb{M}_n(K)$, where $l = |H|$.*

Proof. (i) Denote by $\{v_{bH} : b \in B\}$ a canonical K-basis of $K^{\mu}\overline{B}$ corresponding to μ and by $\{u_b : b \in B\}$ a canonical K-basis of $K^{\lambda}B$ corresponding to λ.

We have $K^{\lambda}H = KH$. If $b \in B$ and $h \in H$ then $\lambda_{b,h} = \mu_{bH,H} = 1$ and $\lambda_{h,b} = 1$. It follows that $u_b u_h = u_h u_b$. Therefore $KH \subset Z(K^{\lambda}B)$. Assume that $u_g \in Z(K^{\lambda}B)$ for certain $g \in B$. Then $u_g u_b = u_b u_g$ for each $b \in B$. Hence $v_{gH}v_{bH} = v_{bH}v_{gH}$ for any $b \in B$. Since $K^{\mu}\overline{B}$ is a central K-algebra, $gH = H$ and consequently $Z(K^{\lambda}B) = KH$. This means that H is the λ-center of B. The field K is a splitting field for KH. It follows, by Proposition 2.1, that $K^{\lambda}B$ can be decomposed into a direct product of central twisted group algebras of $\overline{B}$ over K.

(ii) Denote by F a finite subfield of K which contains a primitive mth root of 1 and all values of the cocycle μ. The algebra $F^{\mu}\overline{B}$ is a central F-algebra. By (i), F is a splitting field for $F^{\lambda}B$, since each finite division algebra is a field. It follows that K is a splitting field for the algebra $K^{\lambda}B \cong K \otimes_F F^{\lambda}B$.

(iii) By Theorem 6.1 in [25, p. 179], $K^{\lambda}B$ can be decomposed into a direct product of mutually isomorphic simple algebras over K. Since $K^{\mu}\overline{B}$ is a simple component of $K^{\lambda}B$, the algebra $K^{\lambda}B$ is K-algebra isomorphic to $K^{\mu}\overline{B} \times \cdots \times K^{\mu}\overline{B}$. ∎

PROPOSITION 2.5 ([7, p. 20]). *Let B be an abelian p'-group. The field K is a splitting field for some K-algebra $K^{\lambda}B$ if and only if B has a subgroup*

H such that B/H is of symmetric type and K contains a primitive mth root of 1, where $m = \max\{\exp(B/H), \exp H\}$.

Proof. Apply Propositions 2.1, 2.2, 2.4 and Lemma 2.3. ∎

PROPOSITION 2.6. *Let K be a finite field of characteristic p, B an abelian p'-group, $\lambda \in Z^2(B, K^*)$ and H the λ-center of B. The field K is a splitting field for $K^\lambda B$ if and only if the restriction of λ to $H \times H$ is a coboundary and K contains a primitive $(\exp H)$th root of 1.*

Proof. Apply Propositions 2.1 and 2.2. ∎

PROPOSITION 2.7. *Let B be a nilpotent p'-group. If K is a splitting field for some twisted group algebra of B over K, then K contains a primitive qth root of 1 for each prime q that divides $|B|$.*

Proof. Assume that K is a splitting field for an algebra $K^\lambda B$ and K does not contain a primitive qth root of 1 for a certain prime q dividing $|B|$. Denote by B_q the Sylow q-subgroup of B. The center of B_q contains an element b of order q. Let $\{u_g : g \in B\}$ be a canonical K-basis of $K^\lambda B$ corresponding to λ. Then u_b lies in the center Z of $K^\lambda B$. Let $\{f_1, \ldots, f_s\}$ be a complete system of pairwise orthogonal primitive idempotents of Z. We have $u_b = \beta_1 f_1 + \ldots + \beta_s f_s$, where $\beta_j \in K$ for every $j \in \{1, \ldots, s\}$. If $u_b^q = \gamma u_e$, $\gamma \in K^*$, then $\gamma = \beta_j^q$ for each j. It follows that $\beta_1 = \cdots = \beta_s$ and $u_b = \beta_1 u_e$. This contradiction proves that K contains a primitive qth root of 1 for each prime q that divides $|B|$. ∎

PROPOSITION 2.8. *Let B be a p'-group.*

(i) *If the field K is a splitting field for all twisted group algebras of B over K, then $K = K^q$ and K contains a primitive qth root of 1 for each prime q that divides $|B : B'|$.*
(ii) *If $K = K^q$ and K contains a primitive qth root of 1 for any prime q that divides $|B|$, then K is a splitting field for every twisted group algebra of B over K.*
(iii) *Assume that every prime divisor of $|B'|$ is also a divisor of $|B : B'|$. Then K is a splitting field for any twisted group algebra of B over K if and only if $K = K^q$ and K contains a primitive qth root of 1 for each prime q that divides $|B|$.*

Proof. (i) Let $B \neq B'$ and q be a prime divisor of $|B : B'|$. Denote by D a normal subgroup of B such that $|B/D| = q$. Let $\widetilde{B} := B/D = \langle xD \rangle$, $\alpha \in K^*$ and

$$K^\mu \widetilde{B} = \bigoplus_{i=0}^{q-1} K v_{xD}^i, \qquad v_{xD}^q = \alpha v_D.$$

Denote $\lambda = \inf(\mu)$. There exists a K-algebra homomorphism of $K^\lambda B$ onto $K^\mu \widetilde{B}$. It follows that K is a splitting field for $K^\mu \widetilde{B}$. Hence $\alpha = \beta^q$ for some $\beta \in K^*$ and K contains a primitive qth root of 1.

(ii) Denote by n the order of a cohomology class $[\lambda] \in H^2(B, K^*)$. It is well known that n divides $|B|$. Arguing as in the proof of Theorem 53.3 in [14, p. 359], we show that $[\lambda]$ contains a cocycle α whose order is equal to n. By Theorem 1.3 in [21, p. 137], there exists a central group extension $1 \to A \to \widehat{B} \to B \to 1$ such that A is a cyclic group of order n and

$$K\widehat{B} \cong \prod_{i=0}^{n-1} K^{\alpha^i} B.$$

Since any prime divisor of $|\widehat{B}|$ is also a divisor of B, the field K contains a primitive mth root of 1, where $m = \exp \widehat{B}$. By the Brauer theorem, K is a splitting field for $K\widehat{B}$. Hence K is a splitting field for $K^\alpha B$.

(iii) Apply (i) and (ii). ∎

Proposition 2.9. *Let B be a p'-group. The field K is a splitting field for all twisted group algebras of B over K if and only if there exists a finite central group extension $1 \to A \to \widehat{B} \to B \to 1$ such that any projective K-representation of B lifts projectively to an ordinary K-representation of $\widehat{B}$ and K is a splitting field for $\widehat{B}$.*

Proof. Assume that K is a splitting field for all twisted group algebras of B over K. By Proposition 2.8, $K^* = (K^*)^m$, where m is the exponent of B/B'. In view of Corollary 2.5 in [21, p. 142], $H^2(B, K^*) = H^2(B, t(K^*))$. Arguing as in the proof of Theorem 2.3 in [21, p. 141], we conclude that there exists a finite central group extension $1 \to A \to \widehat{B} \to B \to 1$ such that the following conditions hold:

(i) If r is the exponent of A, then K^* contains a primitive rth root of 1.
(ii) Every projective K-representation of B lifts projectively to an ordinary K-representation of $\widehat{B}$.

By Theorem 4.2 in [21, p. 80], $K\widehat{B} \cong \prod_i K^{\lambda_i} B$. It follows that K is a splitting field for $K\widehat{B}$. This completes the proof of the necessity.

Let us prove the sufficiency. The group algebra KA lies in the center of $K\widehat{B}$, hence K contains a primitive mth root of 1, where m is the exponent of A. It follows, by Theorem 4.2 in [21, p. 80] and Lemma 2.1 in [21, p. 139], that $K\widehat{B}$ is K-algebra isomorphic to $K^{\sigma_1} B \times \cdots \times K^{\sigma_r} B$ and every algebra $K^\lambda B$ is isomorphic to some $K^{\sigma_i} B$. Hence K is a splitting field for every twisted group algebra of B over K. ∎

Proposition 2.10. *Let B be a p'-group. Assume that either $t(K^*) = t(K^*)^q$ for every prime q that divides $|B'|$, or every prime divisor of $|B'|$ is* *also a divisor of* $|B : B'|$. *Then* K *is a splitting field for any twisted group algebra of* B *over* K *if and only if there exists a covering group* $\widehat{B}$ *of* B *over* K *such that* K *is a splitting field for* $\widehat{B}$.

Proof. Assume that K is a splitting field for any twisted group algebra of B over K. In view of Proposition 2.8, $K = K^q$ for each prime q dividing $|B : B'|$. It follows that $t(K^*) = t(K^*)^q$ for every prime q that divides B. Arguing as in the proof of Theorem 53.3 in [14, p. 359], we show that each cohomology class $[\lambda] \in H^2(B, t(K^*))$ contains a cocycle whose order is equal to the order of $[\lambda]$. In view of Theorem 2.3 in [21, p. 140], there exists a finite central group extension $1 \to A \to \widehat{B} \to B \to 1$ such that $A \cong H^2(B, t(K^*))$ and any projective K-representation of B lifts projectively to an ordinary K-representation of $\widehat{B}$. By Corollary 2.5 in [21, p. 142], we have

$$H^2(B, K^*) \cong H^2(B, t(K^*)).$$

Hence $\widehat{B}$ is a covering group of B over K. Theorem 4.2 in [21, p. 80] yields

$$K\widehat{B} \cong \prod_i K^{\sigma_i} B,$$

since K^* contains a primitive $(\exp A)$th root of 1. It follows that K is a splitting field for $\widehat{B}$. This proves the necessity.

The sufficiency follows from Proposition 2.9. ∎

We note that in [25] Yamazaki proved Theorem 4.2 from [21, p. 80] while Theorem 2.3 from [21, p. 140] and Corollary 2.5 from [21, p. 142] are proved in [26].

3. Groups of OTP projective representation type. We recall that K is a field of characteristic p and $G = G_p \times B$ is a finite group, where G_p is a p-group, B is a p'-group and $|G_p| \neq 1$, $|B| \neq 1$. We assume that if G_p is non-abelian then $[K(\xi) : K]$ is not divisible by p, where ξ is a primitive $(\exp B)$th root of 1.

THEOREM 3.1. *Let* $G = G_p \times B$, s *be the number of invariants of the group* G_p/G'_p *and* D *the subgroup of* G_p *such that* $G'_p \subset D$ *and* $D/G'_p = \operatorname{soc}(G_p/G'_p)$. *Assume that if* $p \neq 2$, $s = i(K) + 1$, $|G'_p| = p$ *and* D *is a non-abelian group of exponent* p, *then* $|D : Z(D)| = p^2$. *The group* G *is of OTP projective* K*-representation type if and only if one of the following conditions is satisfied:*

(i) $s \leq i(K)$ *and* G'_p *is cyclic;*
(ii) $s = i(K) + 1$, G'_p *is cyclic and there exists a cyclic subgroup* T *of* G_p *such that* $G'_p \subset T$ *and* G_p/T *has* $i(K)$ *invariants;*
(iii) K *is a splitting field for some* $K^\nu B$.

Proof. Suppose (ii). Let $\widetilde{G}_p = G_p/T$. There is a cocycle $\sigma \in Z^2(\widetilde{G}_p, K^*)$ such that $K^\sigma \widetilde{G}_p$ is a field. Let $\mu = \inf(\sigma)$. If $V := K^\mu G_p \cdot I(T)$ then V is the radical of $K^\mu G_p$ and $K^\mu G_p/V$ is K-algebra isomorphic to $K^\sigma \widetilde{G}_p$. Therefore $K^\mu G_p$ is a uniserial algebra. Let $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. In view of Theorem 1.3, $K^\lambda G$ is of OTP representation type.

Arguing as in the case (ii) we prove that if (i) holds, then there exists a cocycle $\lambda \in Z^2(G, K^*)$ such that $K^\lambda G$ is of OTP representation type.

Assume that K is a splitting field for some $K^\nu B$. Let $K^\lambda G = KG_p \otimes_K K^\nu B$. By Theorem 1.3, $K^\lambda G$ is of OTP representation type.

If $s \geq i(K) + 2$ or G'_p is non-cyclic then $K^\mu G_p$ is not a uniserial algebra for any $\mu \in Z^2(G_p, K^*)$. Moreover, in the case $s \geq i(K) + 2$, we have $|G_p : G'_p| \geq p^2 d$, where

$$d = \dim_K(K^\mu G_p/\operatorname{rad} K^\mu G_p).$$

Let $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. By Theorem 1.3, an algebra $K^\lambda G$ is of OTP representation type if and only if K is a splitting field for $K^\nu B$.

Assume now that $s = i(K) + 1$, $G'_p = \langle c \rangle$ and G_p does not contain a cyclic subgroup T such that $G'_p \subset T$ and G_p/T has $i(K)$ invariants. Let $H = \langle c^p \rangle$ and $G_p/G'_p = \langle a_1 G'_p \rangle \times \cdots \times \langle a_s G'_p \rangle$, where $|a_j G'_p| = p^{n_j}$ for every $j \in \{1, \dots, s\}$. We have

$$a_j^{p^{n_j}} \in H \quad \text{for each } j \in \{1, \dots, s\}.$$

First, we examine the case $p = 2$. Let $N_{r,t}$ be the subgroup of G_2 generated by the elements a_r, a_t and c, where $r, t \in \{1, \dots, s\}$ and $r \neq t$. If $|N_{r,t} : G'_2| = 4$ and $N'_{r,t} = G'_2$, then $N_{r,t}$ is metacyclic. There exists a cyclic subgroup T of $N_{r,t}$ such that $G'_2 \subset T$ and G_2/T has $i(K)$ invariants, a contradiction. Hence, if $|N_{r,t} : G'_2| = 4$, we have $[a_r, a_t] \in H$ and

$$D/H = \langle cH \rangle \times \langle b_1 H \rangle \times \cdots \times \langle b_s H \rangle,$$

where $b_j = a_j^{2^{n_j - 1}}$ for every $j \in \{1, \dots, s\}$. Each twisted group algebra of the group D/H over the field K is non-uniserial. Consequently, every $K^\mu G_2$ satisfies the Q-condition. By Lemma 1.2, the group $G = G_2 \times B$ is of OTP projective K-representation type if and only if condition (iii) holds.

Now we consider the case $p \neq 2$. By [5, p. 288], $|D'| \leq p$. If $|G'_p| \geq p^2$ then $D/H = \langle cH \rangle \times \langle b_1 H \rangle \times \cdots \times \langle b_s H \rangle$, where

$$b_j = a_j^{p^{n_j - 1}} \quad \text{for each } j \in \{1, \dots, s\}.$$

Arguing as in the case $p = 2$, we conclude that G is of OTP projective K-representation type if and only if condition (iii) holds. Let $|G'_p| = p$. Then $\exp D = p$. If D is abelian then, for any $\mu \in Z^2(G_p, K^*)$, the algebra $K^\mu D$ is not uniserial. Hence in this case every $K^\mu G_p$ satisfies the Q-condition, and Lemma 1.2 applies. Suppose that D is non-abelian. Then $|D : Z(D)| = p^2$

and, for every $\mu \in Z^2(G_p, K^*)$, the algebra $K^\mu G_p/K^\mu G_p \cdot \operatorname{rad} KG'_p$ is not a field. In view of Lemma 1.7 in [8, p. 177], $K^\mu G_p$ is not a uniserial algebra. By Theorem 1.3, G is of OTP projective K-representation type if and only if condition (iii) holds. ∎

Corollary 3.2. *Let $G = G_p \times B$ and K be an arbitrary perfect field of characteristic p. The group G is of OTP projective K-representation type if and only if G_p is cyclic or K is a splitting field for some $K^\nu B$.*

Corollary 3.3. *Let $G = G_p \times B$ and $[K : K^p] = p$. Then G is of OTP projective K-representation type if and only if either G_p is metacyclic or K is a splitting field for some $K^\nu B$.*

Corollary 3.4. *Let $G = G_p \times B$, s be the number of invariants of G_p/G'_p and $[K : K^p] = p^2$. The group G is of OTP projective K-representation type if and only if one of the following conditions is satisfied:*

(i) *$s \leq 2$ and G'_p is cyclic;*
(ii) *$s = 3$ and there exists a cyclic subgroup T of G_p such that $G'_p \subset T$ and G_p/T has two invariants;*
(iii) *K is a splitting field for some $K^\nu B$.*

Proof. Keep the notation of Theorem 3.1. Assume that $p \neq 2$, $s = 3$, $|G'_p| = p$ and D is a non-abelian group of exponent p. Moreover, let $D/G'_p = \langle b_1G'_p\rangle \times \langle b_2G'_p\rangle \times \langle b_3G'_p\rangle$, $G'_p = \langle c\rangle$ and $[b_1, b_2] = c$, $[b_1, b_3] = c^r$, $[b_2, b_3] = c^t$, where $0 \leq r, t < p$. Set $h = b_1^t b_2^{-r} b_3$. Then $b_1h = hb_1$, $b_2h = hb_2$. It follows that $Z(D)$ is generated by h, c. Hence $|D : Z(D)| = p^2$. Applying Theorem 3.1, we conclude that G is of OTP projective K-representation type if and only if one of the present conditions (i)–(iii) is satisfied. ∎

Corollary 3.5. *Let $G = G_p \times B$ and $[K : K^p] = \infty$. The group G is of OTP projective K-representation type if and only if either G'_p is cyclic, or K is a splitting field for some $K^\nu B$.*

Proposition 3.6. *Let $G = G_p \times B$ be an abelian group and s the number of invariants of G_p. The group G is of OTP projective K-representation type if and only if one of the following conditions is satisfied:*

(i) *$s \leq i(K) + 1$;*
(ii) *B has a subgroup H such that B/H is of symmetric type and K contains a primitive mth root of 1, where $m = \max\{\exp(B/H), \exp H\}$.*

Proof. Apply Proposition 2.5 and Theorem 3.1. ∎

Proposition 3.7. *Let G_p be an abelian p-group, s the number of invariants of G_p, B a nilpotent p'-group and $G = G_p \times B$. Assume that K does not contain a primitive qth root of 1 for some prime q dividing $|B|$. The group G is of OTP projective K-representation type if and only if $s \leq i(K) + 1$.*

Proof. Apply Proposition 2.7 and Theorem 3.1. ∎

From now on, K denotes an arbitrary field of characteristic p.

PROPOSITION 3.8. *A group $G = G_p \times B$ is of purely OTP projective K-representation type if and only if either G_p is cyclic, or K is a splitting field for every twisted group algebra of B over K.*

Proof. Let $\nu \in Z^2(B, K^*)$ be an arbitrary cocycle and $K^\lambda G = KG_p \otimes_K K^\nu B$. By Proposition 1.4, $K^\lambda G$ is of OTP representation type if and only if either G_p is cyclic, or K is a splitting field for $K^\nu B$. Assume now that G_p is cyclic, $\mu \in Z^2(G_p, K^*)$ is an arbitrary cocycle and $\lambda = \mu \times \nu$. Since the algebra $K^\mu G_p$ is uniserial, by Lemma 1.1, $K^\lambda G$ is of OTP representation type. ∎

PROPOSITION 3.9. *Let $G = G_p \times B$. Assume that $K = K^q$ and K contains a primitive qth root of 1 for each prime q that divides $|B|$. Then G is of purely OTP projective K-representation type.*

Proof. Apply Propositions 2.8 and 3.8. ∎

COROLLARY 3.10. *If K is a separably closed field then every group $G = G_p \times B$ is of purely OTP projective K-representation type.*

PROPOSITION 3.11. *Let $G = G_p \times B$. Assume that every prime divisor of $|B'|$ is also a divisor of $|B : B'|$. The group G is of purely OTP projective K-representation type if and only if either G_p is cyclic, or $K = K^q$ and K contains a primitive qth root of 1 for each prime q that divides $|B|$.*

Proof. Again apply Propositions 2.8 and 3.8. ∎

THEOREM 3.12. *A group $G = G_p \times B$ is of purely OTP projective K-representation type if and only if either G_p is cyclic, or there exists a finite central group extension $1 \to A \to \widehat{B} \to B \to 1$ such that any projective K-representation of B lifts projectively to an ordinary K-representation of $\widehat{B}$ and K is a splitting field for $\widehat{B}$.*

Proof. Apply Propositions 2.9 and 3.8. ∎

PROPOSITION 3.13. *Let $G = G_p \times B$. Assume that either $t(K^*) = t(K^*)^q$ for any prime q dividing $|B'|$, or every prime divisor of $|B'|$ is also divisor of $|B : B'|$. Then G is of purely OTP projective K-representation type if and only if either G_p is cyclic, or there exists a covering group $\widehat{B}$ of B over K such that K is a splitting field for $\widehat{B}$.*

Proof. Apply Propositions 2.10 and 3.8. ∎

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