COLLOQUIUM MATHEMATICUM

VOL. 126 2012 NO. 1

ON BLOW-UP FOR THE HARTREE EQUATION

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Abstract. We study the blow-up of solutions to the focusing Hartree equation $iu_t + \Delta u + (|x|^{-\gamma} * |u|^2)u = 0$. We use the strategy derived from the almost finite speed of propagation ideas devised by Bourgain (1999) and virial analysis to deduce that the solution with negative energy ($E(u_0) < 0$) blows up in either finite or infinite time. We also show a result similar to one of Holmer and Roudenko (2010) for the Schrödinger equations using techniques from scattering theory.

1. Introduction. We study the blow-up of solutions to the Hartree equation

$$\begin{cases} iu_t + \Delta u + f(u) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u(t,x)$ is a complex-valued function on space-time $\mathbb{R}_t \times \mathbb{R}_x^d$, Δ is the Laplacian in $\mathbb{R}^d$, $f(u)$ is a nonlinearity of Hartree type, $f(u) = \lambda(V * |u|^2)u$ for some fixed constant $\lambda \in \mathbb{R}$ and $0 < \gamma < d$, where $*$ denotes spatial convolution in $\mathbb{R}^d$ and V is a real valued radial function defined in $\mathbb{R}^d$. The case $V(x) = |x|^{-\gamma}$ and $\lambda = 1$ is known as the *focusing case*.

If the solution u of (1.1) has sufficient decay at infinity and smoothness, it conserves mass, energy, and momentum:

$$\begin{aligned} M(u) &= \int_{\mathbb{R}^d} |u(t,x)|^2\,dx = M(u_0), \\ E(u) &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2\,dx - \frac{1}{4}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^\gamma}\,dx\,dy = E(u_0), \\ P(u) &= \operatorname{Im}\int_{\mathbb{R}^d} \bar{u}(t,x)\nabla u(t,x)\,dx = P(u_0). \end{aligned} \tag{1.2}$$

As explained in [4], the above quantities are also conserved for the energy solutions $u \in C_t^0(\mathbb{R}, H^1(\mathbb{R}^d))$.

The equation (1.1) has the scaling invariance property:

$$u_\lambda(x,t) = \lambda^{d/2-s_c}u(\lambda x, \lambda^2 t), \qquad s_c = \gamma/2 - 1, \tag{1.3}$$

2010 *Mathematics Subject Classification*: 35Q40, 35Q55.
Key words and phrases: Hartree equation, blow-up, Hardy–Littlewood–Sobolev type inequality.

DOI: 10.4064/cm126-1-8

in the sense that both the equation and the $\dot{H}^{s_c}$-norm are invariant under the scaling transformation:

$$\|u_\lambda\|_{\dot{H}^{s_c}} = \|u\|_{\dot{H}^{s_c}}.$$

The Hartree equation (1.1) is called *energy-subcritical* when $\gamma < \min\{d, 4\}$, which corresponds to $s_c < 1$; *mass-critical* when $\gamma = 2$, corresponding to $s_c = 0$; *energy-critical* when $d \geq 5$ and $\gamma = 4$, corresponding to $s_c = 1$; and *energy-supercritical* when $4 < \gamma < d$, corresponding to $s_c > 1$.

Numerous papers deal with the Cauchy problem for the Hartree equation. We refer to [4, 11]. A natural question is whether local solutions exist globally.

Now we recall the related results about the focusing Schrödinger equation

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0, \qquad u(0, x) = u_0(x). \tag{1.4}$$

By the scaling analysis, (1.4) is called *mass-subcritical* when $p < 1 + 4/d$, corresponding to $s_c = d/2 - 2/(p-1) < 0$; *mass-critical* when $p = 1 + 4/d$, corresponding to $s_c = 0$; *energy-subcritical* when $p < 1 + 4/(d-2)$, which corresponds to $s_c < 1$; *energy-critical* when $p = 1+4/(d-2)$, corresponding to $s_c = 1$; and *energy-supercritical* when $p > 1 + 4/(d-2)$, corresponding to $s_c > 1$.

It follows from the Gagliardo–Nirenberg inequality that the solution u of (1.4) exists globally in the mass-subcritical case. In the energy-subcritical case, Glassey [5] proved that the solution u of (1.4) blows up in finite time if the initial data satisfies $xu_0 \in L^2(\mathbb{R}^d)$ with negative energy. After this result, many attempts have been made to relax the finite variance assumption. In other words, one wants to know whether any solution corresponding to smooth initial data with negative energy blows up in finite time. In particular, in the energy-critical case, Ogawa and Tsutsumi [14] proved that any solution with initial data $u_0 \in H^1(\mathbb{R}^d)$ radial and $E(u_0) < 0$ must blow up in finite time. The radial or finite variance condition was relaxed to some non-isotropic ones by Martel [10], and was completely removed by Ogawa and Tsutsumi [15] in the mass-critical case in one dimension.

Besides finite time blow-up, there is also another interesting topic. In the mass-critical case, Nawa [13] showed that if the initial data $u_0 \in H^1(\mathbb{R}^d)$ has negative energy, then the solution u of (1.4) blows up in finite or infinite time in the sense that

$$\sup_{t\in(-T_{\min},T_{\max})} \|u(t)\|_{H^1(\mathbb{R}^d)} = +\infty, \tag{1.5}$$

where $(-T_{\min}, T_{\max})$ is the maximal lifespan of the solution with initial data u_0. In the energy-subcritical case, a similar result was established by Holmer and Roudenko [6], by using the concentration-compactness argument developed by Kenig and Merle [7]. By the strategy derived from the finite speed of propagation devised by Bourgain [1] and virial analysis, Du, Wu and Zhang [3] gave a similar result for the energy-critical and energy-supercritical cases. Their method can also be applied to the energy-subcritical case, and this gives another simplified proof for part of the results in [6].

For the Hartree equation (1.1), using the refined Gagliardo–Nirenberg inequality of convolution type and profile decomposition, C. Miao, G. Xu and L. Zhao [12] characterized the dynamics of the finite time blow-up solutions with minimal mass for the mass-critical case with $H^1(\mathbb{R}^4)$ data and $L^2(\mathbb{R}^4)$ data. In this paper, we develop a complete blow-up theory for the Hartree equation with initial data of negative energy.

Now, we state our results:

THEOREM 1.1. (1) *In the energy subcritical case: Assume* $d \geq 3$, $2 \leq \gamma < \min\{d, 4\}$ *and* $u_0 \in H^1(\mathbb{R}^d)$ *with* $E(u_0) < 0$. *Let* u *be a solution of* (1.1) *on the maximal interval* $(-T_{\min}, T_{\max})$.

- *If* $T_{\max} < +\infty$, *then* $\lim_{t \to T_{\max}} \|u(t)\|_{H^1} = +\infty$.
- *If* $T_{\max} = +\infty$, *then there exists a sequence* $t_n \to +\infty$ *such that*

$$\lim_{n \to +\infty} \|u(t_n)\|_{H^1} = +\infty.$$

A similar statement holds for negative time.

(2) *In the energy-critical and energy-supercritical cases: Let* $4 \leq \gamma < d$, $s > s_c = \gamma/2 - 1$ *and* $u_0 \in H^s(\mathbb{R}^d)$ *with* $E(u_0) < 0$. *Let* u *be a solution of* (1.1) *on the maximal interval* $(-T_{\min}, T_{\max})$.

- *If* $T_{\max} < +\infty$, *then* $\lim_{t \to T_{\max}} \|u(t)\|_{H^s} = +\infty$.
- *If* $T_{\max} = +\infty$, *then there exists a sequence* $t_n \to +\infty$ *such that*

$$\lim_{n \to +\infty} \|u(t_n)\|_{H^s} = +\infty.$$

A similar statement holds for negative time.

REMARK 1.1. Roughly speaking, Case 1 refers to finite time blow-up, Case 2 refers to infinite time blow-up. At this stage, it is not clear whether Case 2 can be ruled out, or could indeed happen.

Due to the Galilean transformation, we may relax the negative energy condition to the following

COROLLARY 1.1. *The conclusion of Theorem* 1.1 *still holds true when the condition* $E(u_0) < 0$ *is relaxed to*

$$E(u_0) < \frac{P(u_0)^2}{M(u_0)}.$$

Next we give another restriction similar to one in [6] to show a blow-up result. Let Q be a ground state which satisfies the elliptic equation

$$-\Delta\varphi + \varphi = (|x|^{-\gamma} * |\varphi|^2)\varphi. \tag{1.6}$$

We refer the reader to [9] where a ground state has been constructed as a radial, rapidly decaying function.

Theorem 1.2. *Assume $d \geq 3$, $2 \leq \gamma < \min\{d, 4\}$, and $u_0 \in H^1(\mathbb{R}^d)$ with*

$$\begin{cases} M(u_0)^{1-s_c}E(u_0)^{s_c} < M(Q)^{1-s_c}E(Q)^{s_c}, \\ \|u_0\|_2^{1-s_c}\|\nabla u_0\|_2^{s_c} > \|Q\|_2^{1-s_c}\|\nabla Q\|_2^{s_c}. \end{cases} \tag{1.7}$$

Let u be a solution of (1.1) on the maximal interval $(-T_{\min}, T_{\max})$.

- *If $T_{\max} < +\infty$, then $\lim_{t\to T_{\max}} \|u(t)\|_{H^1} = +\infty$.*
- *If $T_{\max} = +\infty$, then there exists a sequence $t_n \to +\infty$ such that*

$$\lim_{n\to+\infty} \|u(t_n)\|_{H^1} = +\infty.$$

A similar statement holds for negative time.

We will adopt the ideas of Glassey [5] and Du et al. [3] to prove the main theorems. Since in our case, the initial data may not have finite variance, we calculate the second order derivative of the local virial identity. We will apply the strategy of [3] derived from the almost finite speed of propagation ideas devised by Bourgain [1] to obtain Theorem 1.1. And we will borrow some techniques from scattering theory to show Theorem 1.2.

The paper is organized as follows. In Section 2, by using the almost finite speed of propagation and local virial analysis, we prove Theorem 1.1. Section 3 provides the proof of Theorem 1.2 by a technique from scattering theory. We follow the argument of M. Weinstein [16] to show the best constant in the Hardy–Littlewood–Sobolev type inequality in the Appendix.

We conclude the introduction by giving some notation which will be used throughout this paper. For any r, $1 \leq r \leq \infty$, we denote by $\|\cdot\|_r$ the norm in $L^r = L^r(\mathbb{R}^d)$ and by r' the conjugate exponent defined by $1/r + 1/r' = 1$. For any $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^d)$ the usual Sobolev space. If X, Y are nonnegative quantities, we sometimes use $X \lesssim Y$ to denote the estimate $X \leq CY$ for some C.

2. Proof of Theorem 1.1. We only need to prove the part on infinite blow-up, since the proof of the finite blow-up part is standard (see [2]).

Assume the contrary; then

$$\sup_{t\in\mathbb{R}^+} \|u(t)\|_{H^s(\mathbb{R}^d)} \leq C_0 < +\infty, \tag{2.1}$$

where $s = 1$ in the energy-subcritical case and $s > s_c$ in the energy-critical and energy-supercritical cases.

Let

$$V_a(t) = \int_{\mathbb{R}^d} a(x)|u(t,x)|^2\,dx. \tag{2.2}$$

For a solution u satisfying

$$i\partial_t u + \Delta u = \mathcal{N},$$

a further computation establishes that

$$V_a'(t) = 2\,\mathrm{Im} \int_{\mathbb{R}^d} \nabla a(x)\cdot\nabla u(t,x)\bar{u}(t,x)\,dx,$$

and

$$\begin{aligned} V_a''(t) &= 4\,\mathrm{Re} \sum_{1\le j,k\le d} \int_{\mathbb{R}^d} \partial_{jk}a(x)\bar{u}_j(t,x)u_k(t,x)\,dx \\ &\quad + \int_{\mathbb{R}^d} (-\Delta\Delta)a(x)|u(t,x)|^2\,dx + 2\int_{\mathbb{R}^d} \{\mathcal{N},u\}_p(t,x)\nabla a(x)\,dx, \end{aligned} \tag{2.3}$$

where $\{f,g\}$ is the momentum bracket defined as $\mathrm{Re}(f\nabla\bar{g} - g\nabla\bar{f})$.

Now, plugging $\mathcal{N} = -(|x|^{-\gamma} * |u|^2)u$, we have

Lemma 2.1. *For any $a(\cdot) \in C^4(\mathbb{R}^d)$, we have*

$$V_a'(t) = 2\,\mathrm{Im} \int_{\mathbb{R}^d} \nabla a(x)\cdot\nabla u(t,x)\bar{u}(t,x)\,dx,$$

$$\begin{aligned} V_a''(t) &= 4\,\mathrm{Re} \sum_{1\le j,k\le d} \int_{\mathbb{R}^d} \partial_{jk}a(x)\bar{u}_j(t,x)u_k(t,x)\,dx - \int_{\mathbb{R}^d} \Delta^2 a(x)|u(t,x)|^2\,dx \\ &\quad - 2\gamma \iint_{\mathbb{R}^d\times\mathbb{R}^d} (x-y)\cdot\nabla a(x)\frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^{\gamma+2}}\,dx\,dy. \end{aligned}$$

In particular, if $a(\cdot)$ is radial, then setting $r = |x|$ we obtain

$$V_a'(t) = 2\,\mathrm{Im} \int_{\mathbb{R}^d} a'\,\frac{x\cdot\nabla u}{r}\bar{u}\,dx, \tag{2.4}$$

$$\begin{aligned} V_a''(t) &= 4\int_{\mathbb{R}^d} \frac{a'(r)}{r}|\nabla u|^2\,dx + 4\int_{\mathbb{R}^d} \left(\frac{a''(r)}{r^2} - \frac{a'(r)}{r^3}\right)|x\cdot u|^2\,dx \\ &\quad - \int_{\mathbb{R}^d} \Delta^2 a|u|^2\,dx \\ &\quad - \gamma \iint_{\mathbb{R}^d\times\mathbb{R}^d} (x-y)\cdot[\nabla a(x) - \nabla a(y)]\frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^{\gamma+2}}\,dx\,dy, \end{aligned} \tag{2.5}$$

Now we use the above lemma to show a result similar to the almost finite speed of propagation [1]. Let $m_0 = \|u_0\|_2$.

Lemma 2.2. *For any $\eta_0 > 0$ and all $t \le \eta_0 R/(2m_0C_0)$, we have*

$$\int\limits_{|x|\ge 2R} |u(t,x)|^2\,dx \le \eta_0 + o_R(1), \tag{2.6}$$

where $o_R(1) \to 0$ as $R \to +\infty$.

Proof. Take $a_1(\cdot) \in C^4(\mathbb{R}^d)$ radial such that $0 \le a_1 \le 1$, $|a_1'| \le 1/R$, and

$$a_1(r) = \begin{cases} 0, & 0 \le r \le R, \\ 1, & r \ge 2R. \end{cases} \tag{2.7}$$

Then by (2.4) and (2.1), we have

$$|V_{a_1}'(t)| \le 2R^{-1}\|\nabla u\|_2\|u\|_{L^2(|x|\ge R)} \le 2R^{-1}C_0m_0,$$

thus

$$V_{a_1}(t) = V_{a_1}(0) + \int_0^t V_{a_1}'(s)\,ds \le \int\limits_{|x|\ge R} |u_0|^2\,dx + 2R^{-1}C_0m_0t.$$

Therefore

$$\int\limits_{|x|\ge 2R} |u(t,x)|^2\,dx \le V_{a_1}(t) \le \eta_0 + o_R(1)$$

whenever $t \le \eta_0 R/(2m_0C_0)$, and this completes the proof. ∎

Inspired by the virial identity

$$\frac{d^2}{dt^2}\int\limits_{\mathbb{R}^d} |x|^2|u(t)|^2\,dx = 16K(u(t)),$$

where

$$K(u(t)) = \frac{1}{2}\int\limits_{\mathbb{R}^d} |\nabla u|^2\,dx - \frac{\gamma}{8}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^\gamma}\,dx\,dy, \tag{2.8}$$

we can rewrite $V_a''(t)$ in (2.5) as

$$V_a''(t) = 16K(u(t)) + R_1 + R_2 + R_3, \tag{2.9}$$

where

$$R_1 = 4\int\limits_{\mathbb{R}^d}\left(\frac{a'(r)}{r} - 2\right)|\nabla u|^2\,dx + 4\int\limits_{\mathbb{R}^d}\left(\frac{a''(r)}{r^2} - \frac{a'(r)}{r^3}\right)|x\cdot u|^2\,dx,$$

$$R_2 = -\gamma\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}[(x-y)\cdot(\nabla a(x) - \nabla a(y)) - 2|x-y|^2]\frac{|u(x)|^2|u(y)|^2}{|x-y|^{\gamma+2}}\,dx\,dy,$$

$$\begin{aligned} R_3 &= -\int\limits_{\mathbb{R}^d} \Delta^2 a|u|^2\,dx \\ &= -\int\limits_{\mathbb{R}^d}\left(a^{(4)} + \frac{2(d-1)}{r}a^{(3)} + \frac{(d-1)(d-3)}{r^2}\left(a'' - \frac{a'}{r}\right)\right)|u|^2\,dx. \end{aligned}$$

We will show R_1, R_2 and R_3 are error terms, by making use of localization. We take $a(\cdot) \in C^4(\mathbb{R}^d)$ radial satisfying

$$a''(r) = \begin{cases} 2, & 0 \le r \le 2R, \\ 0, & r \ge 4R, \end{cases} \tag{2.10}$$

and $a(0) = a'(0) = 0$, $a'' \le 2$, and $a^{(4)} \le R^{-2}$. Then we have

LEMMA 2.3. *There exists a constant $C_1 = C_1(s, \gamma, d, m_0, c_0)$ such that*

$$V_a''(t) \le 16K(u(t)) + C_1 \|u\|_{L^2(|x|\ge 2R)}^{2-\gamma/(2s)}. \tag{2.11}$$

Proof. First, we prove that $R_1 \le 0$. Indeed, if $a''(r)/r^2 - a'(r)/r^3 \le 0$, then obviously $R_1 \le 0$ since $|a'| \le 2r$. On the other hand, if $a''(r)/r^2 - a'(r)/r^3 \ge 0$, then again

$$R_1 \le 4 \int_{\mathbb{R}^d} (a'' - 2)|\nabla u|^2 \, dx \le 0.$$

Moreover, we have

$$\operatorname{supp}\{(x-y)\cdot(\nabla a(x) - \nabla a(y)) - 2|x-y|^2\} \subset \{(x,y) : |x| \ge 2R\} \cup \{(x,y) : |y| \ge 2R\}.$$

In the region where $|x| \ge 2R$,

$$|(x-y)\cdot(\nabla a(x) - \nabla a(y))| \lesssim |x-y|^2,$$

and we use the Hölder inequality, generalized Young inequality and Sobolev embedding theorem to control the contribution to R_2 from this region by

$$\int_{\mathbb{R}^d} \int_{\{|x|\ge 2R\}} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^\gamma} \, dx\, dy \lesssim \|u\|_{L^{2d/(d-\gamma/2)}(|x|\ge 2R)}^2 \|u\|_{L^{2d/(d-\gamma/2)}}^2$$
$$\lesssim \|u\|_{L^2(|x|\ge 2R)}^{2-\gamma/(2s)} \|u\|_{\dot{H}^s}^{\gamma/(2s)} \|u\|_{L^2}^{2-\gamma/(2s)} \|u\|_{\dot{H}^s}^{\gamma/(2s)} \lesssim C_0^{2+8/(2s)} \|u\|_{L^2(|x|\ge 2R)}^{2-\gamma/(2s)}.$$

Similarly, we have the same control in the region where $|y| \ge 2R$. Thus

$$R_2 \lesssim \|u\|_{L^2(|x|\ge 2R)}^{2-\gamma/(2s)}.$$

Furthermore,

$$R_3 \le CR^{-2} \|u\|_{L^2(|x|\ge 2R)}^2.$$

Hence,

$$V_a''(t) \le 16K(u(t)) + C_1 \|u\|_{L^2(|x|\ge 2R)}^{2-\gamma/(2s)}. \quad \blacksquare$$

Proof of Theorem 1.1. Now we use the above two lemmas to prove Theorem 1.1. By (2.6) and (2.11), we get

$$V_a''(t) \le 16K(u(t)) + C_1(\eta_0^{2-\gamma/(2s)} + o_R(1)) \tag{2.12}$$

for any $t \le T \triangleq \eta_0 R/(2m_0C_0)$. Integrating the above inequality from 0 to T twice, and using the fact that

$$K(u(t)) \le E(u(t)) < 0 \quad \text{for any } t \in \mathbb{R},$$

we obtain

$$\begin{aligned} V_a(T) &\le V_a(0) + V_a'(0)T + \int_0^T\int_0^t g\big(16K(u(s)) + C_1(\eta_0^{2-\gamma/(2s)} + o_R(1))\big)\,ds\,dt \\ &\le V_a(0) + V_a'(0)T + \big(16E(u_0) + C_1(\eta_0^{2-\gamma/(2s)} + o_R(1))\big)T^2. \end{aligned}$$

Taking η_0 such that $C_1\eta_0^{2-\gamma/2s} = -\frac{1}{2}E(u_0)$, and R large enough, one has

$$V_a(T) \le V_a(0) + V_a'(0)\frac{\eta_0 R}{2m_0C_0} + \alpha_0 R^2, \tag{2.13}$$

where $\alpha_0 = E(u_0)\eta_0^2/(4m_0C_0)^2 < 0$. Next we claim that

$$V_a(0) = o_R(1)R^2, \quad V_a'(0) = o_R(1)R. \tag{2.14}$$

In fact,

$$\begin{aligned} V_a(0) &\le \int_{|x|\le\sqrt{R}} |x|^2|u_0(x)|^2\,dx + \int_{\sqrt{R}\le|x|\le 2R} |x|^2|u_0(x)|^2\,dx + R^2\int_{|x|\ge 2R}|u_0(x)|^2\,dx \\ &\le Rm_0^2 + R^2\int_{|x|\ge\sqrt{R}}|u_0(x)|^2\,dx + R^2\int_{|x|\ge 2R}|u_0(x)|^2\,dx \\ &= o_R(1)R^2. \end{aligned}$$

Similarly, we have $V_a'(0) = o_R(1)R^2$.

Taking R large enough, by (2.13) and (2.14), we get

$$V_a(T) \le o_R(1)R^2 + \alpha_0R^2 \le \tfrac{1}{2}\alpha_0R^2 < 0.$$

This contradicts $V_a(T) \ge 0$, and so the proof of Theorem 1.1 is complete.

Proof of Corollary 1.1. Using the Galilean transformation, we define

$$v(x,t) = e^{i\xi_0\cdot(x-t\xi_0)}u(x-2t\xi_0,t). \tag{2.15}$$

If u is a solution of (1.1), then so is v, and also

$$\begin{aligned} E(v) &= E(u_0) + 2\xi_0P(u_0) + |\xi_0|^2M(u_0) \\ &= M(u_0)\left|\xi_0 + \frac{P(u_0)}{M(u_0)}\right|^2 + E(u_0) - \frac{P(u_0)^2}{M(u_0)}. \end{aligned}$$

On the other hand, if we choose $\xi_0 = -P(u_0)/M(u_0)$, then we get

$$E(v) = E(u_0) - \frac{P(u_0)^2}{M(u_0)} < 0. \tag{2.16}$$

Hence, by the same argument as in Theorem 1.1, replacing u by v, we obtain the desired result.

3. Proof of Theorem 1.2. In this section, we use some techniques from scattering theory to prove Theorem 1.2. First, we deduce that the sign of $\|u(t)\|_{L^2}^{1-s_c}\|\nabla u(t)\|_{L^2}^{s_c}$ is invariant along the flow (1.1) under the restriction $M(u_0)^{1-s_c}E(u_0)^{s_c} < M(Q)^{1-s_c}E(Q)^{s_c}$:

PROPOSITION 3.1. *If u_0 satisfies* (1.7), *then*

$$\|u(t)\|_{L^2}^{1-s_c}\|\nabla u(t)\|_{L^2}^{s_c} > \|Q\|_{L^2}^{1-s_c}\|\nabla Q\|_{L^2}^{s_c}, \qquad \forall t \in (-T_{\min}, T_{\max}), \tag{3.1}$$

where $(-T_{\min}, T_{\max})$ *is the maximal lifespan of the solution.*

Proof. For a contradiction, if there exists $t_0 \in (-T_{\min}, T_{\max})$ such that

$$\|u(t_0)\|_{L^2}^{1-s_c}\|\nabla u(t_0)\|_{L^2}^{s_c} = \|Q\|_{L^2}^{1-s_c}\|\nabla Q\|_{L^2}^{s_c}, \tag{3.2}$$

then by the Hardy–Littlewood–Sobolev type inequality,

$$\|(|x|^{-\gamma} * |u|^2)|u|^2\|_{L^1} \le C_{\mathrm{HLS}}\|\nabla u\|_2^{\gamma}\|u\|_2^{4-\gamma}, \tag{3.3}$$

where

$$C_{\mathrm{HLS}} = \left(\frac{4-\gamma}{\gamma}\right)^{\gamma/2}\frac{4}{4-\gamma}\|Q\|_2^{-2} = \frac{\|(|x|^{-\gamma} * |Q|^2)|Q|^2\|_{L^1}}{\|\nabla Q\|_2^{\gamma}\|Q\|_2^{4-\gamma}};$$

this will be proved in the Appendix.

Thus by (3.3), we obtain

$$\begin{aligned}
M(Q)^{\frac{1-s_c}{s_c}}E(Q) &> M(u(t_0))^{\frac{1-s_c}{s_c}}E(u(t_0)) \\
&= \tfrac12\|u(t_0)\|_2^{\frac{2(1-s_c)}{s_c}}\|\nabla u(t_0)\|_2^2 - \tfrac14\|u(t_0)\|_2^{\frac{2(1-s_c)}{s_c}}\|(|x|^{-\gamma} * |u|^2)|u|^2\|_{L^1} \\
&\ge \tfrac12\|u(t_0)\|_2^{\frac{2(1-s_c)}{s_c}}\|\nabla u(t_0)\|_2^2 - \tfrac14 C_{\mathrm{HLS}}(\|u(t_0)\|_2^{\frac{1-s_c}{s_c}}\|\nabla u(t_0)\|_2)^{\gamma} \\
&= \tfrac12 M(Q)^{\frac{1-s_c}{s_c}}\|\nabla Q\|_2^2 - \tfrac14 M(Q)^{\frac{1-s_c}{s_c}}\|(|x|^{-\gamma} * |Q|^2|Q|^2\|_{L^1} \\
&= M(Q)^{\frac{1-s_c}{s_c}}E(Q),
\end{aligned}$$

which gives a contradiction, and this completes the proof. ∎

By a similar argument to that for Theorem 1.1, one can reduce Theorem 1.2 to the following lemma:

LEMMA 3.1. *If u_0 satisfies* (1.7), *then there exists $\beta_0 < 0$ such that*

$$K(u(t)) \le \beta_0, \qquad \forall t \in (-T_{\min}, T_{\max}). \tag{3.4}$$

Proof. First we claim that

$$K(u(t)) < 0, \qquad \forall t \in (-T_{\min}, T_{\max}). \tag{3.5}$$

In fact, by Proposition 3.1 and (1.7), one has

$$\begin{aligned}K(u(t)) &= \frac{\gamma}{2}E(u) - \left(\frac{\gamma}{4}-\frac{1}{2}\right)\|\nabla u\|_2^2 \\ &< \left(\frac{M(Q)}{M(u_0)}\right)^{\frac{1-s_c}{s_c}}\left(\frac{\gamma}{2}E(Q) - \left(\frac{\gamma}{4}-\frac{1}{2}\right)\|\nabla Q\|_2^2\right) = 0,\end{aligned}$$

which means that

$$\|\nabla u\|_2^2 \le \frac{\gamma}{4}\|(|x|^{-\gamma} * |u(t)|^2)|u(t)|^2\|_{L^1}, \qquad \forall t \in (-T_{\min}, T_{\max}). \tag{3.6}$$

This together with (3.3) implies that

$$\frac{4}{\gamma}\|\nabla u(t)\|_2^2 \le \|(|x|^{-\gamma} * |u(t)|^2)|u(t)|^2\|_{L^1} \le C_{\mathrm{HLS}}\|\nabla u\|_2^{\gamma}\|u\|_2^{4-\gamma},$$

and so

$$\|\nabla u(t)\|_2^{\gamma-2} \ge \frac{4}{\gamma C_{\mathrm{HLS}} M(u_0)^{2-\gamma/2}} > 0. \tag{3.7}$$

Next we claim that there exists $\delta_0 > 0$ such that

$$K(u(t)) < -\delta_0\|\nabla u\|_2^2, \qquad \forall t \in (-T_{\min}, T_{\max}). \tag{3.8}$$

In fact, suppose this were not true; then there exist $\{t_n\}$ and $\delta_n \to 0$ such that

$$-\delta_n\left(\frac{\gamma}{4}-\frac{1}{2}\right)\|\nabla u(t_n)\|_2^2 < K(u(t_n)) < 0.$$

Hence

$$\begin{aligned}&M(u(t_n))^{\frac{1-s_c}{s_c}}E(u(t_n)) \\ &\quad= \frac{2}{\gamma}\left(M(u(t_n))^{\frac{1-s_c}{s_c}}\left(K(u(t_n)) + \left(\frac{\gamma}{4}-\frac{1}{2}\right)\|\nabla u(t_n)\|_2^2\right)\right) \\ &\quad\ge \frac{2}{\gamma}\|u(t_n)\|_2^{\frac{2(1-s_c)}{s_c}}\|\nabla u(t_n)\|_2^2(1-\delta_n)\left(\frac{\gamma}{4}-\frac{1}{2}\right) \\ &\quad> \frac{\gamma-2}{2\gamma}(1-\delta_n)\|Q\|_2^{\frac{2(1-s_c)}{s_c}}\|\nabla Q\|_2^2 = (1-\delta_n)M(Q)^{\frac{1-s_c}{s_c}}E(Q);\end{aligned}$$

letting $n \to +\infty$, we have

$$M(u)^{1-s_c}E(u)^{s_c} \ge M(Q)^{1-s_c}E(Q)^{s_c}.$$

This contradicts the inequality $M(u)^{1-s_c}E(u)^{s_c} < M(Q)^{1-s_c}E(Q)^{s_c}$.

Combining (3.7) and (3.8), we obtain

$$K(u(t)) \le \beta_0, \qquad \forall t \in (-T_{\min}, T_{\max}). \ \blacksquare \tag{3.9}$$

4. Appendix. In this Appendix, we find the best constant in the Hardy–Littlewood–Sobolev type inequality (3.3).

First, by the symmetry

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{x\cdot(x-y)}{|x-y|^{\gamma+2}}\phi(x)^2\phi(y)^2\,dx\,dy = -\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{y\cdot(x-y)}{|x-y|^{\gamma+2}}\phi(x)^2\phi(y)^2\,dx\,dy,$$

and a direct computation, we have the following identities:

Lemma 4.1. *If $\phi\in\mathcal{S}(\mathbb{R}^d)$, then*

$$\int\limits_{\mathbb{R}^d} x\cdot\nabla\phi\,\Delta\phi\,dx = \frac{d-2}{2}\int\limits_{\mathbb{R}^d}|\nabla\phi|^2\,dx,$$

$$\int\limits_{\mathbb{R}^d} x\cdot\nabla\phi\ \phi\,dx = -\frac{d}{2}\int\limits_{\mathbb{R}^d}|\phi|^2\,dx,$$

$$\int\limits_{\mathbb{R}^d} x\cdot\nabla\phi\ (|x|^{-\gamma}*|\phi|^2)\phi\,dx = \left(-\frac{d}{2}+\frac{\gamma}{4}\right)\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\phi(x)^2\phi(y)^2}{|x-y|^{\gamma}}\,dx\,dy.$$

Lemma 4.2. *Assume that $2<\gamma<\min\{4,d\}$. Let ϕ be an $H^1(\mathbb{R}^d)$ solution of the equation*

$$-\Delta\phi+\phi = (|x|^{-\gamma}*|\phi|^2)\phi. \tag{4.1}$$

Then the following identities hold:

$$K_1(\phi)\triangleq\int\limits_{\mathbb{R}^d}[|\nabla\phi|^2+|\phi|^2]\,dx - \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\phi(x)^2\phi(y)^2}{|x-y|^{\gamma}}\,dx\,dy = 0,$$

$$K_2(\phi)\triangleq\int\limits_{\mathbb{R}^d}|\nabla\phi|^2\,dx - \frac{\gamma}{4}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\phi(x)^2\phi(y)^2}{|x-y|^{\gamma}}\,dx\,dy = 0,$$

$$K_3(\phi)\triangleq\int\limits_{\mathbb{R}^d}|\nabla\phi|^2\,dx - \frac{\gamma}{4-\gamma}\int\limits_{\mathbb{R}^d}|\phi|^2\,dx = 0.$$

Now we establish the best constant in (3.3):

Lemma 4.3. *Assume that $2<\gamma<\min\{4,d\}$. Let Q be a radially symmetric, positive ground state of the elliptic equation* (4.1). *The best constant in the Hardy–Littlewood–Sobolev type inequality*

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|u(x)|^2|u(y)|^2}{|x-y|^{\gamma}}\,dx\,dy \le C_{\mathrm{HLS}}\|\nabla u\|_2^{\gamma}\|u\|_2^{4-\gamma}, \tag{4.2}$$

is $C_{\mathrm{HLS}} = \left(\frac{4-\gamma}{\gamma}\right)^{\gamma/2}\frac{4}{4-\gamma}\|Q\|_2^{-2}$.

Proof. We follow the argument of M. Weinstein [16]. Consider the Weinstein functional

$$J(u) = \frac{\|u\|_2^{4-\gamma}\|\nabla u\|_2^{\gamma}}{\|(|x|^{-\gamma} * |u|^2)|u|^2\|_{L^1}}, \quad \forall u \in H^1(\mathbb{R}^d). \tag{4.3}$$

We need to show that

$$\inf_{u\in H^1\setminus\{0\}} J(u) = C_{\mathrm{HLS}}^{-1}. \tag{4.4}$$

Let σ be the above infimum, and consider a minimizing sequence $\{u_n\}$. By the Hölder inequality, general Young inequality and Sobolev embedding theorem, we have

$$\|(|x|^{-\gamma} * |u|^2)|u|^2\|_{L^1} \lesssim \|u\|_{L^{4d/(2d-\gamma)}}^4 \lesssim \|u\|_2^{4-\gamma}\|\nabla u\|_2^{\gamma},$$

and so $\sigma > 0$. Set $v_n(x) = \mu_n u_n(\lambda_n x)$ with

$$\mu_n = \frac{\|u_n\|_2^{(d-2)/2}}{\|\nabla u_n\|_2^{d/2}} \quad \text{and} \quad \lambda_n = \frac{\|u_n\|_2}{\|\nabla u_n\|_2},$$

so that $\|v_n\|_2 = \|\nabla v_n\|_2 = 1$ and

$$\|(|x|^{-\gamma} * |v_n|^2)|v_n|^2\|_{L^1}^{-1} = J(v_n) = J(u_n) \to \sigma > 0 \quad \text{as } n \to +\infty.$$

Let v_n^* be the Schwarz symmetrization of v_n, i.e. the radial decreasing rearrangement (see [8]). Then $\|v_n^*\|_2 = \|v_n\|_2 = 1$, $\|\nabla v_n^*\|_2 \le \|\nabla v_n\|_2 = 1$ and

$$\begin{aligned}\|(|x|^{-\gamma} * |v_n^*|^2)|v_n^*|^2\|_{L^1}^{-1} &\le \|(|x|^{-\gamma} * |v_n|^2)|v_n|^2\|_{L^1}^{-1} \\ &= J(v_n) = J(u_n) \to \ \sigma > 0 \quad \text{as } n \to +\infty,\end{aligned}$$

by the general rearrangement inequality. We know that $\{v_n^*\}$ is bounded in H^1, so up to a subsequence, it converges to some v weakly in H^1. By radial symmetry, it also converges strongly in L^p for all $2 < p < 2^* = 2d/(d-2)$. By the Hölder and general Young inequalities, we deduce that

$$\|(|x|^{-\gamma}*|v_n^*|^2)|v_n^*|^2 - (|x|^{-\gamma}*|v|^2)|v|^2\|_{L^1} \lesssim \|v_n^* - v\|_p\|v_n^* + v\|_p(\|v_n^*\|_p^2 + \|v\|_p^2),$$

where $2 < p = 4d/(2d-\gamma) < 2^*$, and so

$$\|(|x|^{-\gamma} * |v|^2)|v|^2\|_{L^1} = \lim_{n\to+\infty} \|(|x|^{-\gamma} * |v_n^*|^2)|v_n^*|^2\|_{L^1} = \sigma^{-1} > 0.$$

By the Fatou Lemma, we have

$$\|v\|_2 \le \varliminf_{n\to+\infty} \|v_n^*\|_2 = 1 \quad \text{and} \quad \|\nabla v\|_2 \le \varliminf_{n\to+\infty} \|v_n^*\|_2 \le 1,$$

and hence

$$\begin{aligned}\sigma \le J(v) = \frac{\|v\|_2^{4-\gamma}\|\nabla v\|_2^{\gamma}}{\|(|x|^{-\gamma} * |v|^2)|v|^2\|_{L^1}} &\le \frac{1}{\|(|x|^{-\gamma} * |v|^2)|v|^2\|_{L^1}} \\ &= \lim_{n\to+\infty} \|(|x|^{-\gamma} * |v_n^*|^2)|v_n^*|^2\|_{L^1}^{-1} = \sigma.\end{aligned}$$

This implies that

$$J(v) = \sigma \quad \text{and} \quad \|v\|_2 = \|\nabla v\|_2 = 1. \tag{4.5}$$

Since v is a minimizer, it satisfies the Euler–Lagrange equation

$$\frac{d}{d\varepsilon} J(v + \varepsilon w)\bigg|_{\varepsilon=0} = 0, \qquad \forall w \in H^1(\mathbb{R}^d). \tag{4.6}$$

Taking into account (4.5), we obtain

$$-\gamma \Delta v + (4-\gamma)v = 4(|x|^{-\gamma} * |v|^2)v. \tag{4.7}$$

Let now u be defined by $v(x) = au(bx)$ with $a = \left(\frac{4-\gamma}{4}\right)^{1/2}\left(\frac{4-\gamma}{\gamma}\right)^{(d-\gamma)/4}$ and $b = \left(\frac{4-\gamma}{4}\right)^{1/2}$, so that u is a solution of (4.1) and

$$J(u) = J(v) = \sigma.$$

Since u satisfies equation (4.1), we deduce from Lemma 4.2 that

$$\|\nabla u\|_2^2 = \frac{\gamma}{4}\|(|x|^{-\gamma} * |u|^2)|u|^2\|_{L^1} \quad \text{and} \quad \|\nabla u\|_2^2 = \frac{\gamma}{4-\gamma}\|u\|_2^2,$$

and so

$$J(u) = \frac{4-\gamma}{4}\left(\frac{\gamma}{4-\gamma}\right)^{\gamma/2}\|u\|_2^2. \tag{4.8}$$

As Q also satisfies equation (4.1), it satisfies the same identity (4.8). Since u minimizes J, we have $J(Q) \geq J(u)$, which implies that $\|u\|_2 \leq \|Q\|_2$. On the other hand, Q being a ground state of (4.1), it is also a solution of (4.1) of minimal L^2-norm (see [9]), so that $\|Q\|_2 \leq \|u\|_2$. Therefore, $\|Q\|_2 = \|u\|_2$, and the result now follows from (4.8). ∎

Acknowledgements. The author thanks the editor for his useful suggestions and comments.

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Received 19 December 2011;
revised 31 January 2012 (5595)