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STANDARD COMMUTING DILATIONS AND LIFTINGS

BY

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Abstract. We identify how the standard commuting dilation of the maximal commuting piece of any row contraction, especially on a finite-dimensional Hilbert space, is associated to the minimal isometric dilation of the row contraction. Using the concept of standard commuting dilation it is also shown that if liftings of row contractions are on finite-dimensional Hilbert spaces, then there are strong restrictions on properties of the liftings.

1. Introduction. We recall that a *row contraction* $\underline{T} = (T_1, \ldots, T_d)$ is a tuple of bounded operators on a common Hilbert space $\mathcal{H}$, i.e., $T_i \in B(\mathcal{H})$ for $i = 1, \ldots, d$, and $\sum_{i=1}^{d} T_i T_i^* \leq \mathbf{1}$. A row contraction $\underline{V} = (V_1, \ldots, V_d)$, with $V_i \in B(\hat{\mathcal{H}})$ for some Hilbert space $\hat{\mathcal{H}} \supset \mathcal{H}$, is said to be a *minimal isometric dilation* of $\underline{T}$ if

- V_i^* leaves $\mathcal{H}$ invariant and $P_{\mathcal{H}} V_i|_{\mathcal{H}} = T_i$ for $i = 1, \ldots, d$,
- $V_i^* V_j = \delta_{ij} \mathbf{1}$,
- $\hat{\mathcal{H}} = \overline{\operatorname{span}}\{V_{\alpha_1} \ldots V_{\alpha_n} h : h \in \mathcal{H},\, 1 \leq \alpha_i \leq d,\, n \in \mathbb{N}\}$.

Every row contraction admits a minimal isometric dilation (mid for short) (cf. [Po89]) and it is unique up to unitary equivalence. An extensive theory of row contractions, given by Popescu (e.g. [Po89], [Po99]), extends part of the harmonic analysis of contractions developed by Sz.-Nagy and Foiaş (cf. [NF70]). There is another type of dilation called the *standard commuting dilation,* whose study was initiated in the works of Drury [Dr78] and Arveson [Ar98] (cf. [BBD04]), for commuting row contractions, i.e., row contractions consisting of mutually commuting bounded operators. This dilation consists of mutually commuting operators, and one nice property is that it splits into a direct sum of a tuple of shift operators and a tuple of normal operators.

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In [BBD04] the question "how the mid is related to the standard commuting dilation" was addressed. In Sections 2 and 3 we carry out this investigation further. Let $\underline{E} = (E_1, \dots, E_d)$ on a Hilbert space $\mathcal{H}_E \supset \mathcal{H}_C$ be a row contraction. If for all $i = 1, \dots, d$ we have block matrices

$$E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix} \tag{1.1}$$

with respect to the decomposition $\mathcal{H}_C \oplus \mathcal{H}_A$ of $\mathcal{H}_E$, where $\mathcal{H}_A = \mathcal{H}_C^\perp$, then we say that $\underline{E}$ is a *lifting* of $\underline{C}$ by $\underline{A}$. The mid is an example of a contractive lifting. An important class of liftings is that of subisometric liftings. In Section 3 we make use of the theory of subisometric liftings to work out an example illustrating a result from the previous section. Some applications of the theory of standard commuting dilations also help us here in understanding some properties of liftings.

We describe below two important tools for studying dilations of row contractions: Beginning with a row contraction $\underline{R} = (R_1, \dots, R_d)$ on a Hilbert space $\mathcal{L}$ define

$$\begin{aligned} \mathcal{C}(\underline{R}) := \{\mathcal{M} : \mathcal{M} \text{ is an invariant subspace for each } R_i^*, \\ R_i^* R_j^* h = R_j^* R_i^* h, \forall h \in \mathcal{M}, \forall 1 \le i, j \le d\}. \end{aligned}$$

Then $\mathcal{C}(\underline{R})$ is a complete lattice with respect to arbitrary intersections and span closures of arbitrary unions. Its maximal element is called the *maximal commuting subspace* and is denoted by $\mathcal{L}^\circ(\underline{R})$ or $\mathcal{L}^\circ$. The tuple $\underline{R}^\circ = (R_1^\circ, \dots, R_d^\circ)$ obtained by compressing $\underline{R}$ to $\mathcal{L}^\circ$ is called the *maximal commuting piece*. The block form of $\underline{R}$ in terms of $\mathcal{L} = \mathcal{L}^\circ \oplus (\mathcal{L}^\circ)^\perp$ is

$$R_i = \begin{pmatrix} R_i^\circ & 0 \\ \tilde{R}_i & R_i^N \end{pmatrix} \tag{1.2}$$

where $\underline{R}^N$ is the compression of $\underline{R}$ to the orthogonal complement of $\mathcal{L}^\circ$. Thus $\underline{R}$ is a lifting of $\underline{R}^\circ$ by $\underline{R}^N$.

The second tool is the *full Fock space* on $\mathbb{C}^d$ which is defined as

$$\Gamma(\mathbb{C}^d) := \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d)^{\otimes 2} \oplus \cdots \oplus (\mathbb{C}^d)^{\otimes n} \oplus \cdots$$

and $e_0 = 1 \oplus 0 \oplus \cdots$ is called the *vacuum vector*. The (left) *shift operator* is given by

$$L_i x := e_i \otimes x \quad \text{for } x \in \Gamma(\mathbb{C}^d),\ i = 1, \dots, d,$$

where $e_1, \dots, e_d$ is the standard basis of $\mathbb{C}^d$. The maximal commuting subspace of the full Fock space $\Gamma(\mathbb{C}^d)$ with respect to $\underline{L} = (L_1, \dots, L_d)$ is denoted by $\Gamma_s(\mathbb{C}^d)$ and is called the *symmetric Fock space.* The maximal commuting piece of $\underline{L}$ is denoted by $\underline{S} = (S_1, \dots, S_d)$.

The following multi-index notation is used. Let $\tilde{\Lambda}$ denote the set of all finite tuples formed using $1, \ldots, d$ (where repetitions are allowed). For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \tilde{\Lambda}$ of length $|\alpha| = m$, e_α will denote the vector $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m}$. For a row contraction $\underline{T}$ the notation T_α stands for $T_{\alpha_1} \ldots T_{\alpha_m}$.

2. Standard commuting dilations. In this section we assume $d < \infty$ because the results of [BBD04] used here have this assumption. Lemma 2.2 is the only exception where this assumption is not needed. It is immediate (cf. [Ar98]) that the unital C^*-algebra $C^*(\underline{S})$ generated by $S_i \in B(\Gamma_s(\mathbb{C}^d))$, $i = 1, \ldots, d$, satisfies

$$C^*(\underline{S}) = \overline{\operatorname{span}}\{S_\alpha S_\beta^* : \alpha, \beta \in \tilde{\Lambda}\}.$$

If $\underline{T}$ is commuting, then there exists a unique unital completely positive map $\phi : C^*(\underline{S}) \to B(\mathcal{H})$ with

$$\phi(S_\alpha S_\beta^*) = T_\alpha T_\beta^*, \qquad \alpha, \beta \in \tilde{\Lambda}$$

(cf. [Ar98]). Hence we have a minimal Stinespring dilation $\pi_1 : C^*(\underline{S}) \to B(\mathcal{H}_1)$ of ϕ such that

$$\phi(X) = P_{\mathcal{H}} \pi_1(X)|_{\mathcal{H}} \quad \forall X \in C^*(\underline{S}) \tag{2.1}$$

and $\overline{\operatorname{span}}\{\pi_1(X)h : X \in C^*(\underline{S}),\, h \in \mathcal{H}\} = \mathcal{H}_1$. The tuple $\underline{\tilde{S}} = (\tilde{S}_1, \ldots, \tilde{S}_d)$, where $\tilde{S}_i = \pi_1(S_i)$, is the *standard commuting dilation* of $\underline{T}$, which is unique up to unitary equivalence (cf. [Ar98], [BBD04]).

We recall a result from [BBD04]:

THEOREM 2.1. *Let $\underline{T}$ be a commuting row contraction on $\mathcal{H}$ and $\underline{V}$ is the mid of $\underline{T}$. Then $\underline{V}^\circ$ is the standard commuting dilation of $\underline{T}$.*

We remark that Popescu's presentation (equation (2.1) of [Po89]) for the mid was employed in [BBD04] to prove the above result. From the easy observation that the maximal commuting subspace of a row contraction $\underline{T}$ is a $\underline{V}$-coinvariant subspace we conclude that $\underline{V}$ is an isometric lifting of $\underline{T}^\circ$. Therefore Theorem 21 implies that there exists a $\underline{V}^\circ$-coinvariant subspace of $\hat{\mathcal{H}}^\circ$ such that the compression of $\underline{V}$ to that subspace is a commuting lifting of $\underline{T}^\circ$. It is natural to ask if $\underline{V}^\circ$ is the standard commuting dilation of $\underline{T}^\circ$. A necessary and sufficient condition for this appears for $*$-stable row contractions in Theorem 9 of [BBD04] and for a more general setup of subproduct systems in Proposition 6.13 of [SS09]. Versions of the above question for the coisometric case have been addressed here in Theorem 2.3 (for finite-dimensional Hilbert spaces) and Proposition 3.3.

Let $\mathcal{G}$ be the (non-selfadjoint unital) weak-operator-topology-closed algebra generated by the $V_i \in B(\hat{\mathcal{H}})$ of the mid $\underline{V}$ of $\underline{T}$. We recall that a row contraction $\underline{T}$ is said to be *coisometric* if $\sum_{i=1}^d T_i T_i^* = \mathbf{1}$.

Lemma 2.2. *Suppose a tuple $\underline{T}$, on a given finite-dimensional Hilbert space $\mathcal{H}$, is coisometric. Let $\mathcal{M}$ be a subspace of* (*the dilation space*) $\hat{\mathcal{H}}$ *which is invariant for both $\mathcal{G}$ and $\mathcal{G}^*$* (*i.e., reducing*). *Denote $\mathcal{M}\cap\mathcal{H}$ by $\mathcal{H}_{\mathcal{M}}$. Then $\mathcal{M} = \mathcal{G}[\mathcal{H}_{\mathcal{M}}]$.*

Proof. Note that $\mathcal{H}_{\mathcal{M}}$ is invariant with respect to T_i^* for $i = 1, \dots, d$ because $\mathcal{H}$ and $\mathcal{M}$ are V_i^*-invariant and $V_i^*|_{\mathcal{H}} = T_i^*$. Lemma 3.4 of [DKS01] states that $\mathcal{G}[\mathcal{L}]$ reduces $\mathcal{G}$ if $\mathcal{L}$ is a T_i^*-invariant subspace. So $\mathcal{G}[\mathcal{H}_{\mathcal{M}}]$ reduces $\mathcal{G}$. Because $\mathcal{H}_{\mathcal{M}} \subset \mathcal{M}$ and $\mathcal{M}$ is $\mathcal{G}$-invariant we also have $\mathcal{G}[\mathcal{H}_{\mathcal{M}}] \subset \mathcal{M}$.

Assume that $\mathcal{H}' = \mathcal{M} \ominus \mathcal{G}[\mathcal{H}_{\mathcal{M}}]$ is non-zero. Corollary 4.2 of [DKS01] shows that any non-zero $\mathcal{G}^*$-invariant subspace intersects $\mathcal{H}$ non-trivially. Hence $\mathcal{H}'$ has a non-trivial intersection with $\mathcal{H}$. This is a contradiction as $(\mathcal{M} \ominus \mathcal{G}[\mathcal{H}_{\mathcal{M}}]) \cap \mathcal{H} = \mathcal{H}_{\mathcal{M}} \cap \mathcal{G}[\mathcal{H}_{\mathcal{M}}]^{\perp} = \{0\}$. Therefore $\mathcal{M} = \mathcal{G}[\mathcal{H}_{\mathcal{M}}]$. ∎

Suppose π is a representation on a Hilbert space $\mathcal{L}$ of the Cuntz algebra $\mathcal{O}_d$ with generators $g_1, \dots, g_d$. The representation π is said to be *spherical* if $\overline{\operatorname{span}}\{\pi(g_\alpha)h : h \in \mathcal{L}^\circ(\pi(g_1), \dots, \pi(g_d)),\ \alpha \in \tilde{\Lambda}\} = \mathcal{L}$.

Now assume that $\mathcal{H}$ is finite-dimensional. Theorem 19 of [BBD04] states that the mid $\underline{V}$ on $\hat{\mathcal{H}}$ of $\underline{T}$ on $\mathcal{H}$ can be decomposed as $\underline{V}^1 \oplus \underline{V}^2$ with respect to the decomposition of $\hat{\mathcal{H}}$ as $\hat{\mathcal{H}}^1 \oplus \hat{\mathcal{H}}^2$ into reducing subspaces where $\underline{V}^1$ is associated to a spherical representation of $\mathcal{O}_d$ and $\underline{V}^2$ has trivial maximal commuting piece. Further it is shown in Theorem 18 of [BBD04] that any spherical representation of $\mathcal{O}_d$ is a direct integral of GNS representations of some Cuntz states. Because $\mathcal{H}$ is finite-dimensional, this direct integral decomposition ([BBD04, Theorem 18]) now tells us that $\hat{\mathcal{H}}^1$ can be further decomposed into irreducible subspaces as $\hat{\mathcal{H}}^1_1 \oplus \dots \oplus \hat{\mathcal{H}}^1_k$ for some $k \in \mathbb{N}$. Let $\mathcal{H}_j := \mathcal{H} \cap \hat{\mathcal{H}}^1_j$. We observe that $\mathcal{H}_j$, $j = 1, \dots, k$, are non-zero T_i^*-invariant subspaces with trivial intersection for $i = 1, \dots, d$ and $\mathcal{G}[\mathcal{H}_j] = \hat{\mathcal{H}}^1_j$ for $j = 1, \dots, k$ from Lemma 2.2. It also follows that the compressions of $\underline{T}$ to the $\mathcal{H}_j$'s are coisometric. But as the restriction of $\underline{V}$ to $\hat{\mathcal{H}}^1_j$ is associated to an irreducible and spherical representation, the related maximal commuting subspace is one-dimensional (cf. [BBD04, Theorem 18 and 19]) and hence is a minimal $\mathcal{G}^*$-invariant subspace for each j. By Lemma 5.8 of [DKS01] such a minimal $\mathcal{G}^*$-invariant subspace is unique, and since the $\mathcal{H}_j$'s are $\mathcal{G}^*$-invariant, the maximal commuting subspace of $\underline{V}$ on $\hat{\mathcal{H}}^1_j$ is contained in $\mathcal{H}_j$.

Consider the case when the maximal commuting subspace of the mid $\underline{V}$ of a row contraction $\underline{T}$ on the Hilbert space $\mathcal{H}$ is contained in $\mathcal{H}$. Proposition 7 of [BBD04] yields $\hat{\mathcal{H}}^\circ \cap \mathcal{H} = \mathcal{H}^\circ$. So the maximal commuting piece of $\underline{T}$ is also the maximal commuting piece of $\underline{V}$ and therefore the standard commuting dilation of itself.

Theorem 2.3. *Suppose the dimension of $\mathcal{H}$ is finite and $\underline{T}$ is a coisometric row contraction on it. Then the maximal commuting subspace of*

$\underline{V}$ is contained in $\mathcal{H}$ and coincides with the maximal commuting subspace of $\underline{T}$.

Proof. Let $\underline{V}$ on $\hat{\mathcal{H}}$ be decomposed as above. From the arguments above, the maximal commuting subspaces of the compressions of $\underline{V}$ on $\hat{\mathcal{H}}_j^1$ are contained in $\mathcal{H}_j$. The linear span of all these subspaces is in fact $\hat{\mathcal{H}}^\circ(\underline{V})$ and hence is also contained in $\mathcal{H}$. The argument for the second assertion has already been given above. ∎

3. Subisometric liftings

DEFINITION 3.1. Let $\underline{C} = (C_1, \ldots, C_d)$ be a row contraction on a Hilbert space $\mathcal{H}_C$. A lifting $\underline{E}$ on a Hilbert space $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$ of $\underline{C}$ is called *subisometric* if the mids $\underline{V}^E$ of $\underline{E}$ and $\underline{V}^C$ of $\underline{C}$ are unitarily equivalent and the corresponding unitary, which intertwines V_i^E and V_i^C for all $i = 1, \ldots, d$, acts as identity on $\mathcal{H}_C$.

A row contraction $\underline{R}$ on a Hilbert space $\mathcal{H}_R$ is said to be $*$-*stable* if $\lim_{n\to\infty} \sum_{|\alpha|=n} \|R_\alpha^* h\|^2 = 0$ for $h \in \mathcal{H}_R$.

EXAMPLE. Here we illustrate Theorem 2.3. Consider the coisometric noncommuting tuple $\underline{T} = (T_1, T_2)$ with

$$T_1 = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 1/(2\sqrt{2}) & 1/2 & 1/(2\sqrt{2}) \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ -1/(2\sqrt{2}) & 1/2 & -1/(2\sqrt{2}) \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}.$$

Then the compression of $\underline{T}$ to the subspace $\{(x_1, x_2, x_3) : x_2 = 0\}$ is the commuting tuple $\underline{Q}$ with

$$Q_1 = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}.$$

So in the notation from (1.2) the maximal commuting piece $\underline{T}^\circ$ is equal to (Q_1, Q_2) and $\underline{T}^N = (1/2, 1/2)$. Alternatively this can be obtained using the formula of Proposition 4 of [BBD04]. Because $\underline{T}^\circ$ is coisometric, commuting and consists of normal operators, the corresponding map $\phi : C^*(\underline{S}) \to B(\mathbb{C}^2)$ of (2.1) satisfies

$$\phi\Big(S_\alpha\Big(\mathbf{1} - \sum_{i=1}^{2} S_i S_i^*\Big) S_\beta^*\Big) = T_\alpha^\circ\Big(\mathbf{1} - \sum_{i=1}^{2} T_i^\circ (T_i^\circ)^*\Big)(T_\beta^\circ)^*.$$

We deduce that $\phi(X) = 0$ for any compact operator X in $C^*(\underline{S})$. Note that the commutators $[S_i^*, S_j]$ are all compact and so ϕ is a unital $*$-homomorphism. Therefore $\underline{T}^\circ$ is the standard commuting dilation of itself. Since $\underline{T}^N = (1/2, 1/2)$ is $*$-stable, $\underline{T}$ is a subisometric lifting of $\underline{T}^\circ$ by Proposition 2.3 of [DG11]. Hence by Theorem 2.1 the maximal commuting piece of mid $\underline{V}$ of $\underline{T}$ is also $\underline{T}^{\circ}$.

For a lifting $\underline{E}$ on a Hilbert space $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$ of $\underline{C}$ by $\underline{A}$, if $\underline{E}$ is a commuting row contraction, then the tuple $\underline{C}$ is also commuting because $\mathcal{H}_C$ is invariant without respect to E_i^*s and hence

$$(C_iC_j - C_jC_i)^* = (E_iE_j - E_jE_i)^*|_{\mathcal{H}_C} = 0 \quad \text{for } 1 \leq i, j \leq d.$$

Similarly if $\underline{E}$ is commuting, then so is $\underline{A}$.

In case $\underline{E}$ is a subisometric lifting of a row contraction $\underline{C}$, clearly unitary equivalence of the mids $\underline{V}^C$ and $\underline{V}^E$ implies unitary equivalence of the maximal commuting pieces of $\underline{V}^C$ and $\underline{V}^E$. In addition when $\underline{E}$ is commuting, by Theorem 2.1 we obtain unitary equivalence of the standard commuting dilations of $\underline{C}$ and $\underline{E}$. We first observe how the maximal commuting piece of any row contraction is related to that of its liftings.

Corollary 3.2. *Let $\underline{E}$, on a finite-dimensional Hilbert space, be a coisometric lifting of $\underline{C}$ by a $*$-stable row contraction $\underline{A}$. Then the maximal commuting pieces of $\underline{E}$ and $\underline{C}$ coincide.*

Proof. Here $\underline{E}$ is a subisometric lifting of $\underline{C}$. So the maximal commuting pieces of $\underline{V}^C$ and $\underline{V}^E$ are unitarily equivalent. By Theorem 2.3 this means that the maximal commuting pieces of $\underline{E}$ and $\underline{C}$ coincide. ∎

Proposition 3.3. *Suppose a row contraction $\underline{T}$ is coisometric and $\underline{T}^N$* (see (1.2)) *is $*$-stable. Then the standard commuting dilation of $\underline{T}^{\circ}$ is the maximal commuting piece of the mid of $\underline{T}$.*

Proof. Since $\underline{T}^N$ is $*$-stable, by using Proposition 2.3 of [DG11] we see that $\underline{T}$ is a subisometric lifting of $\underline{T}^{\circ}$. Therefore the maximal commuting pieces of mids of $\underline{T}$ and $\underline{T}^{\circ}$ are unitarily equivalent. Moreover the maximal commuting piece of the mid of $\underline{T}^{\circ}$ is the standard commuting dilation of $\underline{T}^{\circ}$ by Theorem 2.1. ∎

The next proposition along with Corollary 3.2 shows that if liftings of row contractions are on finite-dimensional Hilbert spaces, then there are strong restrictions on properties of the liftings. For any commuting coisometric row contraction $\underline{T}$ on a Hilbert space $\mathcal{H}$, each T_i^* is subnormal because the standard commuting dilation of $\underline{T}$ consists of normal operators (cf. [At90]). Consequently, if $\mathcal{H}$ is finite-dimensional, this $\underline{T}$ consists of normal operators (cf. [Ha82]).

Proposition 3.4. *A commuting coisometric row contraction on a finite-dimensional Hilbert space cannot be a lifting of another commuting row contraction by a non-zero $*$-stable tuple.*

Proof. Assume that $\underline{E}$ is a commuting coisometric lifting of a commuting row contraction $\underline{C}$ on a finite-dimensional Hilbert space by $\underline{A}$ $*$-stable. Then again by Proposition 2.3 of [DG11], $\underline{E}$ is a subisometric lifting of $\underline{C}$. Consequently, the standard commuting dilations of $\underline{E}$ and $\underline{C}$ are unitarily equivalent. For normal commuting coisometric row contractions it follows that their standard commuting dilations coincide with the row contractions, using the argument in the example at the beginning of this section. By the comment preceding this proposition, $\underline{C}$ consists of normal operators and therefore its standard commuting dilation is equal to itself. In other words $\underline{C}$ is the standard commuting dilation of $\underline{E}$. Since $\underline{E}$ is a compression of the standard commuting dilation of $\underline{E}$ to $\mathcal{H}_E$, this yields $\underline{C} = \underline{E}$. ∎

The following is a related observation for the case $d = 1$, which is an immediate exercise:

Remark 3.5. Let C be a (single) coisometry on a finite-dimensional Hilbert space. Let E be a coisometric lifting of C by A. Then A is coisometric.

REFERENCES

[Ar98] W. Arveson, *Subalgebras of C^*-algebras III. Multivariable operator theory*, Acta Math. 181 (1998), 159–228.

[At90] A. Athavale, *On the intertwining of joint isometries*, J. Operator Theory 23 (1990), 339–350.

[BBD04] B. V. R. Bhat, T. Bhattacharyya and S. Dey, *Standard noncommuting and commuting dilations of commuting tuples*, Trans. Amer. Math. Soc. 356 (2004), 1551–1568.

[DKS01] K. R. Davidson, D. W. Kribs and M. E. Shpigel, *Isometric dilations of non-commuting finite rank n-tuples*, Canad. J. Math. 53 (2001), 506–545.

[De09] S. Dey, *Characteristic functions, liftings and modules*, Habilitationsschrift, 2009, arXiv:0903.4769.

[DG11] S. Dey and R. Gohm, *Characteristic functions of liftings*, J. Operator Theory 65 (2011), 17–45.

[Dr78] S. W. Drury, *A generalization of von Neumann's inequality to the complex ball*, Proc. Amer. Math. Soc. 68 (1978), 300–304.

[Ha82] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Grad. Texts in Math. 19, Springer, New York, 1982.

[NF70] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, 1970.

[Po89] G. Popescu, *Isometric dilations for infinite sequences of noncommuting operators*, Trans. Amer. Math. Soc. 316 (1989), 523–536.

[Po99] —, *Poisson transforms on some C^*-algebras generated by isometries*, J. Funct. Anal. 161 (1999), 27–61.

[SS09] O. M. Shalit and B. Solel, *Subproduct systems*, Documenta Math. 14 (2009), 801–868.

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