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A GENERAL APPROACH TO FINITE-DIMENSIONAL DIVISION ALGEBRAS

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Abstract. We present a short and rather self-contained introduction to the theory of finite-dimensional division algebras, setting out from the basic definitions and leading up to recent results and current directions of research. In Sections 2–3 we develop the general theory over an arbitrary ground field k, with emphasis on the trichotomy of fields imposed by the dimensions in which a division algebra exists, the groupoid structure of the level subcategories $\mathscr{D}_n(k)$, and the role played by the irreducible morphisms. Sections 4–5 deal with the classical case of real division algebras, emphasizing the double sign decomposition of the level subcategories $\mathscr{D}_n(\mathbb{R})$ for $n \in \{2,4,8\}$ and the problem of describing their blocks, along with an account of known partial solutions to this problem.

1. Preface. The present article is a slightly elaborated version of an expository talk given by the author on the Xth Maurice Auslander International Conference in Woods Hole, Massachusetts, April 2011. It intends to introduce the non-specialist reader to the theory of finite-dimensional division algebras.

Since the categories we meet in division algebra theory are never abelian, module theory plays formally no role in this context. But yet, on a deeper level, the view of finite-dimensional division algebras presented here is in fact strongly influenced by the representation-theoretic background of the author. The interested reader will sense the impact of representation-theoretic topics like the Brauer–Thrall theorems, the classification approach, the quiver viewpoint, or the notion of irreducible morphisms (see e.g. [22], [5], [4]).

Apart from these, there is a dialectic influence in the sense that division algebras provide interesting "counterphenomena" to representation theory, i.e. phenomena which in representation theory are known or believed not to occur.

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2. Three guiding problems. Throughout this section, k denotes any field. By a *k-algebra* A we mean a vector space A over k, together with a k-bilinear map

$$A \times A \to A, \qquad (x, y) \mapsto xy,$$

called the *multiplicative structure* of A. Every element a in a k-algebra A determines k-linear operators $L_a : A \to A,\ x \mapsto ax$, and $R_a : A \to A,\ x \mapsto xa$. A non-zero k-algebra A having the *division property* that L_a and R_a are bijective for all $a \in A \setminus \{0\}$ is called a *division algebra* over k.

A *morphism of k-algebras* A and B is a k-linear map $f : A \to B$ satisfying $f(xy) = f(x)f(y)$ for all $x, y \in A$.

LEMMA 2.1. *If $f : A \to B$ is a morphism of k-algebras and A is a division algebra, then f is injective or zero.*

Proof. Assume f is not injective. Then there is an element $a \in \ker(f) \setminus \{0\}$, and $f(ax) = f(a)f(x) = 0f(x) = 0$ for all $x \in A$. Since L_a is surjective, this means that f is zero. ∎

DEFINITION 2.2. A *morphism of division algebras* A and B over k is a non-zero morphism of k-algebras A and B.

If two morphisms of division algebras are composable as maps, then their composed map is again a morphism of division algebras, by Lemma 2.1. Thus the category $\hat{\mathscr{D}}(k)$ of all division algebras over k is well-defined. We denote by $\mathscr{D}(k)$ its full subcategory formed by all finite-dimensional objects, and for each $n \in \mathbb{N}$ by $\mathscr{D}_n(k)$ its full subcategory formed by all n-dimensional objects. The category $\mathscr{D}(k)$ is the subject of the present investigation.

By a *groupoid* we mean a category in which every morphism is an isomorphism. (We do not require the object class of a groupoid to be a set.) The following proposition is an immediate consequence of Lemma 2.1 and Definition 2.2.

PROPOSITION 2.3.

(i) *Every morphism in $\hat{\mathscr{D}}(k)$ is injective.*
(ii) *A morphism in $\mathscr{D}(k)$ is an isomorphism if and only if it is in $\mathscr{D}_n(k)$ for some $n \in \mathbb{N}$.*
(iii) *The category $\mathscr{D}_n(k)$ is a groupoid for every $n \in \mathbb{N}$.*

Note however that some of the groupoids $\mathscr{D}_n(k)$ may be empty! Thus all information about the objects and the isomorphisms in $\mathscr{D}(k)$ is contained in the non-empty groupoids $\mathscr{D}_n(k)$. Regarding the non-isomorphisms in $\mathscr{D}(k)$, the following definition proves to be useful.

DEFINITION 2.4. A non-isomorphism f in $\mathscr{D}(k)$ is called *reducible* if it is composed of two non-isomorphisms in $\mathscr{D}(k)$, and *irreducible* otherwise.

Thus a non-isomorphism f in $\mathscr{D}(k)$ is irreducible if and only if for all morphisms g and h in $\mathscr{D}(k)$ the identity $f = hg$ implies that g is an isomorphism or h is an isomorphism.

Every morphism $f : A \to B$ in $\mathscr{D}(k)$ uniquely determines the ascending sequence $\dim(A) = n_0 < n_1 < \cdots < n_\ell = \dim(B)$ of all natural numbers n_i with $\dim(A) \leq n_i \leq \dim(B)$ such that $\mathscr{D}_{n_i}(k) \neq \emptyset$. In particular, f uniquely determines the natural number ℓ. We call $\ell = \ell(f)$ the *length* of f. Note that $f = hg$ implies $\ell(f) = \ell(h) + \ell(g)$.

Proposition 2.5.

(i) *The isomorphisms in $\mathscr{D}(k)$ are precisely the morphisms of length* 0 *in $\mathscr{D}(k)$.*
(ii) *Every morphism of length* 1 *in $\mathscr{D}(k)$ is irreducible.*
(iii) *Every non-isomorphism in $\mathscr{D}(k)$ is composed of irreducible morphisms.*

Proof. (i) is a reformulation of Proposition 2.3(ii).

(ii) If $\ell(f) = 1$ and $f = hg$, then $1 = \ell(h) + \ell(g)$ implies $\ell(h) = 0$ or $\ell(g) = 0$. The statement now follows from (i).

(iii) We prove the statement for all non-isomorphisms f in $\mathscr{D}(k)$ by induction on $\ell(f) \geq 1$. If $\ell(f) = 1$, then it holds for f, by (ii). Let $\ell(f) \geq 2$. If f is irreducible, then it holds for f. If f is reducible, then there are non-isomorphisms g and h in $\mathscr{D}(k)$ such that $f = hg$. Now $\ell(f) = \ell(h) + \ell(g)$ implies $\ell(h) < \ell(f)$ and $\ell(g) < \ell(f)$. By induction hypothesis, both h and g are composed of irreducible morphisms, and hence so is f. ∎

To summarize, our interest in the category $\mathscr{D}(k)$ is guided by the following three problems.

(A) Describe the set of all $n \in \mathbb{N}$ for which the groupoid $\mathscr{D}_n(k)$ is non-empty.
(B) Describe the categorical structures of all non-empty groupoids $\mathscr{D}_n(k)$.
(C) Describe all irreducible morphisms in $\mathscr{D}(k)$.

Section 3 is devoted to problem (A). Sections 4 and 5 are devoted to problem (B) in case $k = \mathbb{R}$. Problem (C) seems so far not to have been studied explicitly at all, except in [2] where irreducible morphisms of absolute valued algebras are looked at. We conclude this section with a few elementary observations towards (A)–(C).

A k-algebra A is said to have *no zero divisors* if for all $x, y \in A$ the identity $xy = 0$ implies $x = 0$ or $y = 0$.

Lemma 2.6. *Let A be a k-algebra with $0 < \dim(A) < \infty$. Then A is a division algebra if and only if A has no zero divisors.*

Proof. By definition, A is a division algebra if and only if L_a and R_a are bijective for all $a \in A \setminus \{0\}$. Since $\dim(A) < \infty$, this is equivalent to L_a and R_a being injective for all $a \in A \setminus \{0\}$, which in turn is equivalent to A having no zero divisors. ∎

For any k-algebra B we denote by $\mathrm{Ip}(B)$ the set of all non-zero idempotents in B. The proof of the following lemma is straightforward.

LEMMA 2.7. *Let $B \in \mathscr{D}(k)$. If $f : k \to B$ is a morphism in $\mathscr{D}(k)$, then $f(1) \in \mathrm{Ip}(B)$. The map $\mathrm{Mor}_{\mathscr{D}(k)}(k, B) \to \mathrm{Ip}(B),\ f \mapsto f(1)$, is bijective, with inverse map $\mathrm{Ip}(B) \to \mathrm{Mor}_{\mathscr{D}(k)}(k, B),\ e \mapsto f_e$, given by $f_e(\alpha) = \alpha e$ for all $\alpha \in k$.*

Thus the study of morphisms $k \to B$ in $\mathscr{D}(k)$ amounts to the study of non-zero idempotents in B. As a first consequence, if $\dim(B) \geq 2$, then we may distinguish between *irreducible* and *reducible idempotents* $e \in \mathrm{Ip}(B)$, depending on whether the morphism $f_e : k \to B$ is irreducible or not. If $B \in \mathscr{D}_2(k)$, then all morphisms $f : k \to B$ have length 1, which in view of Proposition 2.5(ii) implies that all idempotents $e \in \mathrm{Ip}(B)$ are irreducible. Examples of division algebras $B \in \mathscr{D}_4(\mathbb{R})$ containing both irreducible and reducible idempotents are to be found in [2].

As another consequence of Lemma 2.7 let us solve problem (B) for $n = 1$. We denote by $[k]$ the isomorphism class of k in $\mathscr{D}(k)$.

PROPOSITION 2.8. $\mathrm{Ob}(\mathscr{D}_1(k)) = [k]$.

Proof. Given $B \in \mathscr{D}_1(k)$, choose $b \in B \setminus \{0\}$. Then $b^2 = \beta b$ for some $\beta \in k \setminus \{0\}$, and $e = \beta^{-1} b \in \mathrm{Ip}(B)$. The morphism $f_e : k \to B$ is in $\mathscr{D}_1(k)$, which by Proposition 2.3(ii) means that f_e is an isomorphism. So $B \in [k]$. ∎

The trivial group $\{1\}$ may be viewed as a groupoid $\mathscr{T}$, consisting of precisely one object t and precisely one morphism $1 = \mathbb{I}_t$. By a *trivial category* we mean any category that is equivalent to $\mathscr{T}$.

COROLLARY 2.9. *The groupoid $\mathscr{D}_1(k)$ is trivial.*

Proof. The functor $\mathscr{F} : \mathscr{T} \to \mathscr{D}_1(k)$ defined by $\mathscr{F}(t) = k$ and $\mathscr{F}(1) = \mathbb{I}_k$ is faithful by definition, full because $\mathrm{Mor}_{\mathscr{D}_1(k)}(k, k) = \{\mathbb{I}_k\}$, and dense by Proposition 2.8. So $\mathscr{F}$ is an equivalence of categories. ∎

3. An approach to problem (A). The partial solution to problem (A) presented in this section and stated as Theorem 3.1 below amounts to a trichotomy of fields. It is based on classical results from field theory, topology and logic, which historically emerged independently, and largely not with problem (A) in mind, between 1927 and 1958. The proof of Theorem 3.1 presented here is a condensed version of the original proof, which is found in [15].

Recall that a field k is called *real closed* if it is *formally real* (i.e. -1 is not a sum of squares in k) and algebraically closed within the class of all formally real fields (i.e. if $k \subset \ell$ is an algebraic field extension with ℓ formally real, then $k = \ell$). Fields that are neither algebraically closed nor real closed are briefly called *non-closed*.

Theorem 3.1. *For any field k, the set $\mathscr{N}(k) = \{n \in \mathbb{N} \mid \mathscr{D}_n(k) \neq \emptyset\}$ admits the following description:*

$$\mathscr{N}(k) = \begin{cases} \{1\} & \textit{if } k \textit{ is algebraically closed,} \\ \{1,2,4,8\} & \textit{if } k \textit{ is real closed,} \\ \textit{unbounded} & \textit{if } k \textit{ is non-closed.} \end{cases}$$

Proof. Let k be non-closed. Then the set

$$\mathscr{M}(k) = \{\deg(p) \mid p \in k[X] \text{ is irreducible}\}$$

is unbounded [3]. If $p \in k[X]$ is irreducible and $\deg(p) = n$, then $k[X]/(p)$ is an object in $\mathscr{D}_n(k)$. So $\mathscr{M}(k) \subset \mathscr{N}(k)$, and hence $\mathscr{N}(k)$ is unbounded.

Let $k = \mathbb{R}$. The four classical examples of real division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ show that $\{1,2,4,8\} \subset \mathscr{N}(\mathbb{R})$. Hopf [23] proved that $\mathscr{N}(\mathbb{R}) \subset \{2^m \mid m \in \mathbb{N}\}$. Bott and Milnor [7], and independently Kervaire [24], sharpened Hopf's inclusion to $\mathscr{N}(\mathbb{R}) \subset \{2^m \mid m \in \mathbb{N} \text{ and } m \leq 3\}$, thus accomplishing the statement $\mathscr{N}(\mathbb{R}) = \{1,2,4,8\}$.

Let k be real closed. Then a theorem of Tarski's [26–28] asserts that k and $\mathbb{R}$ satisfy the same first order sentences in the language of rings. For each $n \in \mathbb{N} \setminus \{0\}$ we set $\underline{n} = \{1, \dots, n\}$ and introduce the triple sequence of variables $\overline{a} = (a_{hij})_{hij \in \underline{n}^3}$ and the sequences of variables $\overline{x} = (x_i)_{i \in \underline{n}}$ and $\overline{y} = (y_j)_{j \in \underline{n}}$. Then

$$\Big(\bigwedge_{h=1}^{n} \Big(\sum_{i,j=1}^{n} a_{hij} x_i y_j = 0 \Big) \Big) \to \Big(\Big(\bigwedge_{i=1}^{n} (x_i = 0) \Big) \vee \Big(\bigwedge_{j=1}^{n} (y_j = 0) \Big) \Big)$$

is a first order formula in the language of rings with free variables a_{hij}, x_i, y_j. We denote it by $\varphi_n(\overline{a}, \overline{x}, \overline{y})$ and form

$$\sigma_n = \exists \overline{a} \ \forall \overline{x}, \overline{y} \ \varphi_n(\overline{a}, \overline{x}, \overline{y}),$$

which is a first order sentence in the language of rings. The notation $k \models \sigma_n$ expresses that k satisfies σ_n, which means the existence of n^3 structure constants in k such that the corresponding algebra structure on k^n admits no zero divisors. In view of Lemma 2.6 and Tarski's theorem we obtain the chain of equivalences

$$n \in \mathscr{N}(k) \ \Leftrightarrow \ k \models \sigma_n \ \Leftrightarrow \ \mathbb{R} \models \sigma_n \ \Leftrightarrow \ n \in \mathscr{N}(\mathbb{R}).$$

Accordingly $\mathscr{N}(k) = \mathscr{N}(\mathbb{R}) = \{1,2,4,8\}$.

Let k be algebraically closed. Then, as Gabriel observed (oral communication, Zürich University, 1994), every k-algebra A with $1 < \dim(A) < \infty$ has zero divisors. Indeed, choose non-proportional vectors $v, w \in A$. If L_v is not bijective, then L_v is not injective, hence $vy = 0$ for some $y \in A\backslash\{0\}$. If L_v is bijective, then the linear operator $L_v^{-1}L_w : A \to A$ has an eigenvalue $\lambda \in k$. Every eigenvector y of $L_v^{-1}L_w$ with eigenvalue λ satisfies $(w - \lambda v)y = 0$.

Now $\{1\} \subset \mathscr{N}(k)$ holds trivially as $k \in \mathscr{D}_1(k)$, and $\mathscr{N}(k) \subset \{1\}$ holds by Lemma 2.6 and Gabriel's observation. So $\mathscr{N}(k) = \{1\}$. ∎

In conclusion of this section we note the following immediate consequence of Theorem 3.1, Proposition 2.8 and Corollary 2.9.

COROLLARY 3.2. *For every field k, the following assertions are equivalent:*

(i) *The category $\mathscr{D}(k)$ is trivial.*
(ii) *Every finite-dimensional division algebra over k is isomorphic to k.*
(iii) *k is algebraically closed.*

Compare this corollary to the mantra frequently heard at the outset of mathematical talks (but hardly ever justified), maintaining that the assumption "$k = \overline{k}$" is inessential!

4. An approach to problem (B) **in the real case.** We now turn to problem (B) in the classical case $k = \mathbb{R}$. Using the brief notation $\mathscr{D} = \mathscr{D}(\mathbb{R})$ and $\mathscr{D}_n = \mathscr{D}_n(\mathbb{R})$, we know by Theorem 3.1 and Proposition 2.8 that

$$\mathrm{Ob}(\mathscr{D}) = [\mathbb{R}] \cup \mathrm{Ob}(\mathscr{D}_2) \cup \mathrm{Ob}(\mathscr{D}_4) \cup \mathrm{Ob}(\mathscr{D}_8).$$

In this section we present a general approach to the non-empty groupoids $\mathscr{D}_2$, $\mathscr{D}_4$ and $\mathscr{D}_8$ which was in the air for quite a while, but was made explicit only recently in [14].

With any $A \in \mathrm{Ob}(\mathscr{D}) \setminus [\mathbb{R}]$ we associate the diagram of maps

$$\begin{tikzcd} \mathrm{GL}_{\mathbb{R}}(A) \arrow[r, "\det"] & \mathbb{R} \setminus \{0\} \arrow[d, "\mathrm{sign}"] \\ A \setminus \{0\} \arrow[u, shift left, "L"] \arrow[u, shift right, "R"'] & \mathrm{C}_2 \end{tikzcd}$$

where $\mathrm{C}_2 = \{\pm 1\}$ denotes the cyclic group of order 2, and L, R and sign are defined by $L(a) = L_a$, $R(a) = R_a$ and $\mathrm{sign}(x) = x/|x|$. By composition we obtain the maps $\ell : A\backslash\{0\} \to \mathrm{C}_2$, $\ell(a) = \mathrm{sign}(\det(L_a))$ and $r : A\backslash\{0\} \to \mathrm{C}_2$, $r(a) = \mathrm{sign}(\det(R_a))$.

LEMMA 4.1. *For every $A \in \mathrm{Ob}(\mathscr{D})\backslash[\mathbb{R}]$, both maps ℓ and r are constant.*

Proof. We equip $A \setminus \{0\}$, $\mathrm{GL}_{\mathbb{R}}(A)$ and $\mathbb{R} \setminus \{0\}$ with the Euclidean topology and C_2 with the discrete topology. Then all maps L, R, det and sign are continuous. Hence so are ℓ and r. The topological space $A\setminus\{0\}$ is connected, as $\dim(A) > 1$. Every continuous map from a connected space to a discrete space is constant. ∎

The map $p : \mathrm{Ob}(\mathscr{D}) \setminus [\mathbb{R}] \to \mathrm{C}_2 \times \mathrm{C}_2$, $p(A) = (\ell(A), r(A))$, associating with A the unique values $\ell(A)$ and $r(A)$ of the maps ℓ and r is thus well-defined. For each $n \in \{2, 4, 8\}$ it restricts to $p_n : \mathrm{Ob}(\mathscr{D}_n) \to \mathrm{C}_2 \times \mathrm{C}_2$. For every $(\alpha, \beta) \in \mathrm{C}_2\times\mathrm{C}_2$ the fibre $p_n^{-1}(\alpha, \beta)$ forms a full subcategory $\mathscr{D}_n^{\alpha\beta} \subset \mathscr{D}_n$. It is easy to see that p is constant on all isomorphism classes [14, Proposition 2.2]. Together with Proposition 2.3(ii) this yields the following result.

Proposition 4.2. *For each $n \in \{2, 4, 8\}$, the category $\mathscr{D}_n$ decomposes in accordance with*

$$\mathscr{D}_n = \coprod_{(\alpha,\beta)\in \mathrm{C}_2\times \mathrm{C}_2} \mathscr{D}_n^{\alpha\beta}.$$

Hence problem (B) for $k = \mathbb{R}$ splits into the twelve subproblems of describing the structures of the blocks $\mathscr{D}_n^{\alpha\beta}$ for all $n \in \{2, 4, 8\}$ and $(\alpha, \beta) \in \mathrm{C}_2 \times \mathrm{C}_2$. Complete solutions to these are at present only known for the four blocks $\mathscr{D}_2^{\alpha\beta}$ [17]. See Subsection 5.2 for a streamlined version.

If $\mathscr{C}_n \subset \mathscr{D}_n$ is any full subcategory and $\mathscr{C}_n^{\alpha\beta} \subset \mathscr{C}_n$ denotes the full subcategory with $\mathrm{Ob}(\mathscr{C}_n^{\alpha\beta}) = \mathrm{Ob}(\mathscr{C}_n) \cap p_n^{-1}(\alpha, \beta)$, then even $\mathscr{C}_n$ decomposes in accordance with

$$\mathscr{C}_n = \coprod_{(\alpha,\beta)\in \mathrm{C}_2\times \mathrm{C}_2} \mathscr{C}_n^{\alpha\beta},$$

and a description of the structure of $\mathscr{C}_n^{\alpha\beta}$ may be considered as a step towards the desired description of $\mathscr{D}_n^{\alpha\beta}$. Such partial solutions to the eight remaining subproblems concerning the blocks $\mathscr{D}_4^{\alpha\beta}$ and $\mathscr{D}_8^{\alpha\beta}$ are known for a sample of full subcategories $\mathscr{C}_4 \subset \mathscr{D}_4$ and $\mathscr{C}_8 \subset \mathscr{D}_8$. One of these, concerning the full subcategory $\mathscr{C}_4 = \mathscr{A}_4$ of all 4-dimensional absolute valued algebras, is presented in detail in Subsection 5.3. A brief guide to further partial solutions is included in Subsection 5.4.

5. Partial solutions to problem (B) in the real case

5.1. Prerequisites. Every left group action $G \times M \to M$ gives rise to a groupoid ${}_GM$, with object set $\mathrm{Ob}({}_GM) = M$ and morphism sets

$$\mathrm{Mor}_{{}_GM}(x, y) = \{(g, x, y) \mid g \in G \text{ with } gx = y\}$$

for all $x, y \in M$. Morphisms $(g, x, y) \in \mathrm{Mor}_{{}_GM}(x, y)$ may briefly be denoted by g, provided that the objects x and y are specified in some other way.

Let $n \in \{2, 4, 8\}$, $\mathscr{C}_n \subset \mathscr{D}_n$ be a full subcategory, and $(\alpha, \beta) \in \mathrm{C}_2 \times \mathrm{C}_2$. By a *description of the block* $\mathscr{C}_n^{\alpha\beta}$ we mean the display of a group action

$G \times M \to M$, together with an equivalence of categories $\mathscr{F} : {}_G M \to \mathscr{C}_n^{\alpha\beta}$. We consider any such description as a *partial solution to problem* (B).

The *isotope* of a k-algebra A with respect to $(\sigma, \tau) \in \mathrm{GL}_k(A) \times \mathrm{GL}_k(A)$ is the k-algebra $A_{\sigma\tau}$ with underlying vector space A, and multiplication $x \circ y = \sigma(x)\tau(y)$. It follows from Lemma 2.6 that $A_{\sigma\tau} \in \mathscr{D}(k)$ if $A \in \mathscr{D}(k)$.

Regarding the blocks $\mathscr{C}_n^{\alpha\beta}$ of $\mathscr{C}_n$ we sometimes prefer the more intuitive notation $\mathscr{C}_n^{++} = \mathscr{C}_n^{1,1}$, $\mathscr{C}_n^{+-} = \mathscr{C}_n^{1,-1}$, $\mathscr{C}_n^{-+} = \mathscr{C}_n^{-1,1}$, and $\mathscr{C}_n^{--} = \mathscr{C}_n^{-1,-1}$.

5.2. Description of the blocks $\mathscr{D}_2^{\alpha\beta}$. With reference to the standard basis $(1, i)$ of the real vector space $\mathbb{C}$, we identify complex numbers $x_1 + ix_2$ with their coordinate columns $\binom{x_1}{x_2}$, and linear operators $\sigma \in \mathrm{GL}_{\mathbb{R}}(\mathbb{C})$ with their matrices $S = (\sigma(1)\ \sigma(i)) \in \mathrm{GL}(2)$. In particular, complex conjugation and rotation in the complex plane by $2\pi/3$ are described by the matrices

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad R = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

respectively. They generate the cyclic group $\mathrm{C}_2 = \langle K \rangle$ of order 2 and the dihedral group $\mathrm{D}_3 = \langle R, K \rangle$ of order 6.

By $\mathscr{S}$ we denote the set of all real 2×2-matrices that are positive definite symmetric and have determinant 1. The left actions of C_2 and D_3 on $\mathscr{S}^2$ by simultaneous conjugation give rise to the groupoids ${}_{\mathrm{C}_2}\mathscr{S}^2$ and ${}_{\mathrm{D}_3}\mathscr{S}^2$.

The following proposition is an immediate consequence of [17, Propositions 3.1 and 3.2]). It describes the blocks $\mathscr{D}_2^{\alpha\beta}$ of $\mathscr{D}_2$.

PROPOSITION 5.1.

(i) *For each* $(i, j) \in \{(0,0), (0,1), (1,0)\}$, *an equivalence of categories* $\mathscr{F}_{ij} : {}_{\mathrm{C}_2}\mathscr{S}^2 \to \mathscr{D}_2^{(-1)^j,(-1)^i}$ *is given on objects by* $\mathscr{F}_{ij}(A, B) = \mathbb{C}_{AK^i, BK^j}$ *and on morphisms by* $\mathscr{F}_{ij}(F, (A, B), (C, D)) = F$.

(ii) *An equivalence of categories* $\mathscr{F}_{11} : {}_{\mathrm{D}_3}\mathscr{S}^2 \to \mathscr{D}_2^{--}$ *is given on objects by* $\mathscr{F}_{11}(A, B) = \mathbb{C}_{KA, KB}$ *and on morphisms by* $\mathscr{F}_{11}(F, (A, B), (C, D)) = F$.

The effectiveness of this description of all four blocks $\mathscr{D}_2^{\alpha\beta}$ of $\mathscr{D}_2$ is demonstrated in [17], where a classification of $\mathrm{Ob}(\mathscr{D}_2)$ is derived from it, and the automorphism groups of all objects in the classifying list are displayed.

5.3. Description of the blocks $\mathscr{A}_4^{\alpha\beta}$. An *absolute valued algebra* $A = (A, \|\cdot\|)$ is a non-zero real algebra A, together with a norm $\|\cdot\| : A \to \mathbb{R}$ satisfying $\|xy\| = \|x\|\,\|y\|$ for all $x, y \in A$. A *morphism of absolute valued algebras* $(A, \|\cdot\|)$ and $(B, \|\cdot\|')$ is an algebra morphism $f : A \to B$. Thus the category $\hat{\mathscr{A}}$ of all absolute valued algebras is well-defined. We denote by $\mathscr{A}$ its full subcategory formed by all finite-dimensional objects, and for each $n \in \mathbb{N}$ by $\mathscr{A}_n$ its full subcategory formed by all n-dimensional objects.

Since every absolute valued algebra has no zero divisors, it follows by Lemma 2.6 that every finite-dimensional absolute valued algebra is a real division algebra. Moreover, the norm $\|\cdot\|$ of a finite-dimensional absolute valued algebra $(A, \|\cdot\|)$ is uniquely determined by A [1]. Thus $\mathscr{A}$ may be viewed as a full subcategory of $\mathscr{D}$. In particular, $\mathscr{A}_4 \subset \mathscr{D}_4$ is a full subcategory. We proceed to describe the blocks $\mathscr{A}_4^{\alpha\beta}$ of $\mathscr{A}_4$.

The left action of the classical group $\mathrm{SO}(3)$ on the set $(\mathrm{SO}(3))^2$ by simultaneous conjugation,

$$P \cdot (A, B) = (PAP^{-1}, PBP^{-1}),$$

determines the groupoid ${}_{\mathrm{SO}(3)}(\mathrm{SO}(3))^2$. We aim to exhibit for each $(\alpha, \beta) \in \mathrm{C}_2 \times \mathrm{C}_2$ an equivalence of categories

$$\mathscr{F}_{\alpha\beta} : {}_{\mathrm{SO}(3)}(\mathrm{SO}(3))^2 \to \mathscr{A}_4^{\alpha\beta}.$$

To this end we need to recollect a few established results. Let e, i, j, k be Hamilton's standard basis of the quaternion algebra $\mathbb{H}$, and denote by $\mathbb{S}^3$ the group of all unit quaternions. Then i, j, k span the purely imaginary hyperplane V in $\mathbb{H}$. Every $a \in \mathbb{S}^3$ determines a special orthogonal operator

$$\kappa_a : \mathbb{H} \to \mathbb{H}, \qquad \kappa_a(x) = axa^{-1},$$

inducing a special orthogonal operator

$$\kappa_a^V : V \to V, \qquad \kappa_a^V(x) = axa^{-1}.$$

A classical theorem of Hamilton's asserts that the map

$$\kappa^V : \mathbb{S}^3 \to \mathrm{SO}(V), \qquad \kappa^V(a) = \kappa_a^V,$$

is a surjective group homomorphism with kernel $\{\pm 1\}$. Passing from κ_a^V to its matrix in the standard basis (i, j, k) of V, we obtain the surjective group homomorphism

$$\mu : \mathbb{S}^3 \to \mathrm{SO}(3), \qquad \mu(a) = [\kappa_a^V]_{(i,j,k)},$$

with kernel $\{\pm 1\}$. Hamilton's group homomorphism μ turns out to interact nicely with results of Ramírez Álvarez [25] which we proceed to recall.

Quaternion multiplication by fixed unit quaternions $a, b \in \mathbb{S}^3$ gives rise to special orthogonal operators L_a and R_b in $\mathrm{SO}(\mathbb{H})$, while quaternion conjugation K belongs to $\mathrm{O}^-(\mathbb{H})$. For any $(\sigma, \tau) \in \mathrm{O}(\mathbb{H}) \times \mathrm{O}(\mathbb{H})$, the isotope $\mathbb{H}_{\sigma\tau}$ is in $\mathscr{A}_4$ and $p(\mathbb{H}_{\sigma\tau}) = (\det(\tau), \det(\sigma))$. Introducing the notation

$$\mathbb{H}^{++}(a, b) = \mathbb{H}_{L_a, R_b}, \qquad \mathbb{H}^{+-}(a, b) = \mathbb{H}_{R_a K, R_b},$$
$$\mathbb{H}^{-+}(a, b) = \mathbb{H}_{L_a, L_b K}, \qquad \mathbb{H}^{--}(a, b) = \mathbb{H}_{L_a K, R_b K}$$

we find that $\mathbb{H}^{\alpha\beta}(a, b) \in \mathscr{A}_4^{\alpha\beta}$ for all $(\alpha, \beta) \in \mathrm{C}_2 \times \mathrm{C}_2$ and $(a, b) \in \mathbb{S}^3 \times \mathbb{S}^3$. Thus for each $(\alpha, \beta) \in \mathrm{C}_2 \times \mathrm{C}_2$, the object set $\{\mathbb{H}^{\alpha\beta}(a, b) \mid (a, b) \in \mathbb{S}^3 \times \mathbb{S}^3\}$ forms a full subcategory $\mathscr{R}_4^{\alpha\beta}$ of $\mathscr{A}_4^{\alpha\beta}$.

Every pair of pairs $((a,b),(c,d)) \in (\mathbb{S}^3 \times \mathbb{S}^3) \times (\mathbb{S}^3 \times \mathbb{S}^3)$ determines a subset $M((a,b),(c,d)) \subset \mathbb{S}^3 \times \mathrm{C}_2 \times \mathrm{C}_2$, defined by

$$M((a,b),(c,d)) = \{(p,\gamma,\delta) \mid (\kappa_p(a),\kappa_p(b)) = (\gamma c, \delta d)\}.$$

The following proposition summarizes in our terminology those results from [25] which are of interest to our setting.

Proposition 5.2. *For each $(\alpha,\beta) \in \mathrm{C}_2 \times \mathrm{C}_2$ the following hold true.*

(i) *The full subcategory $\mathscr{R}_4^{\alpha\beta} \subset \mathscr{A}_4^{\alpha\beta}$ is dense.*
(ii) *For all $((a,b),(c,d)) \in (\mathbb{S}^3 \times \mathbb{S}^3) \times (\mathbb{S}^3 \times \mathbb{S}^3)$,*

$$\mathrm{Mor}_{\mathscr{A}_4^{\alpha\beta}}(\mathbb{H}^{\alpha\beta}(a,b), \mathbb{H}^{\alpha\beta}(c,d)) = \{\gamma\delta\kappa_p \mid (p,\gamma,\delta) \in M((a,b),(c,d))\}.$$

Based on the choice of a map $\sigma : \mathrm{SO}(3) \to \mathbb{S}^3$ such that $\mu\sigma = \mathbb{I}_{\mathrm{SO}(3)}$, we now define for each $(\alpha,\beta) \in \mathrm{C}_2 \times \mathrm{C}_2$ a functor $\mathscr{F}_{\alpha\beta} : {}_{\mathrm{SO}(3)}(\mathrm{SO}(3))^2 \to \mathscr{A}_4^{\alpha\beta}$ as follows. Given any morphism $P : (A,B) \to (C,D)$ in ${}_{\mathrm{SO}(3)}(\mathrm{SO}(3))^2$, we set

$$(a,b,c,d,p) = (\sigma(A),\sigma(B),\sigma(C),\sigma(D),\sigma(P)),$$

we observe that $(p,\gamma,\delta) \in M((a,b),(c,d))$ for a unique pair $(\gamma,\delta) \in \mathrm{C}_2 \times \mathrm{C}_2$, and we define $\mathscr{F}_{\alpha\beta}(A,B) = \mathbb{H}^{\alpha\beta}(a,b)$, $\mathscr{F}_{\alpha\beta}(C,D) = \mathbb{H}^{\alpha\beta}(c,d)$, and $\mathscr{F}_{\alpha\beta}(P) = \gamma\delta\kappa_p$. It is easily checked that $\mathscr{F}_{\alpha\beta}$ is a functor.

Proposition 5.3. *For each $(\alpha,\beta) \in \mathrm{C}_2 \times \mathrm{C}_2$, the functor*

$$\mathscr{F}_{\alpha\beta} : {}_{\mathrm{SO}(3)}(\mathrm{SO}(3))^2 \to \mathscr{A}_4^{\alpha\beta}$$

is an equivalence of categories.

Proof. For every $A \in \mathscr{A}_4^{\alpha\beta}$ there is a pair $(c,d) \in \mathbb{S}^3 \times \mathbb{S}^3$ such that $\mathbb{H}^{\alpha\beta}(c,d) \xrightarrow{\sim} A$, by Proposition 5.2(i). Setting $(C,D) = (\mu(c),\mu(d))$ we find that $(\sigma(C),\sigma(D)) = (\gamma c, \delta d)$ for some $(\gamma,\delta) \in \mathrm{C}_2 \times \mathrm{C}_2$. Accordingly

$$\mathscr{F}_{\alpha\beta}(C,D) = \mathbb{H}^{\alpha\beta}(\sigma(C),\sigma(D)) = \mathbb{H}^{\alpha\beta}(\gamma c, \delta d) \xrightarrow{\sim} \mathbb{H}^{\alpha\beta}(c,d) \xrightarrow{\sim} A,$$

where the first isomorphism in this sequence holds by Proposition 5.2(ii). Thus $\mathscr{F}_{\alpha\beta}$ is dense.

Let objects (A,B) and (C,D) in ${}_{\mathrm{SO}(3)}(\mathrm{SO}(3))^2$ and a morphism

$$f : \mathscr{F}_{\alpha\beta}(A,B) \to \mathscr{F}_{\alpha\beta}(C,D)$$

in $\mathscr{A}_4^{\alpha\beta}$ be given. Setting $(a,b,c,d) = (\sigma(A),\sigma(B),\sigma(C),\sigma(D))$ we have $\mathscr{F}_{\alpha\beta}(A,B) = \mathbb{H}^{\alpha\beta}(a,b)$, $\mathscr{F}_{\alpha\beta}(C,D) = \mathbb{H}^{\alpha\beta}(c,d)$, and $f = \gamma\delta\kappa_p$ for some $(p,\gamma,\delta) \in M((a,b),(c,d))$, due to Proposition 5.2(ii). Setting $P = \mu(p)$ one finds that $P : (A,B) \to (C,D)$ is a morphism in ${}_{\mathrm{SO}(3)}(\mathrm{SO}(3))^2$. Moreover $\sigma(P) = \eta p$ for some $\eta \in \mathrm{C}_2$. This implies $\kappa_{\sigma(P)} = \kappa_p$, so $(\sigma(P),\gamma,\delta) \in M((a,b),(c,d))$, and hence $\mathscr{F}_{\alpha\beta}(P) = \gamma\delta\kappa_{\sigma(P)} = \gamma\delta\kappa_p = f$. Thus $\mathscr{F}_{\alpha\beta}$ is full.

Let objects (A, B) and (C, D) and two morphisms $P, Q : (A, B) \to (C, D)$ in ${}_{\mathrm{SO}(3)}(\mathrm{SO}(3))^2$ be given, such that $\mathscr{F}_{\alpha\beta}(P) = \mathscr{F}_{\alpha\beta}(Q)$. We set

$$(a, b, c, d, p, q) = (\sigma(A), \sigma(B), \sigma(C), \sigma(D), \sigma(P), \sigma(Q)).$$

Then there are unique pairs $(\gamma, \delta), (\varepsilon, \zeta) \in \mathrm{C}_2 \times \mathrm{C}_2$ such that $(p, \gamma, \delta), (q, \varepsilon, \zeta) \in M((a, b), (c, d))$. Now $\gamma\delta\kappa_p = \mathscr{F}_{\alpha\beta}(P) = \mathscr{F}_{\alpha\beta}(Q) = \varepsilon\zeta\kappa_q$ implies $\gamma\delta 1 = \gamma\delta\kappa_p(1) = \varepsilon\zeta\kappa_q(1) = \varepsilon\zeta 1$, hence $\gamma\delta = \varepsilon\zeta$, and so $\kappa_p = \kappa_q$. Equivalently $\kappa_{q^{-1}p} = \mathbb{I}_{\mathbb{H}}$. So $q^{-1}p$ is a unit quaternion belonging to the centre of $\mathbb{H}$. Since $Z(\mathbb{H}) = \mathbb{R}1$, we conclude that $q^{-1}p = \vartheta 1$ for some $\vartheta \in \mathrm{C}_2$. Hence $p = \vartheta q$, and finally $P = \mu(p) = \mu(\vartheta q) = \mu(q) = Q$. Thus $\mathscr{F}_{\alpha\beta}$ is faithful. ∎

The equivalence of all four groupoids $\mathscr{A}_4^{\alpha\beta}$ to ${}_{\mathrm{SO}(3)}(\mathrm{SO}(3))^2$ was first observed by Forsberg in [21] where he also deduces it from [25], yet in a less streamlined way than in Proposition 5.3 above. It reappears in different disguise in [14], as a special case of Darpö's description of all isotopes of the quaternion algebra [12]. The effectiveness of our description of the blocks $\mathscr{A}_4^{\alpha\beta}$ is also demonstrated by Forsberg in [21], in so far as he derives from it a classification of $\mathrm{Ob}(\mathscr{A}_4)$ (cf. [8]), along with a description of the automorphism groups of all objects in the classifying list in terms of subgroups of SO(3).

As an immediate consequence of Propositions 5.3 and 5.1 we observe that all four blocks $\mathscr{A}_4^{\alpha\beta}$ are equivalent, while $\mathscr{D}_2^{--}$ is inequivalent to each of the equivalent blocks $\mathscr{D}_2^{++}$, $\mathscr{D}_2^{+-}$ and $\mathscr{D}_2^{-+}$. The question of equivalence of the blocks $\mathscr{C}_n^{\alpha\beta}$ of a full subcategory $\mathscr{C}_n \subset \mathscr{D}_n$ is investigated in greater generality in [14].

5.4. Guide to further partial solutions to problem (B). For the convenience of the interested reader we include a brief guide to research articles containing further partial solutions to problem (B) in the real case. This guide is most probably incomplete, as it only mentions those articles which the author happens to know. In some of them, the asserted "description of blocks" is not given in the rigorous sense defined in Subsection 5.1, and additional work may be required to mould it into that shape.

Every finite-dimensional real division algebra which is commutative has dimension at most 2 [23]. Let $\mathscr{C}_2 \subset \mathscr{D}_2$ be the full subcategory formed by all commutative 2-dimensional real division algebras. Commutativity implies that $\mathscr{C}_2^{+-} = \mathscr{C}_2^{-+} = \emptyset$. The diagonal blocks $\mathscr{C}_2^{++}$ and $\mathscr{C}_2^{--}$ are described in [13].

A non-zero k-algebra A is called *quadratic* if it contains a unity 1 and the sequence $1, x, x^2$ is linearly dependent for each $x \in A$. Let $\mathscr{Q}_n \subset \mathscr{D}_n$ be the full subcategory formed by all quadratic n-dimensional real division algebras. The existence of a unity implies that $\mathscr{Q}_n^{+-} = \mathscr{Q}_n^{-+} = \mathscr{Q}_n^{--} = \emptyset$.

The block $\mathcal{Q}_4^{++} = \mathcal{Q}_4$ is described in [19]. In [20] the block $\mathcal{Q}_8^{++} = \mathcal{Q}_8$ is shown to decompose in accordance with

$$\mathcal{Q}_8^{++} = \mathcal{Q}_8^1 \amalg \mathcal{Q}_8^3 \amalg \mathcal{Q}_8^5,$$

where the blocks $\mathcal{Q}_8^d$ are formed by the non-empty fibres $\deg^{-1}(d)$ of the degree map $\deg : \mathcal{Q}_8 \to \{1,3,5\}$ introduced in [18]. The block $\mathcal{Q}_8^1$ is described in [18].

A k-algebra A is called *flexible* if $(xy)x = x(yx)$ for all $x, y \in A$. Let $\mathscr{F}_n \subset \mathscr{D}_n$ be the full subcategory formed by all flexible n-dimensional real division algebras. In [6] it is proved that $\mathscr{F}_2 = \mathscr{C}_2$, $\mathscr{F}_4$ is formed by the scalar isotopes of flexible quadratic 4-dimensional real division algebras, and $\mathscr{F}_8$ is formed by the scalar isotopes of flexible quadratic 8-dimensional real division algebras together with the generalized pseudo-octonion algebras. It follows that $\mathscr{F}_4^{+-} = \mathscr{F}_4^{-+} = \emptyset$ and $\mathscr{F}_8^{+-} = \mathscr{F}_8^{-+} = \emptyset$. On the basis of [6] and [10], the diagonal blocks $\mathscr{F}_4^{++}$, $\mathscr{F}_4^{--}$ and $\mathscr{F}_8^{++}$, $\mathscr{F}_8^{--}$ are described in [11].

A k-algebra A is called *power-commutative* if every subalgebra generated by one element is commutative. Let $\mathscr{P}_n \subset \mathscr{D}_n$ be the full subcategory formed by all power-commutative n-dimensional real division algebras. In [16] it is proved that $\mathscr{P}_4$ is formed by all planar isotopes of quadratic 4-dimensional real division algebras. This implies $\mathscr{P}_4^{+-} = \mathscr{P}_4^{-+} = \emptyset$. The diagonal blocks $\mathscr{P}_4^{++}$ and $\mathscr{P}_4^{--}$ are described in [16].

Let $\mathscr{H}_4 \subset \mathscr{D}_4$ be the full subcategory formed by all isotopes of $\mathbb{H}$. The blocks $\mathscr{H}_4^{\alpha\beta}$ are described in [12].

Let $\mathscr{A}_{8c}, \mathscr{A}_{8l}, \mathscr{A}_{8r}$ be the full subcategories of $\mathscr{D}_8$ that are formed by all 8-dimensional absolute valued algebras having a non-zero central idempotent, a left unity, or a right unity respectively. The existence of a non-zero central idempotent implies $\mathscr{A}_{8c}^{+-} = \mathscr{A}_{8c}^{-+} = \emptyset$, the existence of a left unity implies $\mathscr{A}_{8l}^{-+} = \mathscr{A}_{8l}^{--} = \emptyset$, and the existence of a right unity implies $\mathscr{A}_{8r}^{+-} = \mathscr{A}_{8r}^{--} = \emptyset$. The blocks $\mathscr{A}_{8c}^{++}$, $\mathscr{A}_{8c}^{--}$, $\mathscr{A}_{8l}^{++}$, $\mathscr{A}_{8l}^{+-}$, $\mathscr{A}_{8r}^{++}$ and $\mathscr{A}_{8r}^{-+}$ are described in [9].

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