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INTEGRAL OPERATORS GENERATED BY MERCER-LIKE KERNELS ON TOPOLOGICAL SPACES

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Abstract. We analyze some aspects of Mercer's theory when the integral operators act on $L^2(X,\sigma)$, where X is a first countable topological space and σ is a non-degenerate measure. We obtain results akin to the well-known Mercer's theorem and, under a positive definiteness assumption on the generating kernel of the operator, we also deduce series representations for the kernel, traceability of the operator and an integration formula to compute the trace. In this way, we upgrade considerably similar results found in the literature, in which X is always metrizable and compact and the measure σ is finite.

1. Introduction. It is well-known that if $(X,\mathcal{M},\sigma)$ is a measure space and $K : X\times X\to\mathbb{C}$ is an element of $L^2(X\times X,\sigma\times\sigma)$ then the formula

$$\mathcal{K}(f) := \int_X K(\cdot,y)f(y)\,d\sigma(y),\qquad f\in L^2(X,\sigma),\tag{1.1}$$

defines a continuous linear operator $\mathcal{K} : L^2(X,\sigma)\to L^2(X,\sigma)$, the norm of which is bounded as

$$\|\mathcal{K}\| \le \Big(\int_X\int_X |K(x,y)|^2\,d\sigma(x)d\sigma(y)\Big)^{1/2}.\tag{1.2}$$

If either K is a Hilbert–Schmidt kernel or $L^2(X,\sigma)$ is separable, then $\mathcal{K}$ is a typical example of a compact operator ([L, p. 247], [Y, p. 277]). If K is hermitian $(\sigma\times\sigma)$-a.e., Fubini's Theorem implies that

$$\int_X \mathcal{K}(f)(x)\overline{g(x)}\,d\sigma(x) = \int_X f(x)\overline{\mathcal{K}(g)(x)}\,d\sigma(x),\qquad f,g\in L^2(X,\sigma),\tag{1.3}$$

that is, $\mathcal{K}$ is self-adjoint. Hence, the spectral theorem for self-adjoint compact operators is applicable and $\mathcal{K}$ can be represented in the form

$$\mathcal{K}(f) = \sum_{n=1}^{\infty}\lambda_n\langle f,f_n\rangle f_n,\qquad f\in L^2(X,\sigma),\tag{1.4}$$

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where $\{\lambda_n\}$ is a sequence of real numbers decreasing to 0 (or finite) and $\{f_n\}$ is an orthonormal sequence in $L^2(X,\sigma)$. The symbol $\langle\cdot,\cdot\rangle$ stands for the usual inner product of $L^2(X,\sigma)$.

In general, the positive definiteness of K is a convenient assumption added to the setting in order to guarantee that K is hermitian $(\sigma\times\sigma)$-a.e. That forces the sequence $\{\lambda_n\}$ above to consist of non-negative numbers. A brief discussion of two notions of positive definiteness can be found in [FM]. If X has a metric structure, a basic question becomes to establish the right setting in order that $\mathcal{K}$ be a trace-class (nuclear) operator, that is,

$$\sum_{f\in\mathfrak{B}}\langle\mathcal{K}^*\mathcal{K}(f),f\rangle^{1/2}<\infty \tag{1.5}$$

for every orthonormal basis $\mathfrak{B}$ of the space $(L^2(X,\sigma),\langle\cdot,\cdot\rangle)$. Here, $\mathcal{K}^*$ is the adjoint of $\mathcal{K}$. Depending on X, the continuity of K may be needed in order to obtain additional desirable spectral properties for $\mathcal{K}$. For instance, a natural question that arises when $\mathcal{K}$ is trace-class is whether the sum of all eigenvalues of $\mathcal{K}$ can be computed via an integration formula.

The results and questions listed above and results on series representations for K and $\mathcal{K}$ belong to the scope of what we call Mercer's theory. It encompasses Mercer's Theorem and its extensions, generalizations and direct consequences.

Mercer's theory in the case where X is a compact interval and σ is the Lebesgue measure is well-established in the literature. For compact intervals we refer the reader to [GGK, GK, HT] and references therein. Extension to unbounded intervals is analyzed in [DG-B, KLR, St] and, more recently, in some papers authored by Buescu and Paixão (see [B] for example). Extensions to other domains can be found in [N, NR]. In [Br1], Brislawn discusses the case in which X is a subset of $\mathbb{R}^n$, presenting equivalences for the trace-class property in terms of the Hardy–Littlewood maximal function of the kernel. In [Br2], he deals with the nuclearity of operators defined in countably generated measure spaces replacing the Hardy–Littlewood theory with martingales. A quite general Mercer's theory is developed in [FMO, FM], while Sato's paper ([S]) characterizes traceability of the integral operators assuming that (X,σ) possesses a separable metric structure. The use of reproducing kernel Hilbert spaces in order to guarantee the validity of Mercer's Theorem for non-compact domains in both metric and non-metric cases can be found in [CVT, Su].

The main goal of the present paper is to go one step further. Precisely, we intend to establish results within Mercer's theory in the case when the separable metric structure of (X,σ) is replaced with a plain topological one, as far as we know a case not considered yet. More specifically, we will assume X is a first countable space and σ is a non-degenerate, Borel and locally finite measure. Further, we shall deduce some more refined results for X locally compact.

A few comments are in order. The assumptions on X and σ we will adopt do not coincide with those considered elsewhere (see references mentioned before) while the conclusions in several of our results are exactly the same as found in other sources. So, the reader should be alert to the proofs where the steps allowing one to go from metric to topological assumptions are explained in detail. Nonetheless, some arguments in our proofs resemble known arguments already used in the literature.

The setting we adopt was intended to include the important case in which X is the unit sphere S^m in $\mathbb{R}^{m+1}$ and σ is the usual surface measure on S^m. For instance, in [FMO] where a similar development is done, the spherical setting is not covered due to the fact that the surface measure does not satisfy the basic assumptions adopted there. The interested reader may proceed with a formal comparison.

An outline of the paper is as follows. Section 2 offers results along the lines of the so-called Mercer's Theorem: under some basic assumptions on the integral operator, we derive a series representation of the generating kernel and properties attached to them. In Section 3, we introduce a concept of smoothness of the generating kernel. We then discuss a connection between smoothness and $L^2(X,\sigma)$-positive definiteness. In Section 4, we apply the results of the previous sections to traceability of operators and some other properties.

2. Series representation for the generating kernel. We begin with some basic terminology. If X is a nonempty set, we write Δ_X for the *diagonal* of $X \times X$, that is,

$$\Delta_X := \{(x,x) : x \in X\}. \tag{2.1}$$

The dual (X,σ) will denote a (Borel) measure space. A measure σ on a topological space $(X,\mathfrak{F})$ will be termed *non-degenerate* when all open subsets of X are σ-measurable and the following condition holds:

$$\sigma(A) > 0, \qquad A \in \mathfrak{F} \setminus \{\emptyset\}. \tag{2.2}$$

We will deal with positive definiteness in the L^2-sense. Precisely, we will say that a kernel $K : X \times X \to \mathbb{C}$ is $L^2(X,\sigma)$-*positive definite* when $K \in L^2(X \times X, \sigma \times \sigma)$ and the corresponding integral operator $\mathcal{K}$, as defined in (1.1), is positive:

$$\langle \mathcal{K}(f), f \rangle \geq 0, \qquad f \in L^2(X,\sigma). \tag{2.3}$$

Standard arguments show that an integral operator $\mathcal{K}$ generated by an $L^2(X,\sigma)$-positive definite kernel K is self-adjoint (when the space $L^2(X,\sigma)$ is real, the self-adjointness of $\mathcal{K}$ needs to be added).

The following lemma is known ([FMO]).

LEMMA 2.1. *If $K \in L^2(X \times X, \sigma \times \sigma)$ is hermitian and $f \in L^2(X,\sigma)$ then*

$$\int_X \int_X K(x,y) f(x)\overline{f(y)}\, d\sigma(x) d\sigma(y) = \int_X \int_X \operatorname{Re}\big(K(x,y) f(x)\overline{f(y)}\big)\, d\sigma(x)\, d\sigma(y).$$

Below, χ_A denotes the characteristic function of a subset A of X.

LEMMA 2.2. *Let X be a first countable topological space endowed with a non-degenerate measure σ. Assume that K is $L^2(X,\sigma)$-positive definite and continuous at any point of Δ_X. Then K is non-negative on Δ_X.*

Proof. As K is $L^2(X,\sigma)$-positive definite, it is hermitian $(\sigma \times \sigma)$-a.e., and consequently $K(x,x) \in \mathbb{R}$ a.e. Since K is also continuous on Δ_X, in fact, $K(x,x) \in \mathbb{R}$ for all $x \in X$. To show that K is nonnegative on Δ_X, fix $x_0 \in X$. Given $\epsilon > 0$, we can use the first countability axiom to select a collection $\{V_1, V_2, \ldots\}$ of neighborhoods of x_0 and a positive integer n_0 so that $|K(x,y) - K(x_0,x_0)| < \epsilon$ whenever $x \in V_n$, $y \in V_m$ and $m, n \geq n_0$. It follows that $\operatorname{Re} K(x,y) < \epsilon + K(x_0,x_0)$. If $K(x_0,x_0) < 0$, we could select $\epsilon \in (0, -K(x_0,x_0))$ and conclude that

$$\operatorname{Re} K(x,y) < 0, \qquad x, y \in V_n, \tag{2.4}$$

for n arbitrarily large. Recalling the non-degeneracy of σ, an application of Lemma 2.1 would lead to

$$\langle \mathcal{K}(\chi_{V_n}), \chi_{V_n} \rangle = \int_{V_n} \int_{V_n} \operatorname{Re} K(x,y)\, d\sigma(x)\, d\sigma(y) < 0, \tag{2.5}$$

for n arbitrarily large, a contradiction to the $L^2(X,\sigma)$-positive definiteness of K. ∎

Theorem 2.3 (proved in [FMO]) describes the basic properties of operators defined by summable expansions with non-negative coefficients.

THEOREM 2.3. *Let (X,σ) be a measure space and $\{f_n\}$ an orthonormal sequence in $L^2(X,\sigma)$. If $\{a_n\}$ is a bounded sequence of non-negative real numbers then the formula*

$$T(f) = \sum_{n=1}^{\infty} a_n \langle f, f_n \rangle f_n, \qquad f \in L^2(X,\sigma), \tag{2.6}$$

defines a bounded linear operator on $L^2(X,\sigma)$ with the following properties:

(i) *If $a_n > 0$ then f_n is an eigenvector of T with eigenvalue a_n.*
(ii) *$\langle T(f), f \rangle \geq 0$ for all $f \in L^2(X,\sigma)$.*

(iii) *If $T = \mathcal{K}$ for some kernel $K \in L^2(X\times X, \sigma\times\sigma)$, then K is $L^2(X,\sigma)$-positive definite.*

If we refine the assumptions on X and σ, then with a little extra effort, we obtain the following improvement of Theorem 2.3 in the case when $\mathcal{K}$ is an integral operator.

THEOREM 2.4. *Let X be a first countable topological space endowed with a non-degenerate measure σ. Let $\{f_n\}$ be an orthonormal sequence of continuous functions from $L^2(X,\sigma)$, $\{a_n\}$ a bounded sequence of non-negative real numbers and consider the operator T defined in* (2.6). *Assume that $T = \mathcal{K}$ for some $K \in L^2(X \times X, \sigma \times \sigma)$ which is continuous on Δ_X. Then*

$$\sum_{n=1}^{\infty} a_n|f_n(x)|^2 \le K(x,x), \qquad x \in X. \tag{2.7}$$

Proof. Let p be a fixed positive integer. Due to our assumption on $\{a_n\}$ and Bessel's inequality we have, for $f \in L^2(X,\sigma)$,

$$\int_X \sum_{n=p+1}^{\infty} a_n\langle f, f_n\rangle f_n(x)\overline{f(x)}\,d\sigma(x) = \sum_{n=p+1}^{\infty} a_n\langle f, f_n\rangle \int_X f_n(x)\overline{f(x)}\,d\sigma(x),$$

that is,

$$\int_X \sum_{n=p+1}^{\infty} a_n\langle f, f_n\rangle f_n(x)\overline{f(x)}\,d\sigma(x) = \sum_{n=p+1}^{\infty} a_n|\langle f, f_n\rangle|^2. \tag{2.8}$$

Clearly, the kernel K_p given by

$$K_p(x,y) := K(x,y) - \sum_{n=1}^{p} a_n f_n(x)\overline{f_n(y)}, \qquad x, y \in X, \tag{2.9}$$

is an element of $L^2(X \times X, \sigma \times \sigma)$, and is continuous on Δ_X since K is. Using (2.8), we can see that

$$\begin{aligned}\langle \mathcal{K}_p(f), f\rangle &= \int_X \Big[\mathcal{K}(f)(x) - \sum_{n=1}^{p} a_n\langle f, f_n\rangle f_n(x)\Big]\overline{f(x)}\,d\sigma(x)\\ &= \sum_{n=p+1}^{\infty} a_n|\langle f, f_n\rangle|^2.\end{aligned}$$

In particular, K_p is $L^2(X,\sigma)$-positive definite. Thus, Lemma 2.2 yields $K_p(x,x) \ge 0$, $x \in X$. Since p was arbitrary, the inequality in the statement follows. ■

We now take up the topic of paramount interest in Mercer's theory, namely, a series representation for K. Theorem 2.5 below is the first step in this direction.

THEOREM 2.5. *Under the assumptions Theorem* 2.4, *the series* $\sum_{n=1}^{\infty} a_n f_n(x)\overline{f_n(y)}$ *is absolutely and uniformly convergent on compact subsets of* X *with respect to each variable, when the other is fixed.*

Proof. Let $x \in X$ be fixed and Y a compact subset of X. The Cauchy–Schwarz inequality implies that

$$\Big|\sum_{n=p}^{q} a_n f_n(x)\overline{f_n(y)}\Big|^2 \leq \sup_{\zeta\in Y} K(\zeta,\zeta)\sum_{n=p}^{q} a_n|f_n(x)|^2, \qquad y \in Y, \tag{2.10}$$

whenever $q \geq p \geq 1$. An application of the Cauchy criterion for uniform convergence, along with inequality (2.7), implies that $\sum_{n=1}^{\infty} a_n f_n(x)\overline{f_n(y)}$ is absolutely and uniformly convergent in Y. Similarly, if y is fixed, the same series is absolutely and uniformly convergent for $x \in Y$. ∎

If X has finite measure, the integral operator $\mathcal{K}$ generated by a bounded and continuous kernel K has the additional property that its range contains continuous functions only. In our context the corresponding result is as follows.

THEOREM 2.6. *Under the assumptions of Theorem* 2.4, *the range of* T *contains continuous functions only.*

Proof. Fix $f \in L^2(X,\sigma)$ and $x \in X$. Consider a sequence $\{x_n\} \subset X$ converging to x and write $Y = \{x\} \cup \{x_n : n = 1, 2, \ldots\}$. The Cauchy–Schwarz inequality and Theorem 2.4 lead to

$$\begin{aligned}\Big|\sum_{n=p}^{q} a_n\langle f, f_n\rangle f_n(y)\Big|^2 &\leq \sum_{n=p}^{q} a_n|\langle f, f_n\rangle|^2 \sum_{n=p}^{q} a_n|f_n(y)|^2\\ &\leq K(y,y)\sum_{n=p}^{q} a_n|\langle f, f_n\rangle|^2, \qquad y \in Y.\end{aligned}$$

whenever $q \geq p \geq 1$. As K is continuous on Δ_X, the compactness of Y implies that

$$\begin{aligned}\Big|\sum_{n=p}^{q} a_n\langle f, f_n\rangle f_n(y)\Big|^2 &\leq \sup_{\zeta\in Y} K(\zeta,\zeta)\sum_{n=p}^{q} a_n|\langle f, f_n\rangle|^2\\ &\leq \|T\| \sup_{\zeta\in Y} K(\zeta,\zeta)\sum_{n=p}^{q} |\langle f, f_n\rangle|^2, \qquad y \in X,\ q \geq p \geq 1.\end{aligned}$$

The series $\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2$ is convergent by Bessel's inequality. Hence, the Cauchy criterion for uniform convergence shows that $\sum_{n=1}^{\infty} a_n\langle f, f_n\rangle f_n(y)$ is uniformly convergent for all $y \in Y$. Hence, it defines a continuous function on Y, and in particular, $T(f)(x_n)$ converges to $T(f)(x)$. Since X is first countable, this implies continuity at x. ∎

Another related result is the $L^2(X,\sigma)$-convergence of the series in (2.6). To achieve this, we assume local compactness of X and integrability of $x \in X \mapsto K(x,x)$.

THEOREM 2.7. *Under the assumptions of Theorem* 2.4, *suppose moreover that X is locally compact and $x \in X \mapsto K(x,x)$ is integrable. Then $\sum_{n=1}^{\infty} a_n |f_n(x)|^2$ is $L^2(X,\sigma)$-convergent to $K(x,x)$.*

Proof. Recalling (2.7), we see that

$$\sum_{n=1}^{\infty} |a_n f_n(x)|^2 \le \|T\| \sum_{n=1}^{\infty} a_n |f_n(x)|^2 \le \|T\| K(x,x), \qquad x \in X. \tag{2.11}$$

By the Riesz–Fisher Theorem ([R, p. 330]), for every $x \in X$, the sum $\sum_{n=1}^{\infty} a_n f_n(x)\overline{f_n}$ is $L^2(X,\sigma)$-convergent to a function $K_x \in L^2(X,\sigma)$. As a consequence,

$$\begin{aligned}\int_X K_x(y) f(y)\, d\sigma(y) &= \sum_{n=1}^{\infty} a_n \langle f, f_n\rangle f_n(x) = T(f)(x)\\ &= \mathcal{K}(f)(x) = \int_X K(x,y) f(y)\, d\sigma(y), \qquad x \in X,\end{aligned}$$

whenever $f \in L^2(X,\sigma)$. In particular, $K_x = K(x,\cdot)$ a.e. By Theorem 2.5, for every $y \in X$ there is a compact subset U_y of X in which the convergence of $\sum_{n=1}^{\infty} a_n f_n(x)\overline{f_n}$ to K_x is uniform. Hence, K_x is continuous in U_y, $y \in X$. In particular, $K_x = K(x,\cdot)$ everywhere. It is now clear that

$$K(x,x) = K_x(x) = \sum_{n=1}^{\infty} a_n |f_n(x)|^2, \qquad \text{a.e. } x \in X. \tag{2.12}$$

Since

$$\sum_{n=1}^{N} a_n |f_n(x)|^2 \le K(x,x), \qquad \text{a.e. } x \in X, \tag{2.13}$$

the continuity of the functions involved implies that the inequality holds in fact everywhere. As $x \mapsto K(x,x)$ is integrable, we thus get a uniform bound in $L^1(X,\sigma)$ for the sequence of partial sums appearing above. That suffices for the convergence of the series to $K(x,x)$ in $L^2(X,\sigma)$. ∎

The last result of the section deals with the convergence of the series $\sum_{n=1}^{\infty} a_n f_n(x)\overline{f_n(y)}$. The integrability of $K(x,x)$ is no longer needed.

THEOREM 2.8. *Under the assumptions of Theorem* 2.4, *suppose moreover that X is locally compact. Then the series $\sum_{n=1}^{\infty} a_n f_n(x)\overline{f_n(y)}$ converges absolutely and uniformly on compact subsets of $X \times X$.*

Proof. Formula (2.13) and Dini's theorem for compact topological spaces imply that the convergence of $\sum_{n=1}^{\infty} a_n |f_n(x)|^2$ to $K(x,x)$ is uniform on compact subsets of X. Since the Cauchy–Schwarz inequality implies that

$$(2.14)\qquad \Big|\sum_{n=p}^{q} a_n f_n(x)\overline{f_n(y)}\Big|^2 \le \sum_{n=p}^{q} a_n|f_n(x)|^2 \sum_{n=p}^{q} a_n|f_n(y)|^2, \qquad x,y\in X,$$

whenever $q \ge p \ge 1$, an automatic consequence is the uniform and absolute convergence of $\sum_{n=1}^{\infty} a_n f_n(x)\overline{f_n(y)}$ on compact subsets of $X \times X$. ∎

3. Smoothness of the generating kernel. In this section we present kernels that fit into the results described in Section 2. They are smooth in the sense implied by Lemma 3.1 below. The final result of the section describes a context under which the kernel is automatically smooth.

As previously observed, if (X,σ) is a measure space and K is $L^2(X,\sigma)$-positive definite then the corresponding integral operator $\mathcal{K}$ is compact and self-adjoint and the spectral theorem for such operators can be applied. In particular, there exist an orthonormal sequence $\{f_n\}$ in $L^2(X,\sigma)$ and a non-increasing sequence $\{a_n\} \subset [0,\infty)$ such that if $f \in L^2(X,\sigma)$ then the series $\sum_{n=1}^{\infty} a_n\langle f, f_n\rangle f_n$ is $L^2(X,\sigma)$-convergent to $\mathcal{K}(f)$. If $L^2(X,\sigma)$ is separable, then the sequence can be chosen to be a complete set. Lemma 3.1 below complements the information provided by that theorem when a continuity assumption is added.

LEMMA 3.1. *Let X be a topological space endowed with a measure σ and K an $L^2(X,\sigma)$-positive definite kernel. If $K(x,\cdot)$ belongs to $L^2(X,\sigma)$ for every $x \in X$ and the function $x \in X \mapsto K(x,\cdot) \in L^2(X,\sigma)$ is continuous then the functions f_n above are continuous when $a_n > 0$.*

Proof. Since $\mathcal{K}(f_n) = a_n f_n$, $n = 1,2,\ldots$, it suffices to show that the range of $\mathcal{K}$ contains continuous functions only. But that follows from the inequality

$$|\mathcal{K}(f)(x) - \mathcal{K}(f)(y)| \le \|K(x,\cdot) - K(y,\cdot)\|_2\|f\|_2, \qquad f \in L^2(X,\sigma),\ x,y \in X,$$

and the assumptions of the lemma. ∎

The definition of smoothness we intend to explore can now be introduced. A kernel $K : X\times X \to \mathbb{C}$ is *smooth* when the following three conditions hold:

(i) K is continuous on Δ_X.
(ii) For every $x \in X$, the function $K(x,\cdot)$ belongs to $L^2(X,\sigma)$.
(iii) The function $x \in X \mapsto K(x,\cdot) \in L^2(X,\sigma)$ is continuous.

The reader should compare the definition above with that in [FMO] where (i) is replaced with continuity in $X \times X$. Theorems 2.5 and 2.8 can be restated as follows, when smoothness is added to the setting.

THEOREM 3.2. *Let X be a first countable topological space endowed with a non-degenerate measure σ, and K a smooth $L^2(X,\sigma)$-positive definite kernel. Then the conclusions in Lemma* 3.1 *hold and, in addition, the series $\sum_{n=1}^{\infty} a_n f_n(x)\overline{f_n(y)}$ converges absolutely and uniformly on compact subsets of X with respect to one variable, when the other is fixed.*

THEOREM 3.3. *Let X be a first countable locally compact topological space endowed with a non-degenerate measure σ and K a smooth $L^2(X,\sigma)$-positive definite kernel. Then the conclusions in Lemma* 3.1 *hold and, in addition, the series $\sum_{n=1}^{\infty} a_n f_n(x)\overline{f_n(y)}$ converges absolutely and uniformly on compact subsets of $X \times X$.*

Next, we intend to go from positive definiteness to smoothness, still keeping the non-metric setting we have adopted. One set of assumptions that allows one to prove such an implication is described in Theorem 3.4 below.

THEOREM 3.4. *Let X be a first countable topological space endowed with a non-degenerate and locally finite measure σ, and let K be a continuous $L^2(X,\sigma)$-positive definite kernel. Then*

$$|K(x,y)|^2 \leq K(x,x)K(y,y), \qquad x,y \in X. \tag{3.1}$$

Proof. The assertion holds trivially when $K(x,y)=0$. So, let $x,y \in X$ with $K(x,y) \neq 0$ and fix $\epsilon \in (0,|K(x,y)|)$. We can select open sets U and V in X so that $0<\sigma(U)<\infty$, $0<\sigma(V)<\infty$,

$$\begin{aligned} |K(x',x'')-K(x,x)| &< \epsilon, \quad x',x'' \in U,\\ |K(x',y')-K(x,y)| &< \epsilon, \quad x' \in U,\ y' \in V,\\ |K(y',y'')-K(y,y)| &< \epsilon, \quad y',y'' \in V. \end{aligned}$$

Since $\operatorname{Re} K$ is $L^2(X,\sigma)$-positive definite, we have

$$I := \int_{X\times X} \operatorname{Re} K(x',y') f_\lambda(x')\overline{f_\lambda(y')}\, d\sigma(x')\, d\sigma(y') \geq 0, \tag{3.2}$$

where

$$f_\lambda := \frac{1}{\sigma(U)}\chi_U + \frac{\lambda}{\sigma(V)}\chi_V, \qquad \lambda \in \mathbb{C}.$$

We split I as $I = I_1+I_2+I_3+I_4$ in accordance with the four summands in

$$\begin{aligned} f_\lambda(x')\overline{f_\lambda(y')} = {}& \frac{1}{\sigma(U)^2}\chi_U(x')\chi_U(y') + \frac{|\lambda|^2}{\sigma(V)^2}\chi_V(x')\chi_V(y')\\ &+ \frac{\overline{\lambda}}{\sigma(U)\sigma(V)}\chi_U(x')\chi_V(y') + \frac{\lambda}{\sigma(V)\sigma(U)}\chi_V(x')\chi_U(y'). \end{aligned}$$

The first two can be estimated as follows:

$$I_1 \leq \int_{X\times X} |K(x',y')| \frac{1}{\sigma(U)^2}\chi_U(x')\chi_U(y')\, d\sigma(x')\, d\sigma(y') \leq \epsilon + K(x,x),$$

$$I_2 \leq \int_{X\times X} |K(x',y')| \frac{\lambda\overline{\lambda}}{\sigma(V)^2}\chi_V(x')\chi_V(y')\, d\sigma(x')\, d\sigma(y') \leq |\lambda|^2(\epsilon + K(y,y)).$$

On the other hand,

$$\begin{aligned} I_3 + I_4 &= (\lambda+\overline{\lambda})\operatorname{Re} K(x,y) \\ &\quad + (\lambda+\overline{\lambda}) \int_{X\times X} \operatorname{Re}(K(x',y') - K(x,y)) \frac{\chi_U(x')\chi_V(y')}{\sigma(U)\sigma(V)}\, d\sigma(x')\, d\sigma(y'). \end{aligned}$$

Hence, we conclude that

$$I_3 + I_4 \leq 2|\lambda|\epsilon + (\lambda+\overline{\lambda})\operatorname{Re} K(x,y), \qquad \lambda \in \mathbb{C}.$$

Thus, (3.2) implies that

$$\epsilon + K(x,x) + |\lambda|^2(\epsilon + K(y,y)) + 2|\lambda|\epsilon + (\lambda+\overline{\lambda})\operatorname{Re} K(x,y) \geq 0, \qquad \lambda \in \mathbb{C},$$

and consequently

$$(\epsilon + K(y,y))|\lambda|^2 - |K(x,y)|(\lambda+\overline{\lambda}) + 2\epsilon|\lambda| + \epsilon + K(x,x) \geq 0, \qquad \lambda \in \mathbb{C}.$$

Clearly, this yields

$$(\epsilon + K(y,y))\lambda^2 + 2(\epsilon - |K(x,y)|)\lambda + \epsilon + K(x,x) \geq 0, \qquad \lambda \geq 0,$$

while our choice for ϵ leads to

$$(\epsilon + K(y,y))\lambda^2 + 2(\epsilon - |K(x,y)|)\lambda + \epsilon + K(x,x) \geq 0, \qquad \lambda \in \mathbb{R}.$$

In particular,

$$4(\epsilon + |K(x,y)|)^2 - 4(\epsilon + K(x,x))(\epsilon + K(y,y)) \leq 0,$$

and that reduces to the inequality in the statement when we let $\epsilon \to 0^+$. ∎

A bonus from Theorem 3.4 is a setting in which smoothness follows from $L^2(X,\sigma)$-positive definiteness.

THEOREM 3.5. *Let X be a first countable topological space endowed with a non-degenerate and locally finite measure σ. Assume that K is a continuous $L^2(X,\sigma)$-positive definite kernel and $x \in X \mapsto K(x,x)$ is integrable. Then K is smooth.*

Proof. The previous theorem implies that

$$\int_X |K(x,y)|^2\, d\sigma(y) < \infty, \qquad x \in X. \tag{3.4}$$

This is condition (ii) in the definition of smoothness. To show the continuity of $x \in X \mapsto K(x,\cdot) \in L^2(X,\sigma)$ at $x_0 \in X$, let $\{x_n\}$ be a sequence in X

converging to x_0. Since K is continuous, the sequence $\{K(x_n, y)\}$ converges to $K(x_0, y)$, for every $y \in X$ fixed. From Theorem 3.4, we deduce that

$$\begin{aligned}|K(x_n, y) - K(x_0, y)|^2 &\leq 2\,|K(x_n, y)|^2 + 2\,|K(x_0, y)|^2 \\ &\leq 2\,K(y, y)^2\,[K(x_n, x_n)^2 + K(x_0, x_0)^2], \qquad y \in X.\end{aligned}$$

It is now clear that

$$|K(x_n, y) - K(x_0, y)| \leq 2K(y, y) \sup_m K(x_m, x_m), \qquad y \in X. \tag{3.5}$$

Since $|K(x_n, \cdot) - K(x, \cdot)|^2 \in L^1(X, \sigma)$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} |K(x_n, y) - K(x, y)|^2 = 0$, the Dominated Convergence Theorem implies the continuity in condition (iii). ∎

The following consequence deserves no explanation.

Corollary 3.6. *Let X be a first countable topological space endowed with a non-degenerate and finite measure σ. If K is a continuous $L^2(X, \sigma)$-positive definite kernel then K is smooth.*

4. Traceability and other results. Before embarking on the results of the section, we recall some elementary facts about trace-class operators and introduce notation.

Let $\mathcal{B}(\mathcal{H})$ denote the space of all bounded linear operators on a separable Hilbert space $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$ is compact, the operator $|T| := (T^*T)^{1/2}$ is positive and compact. Therefore, denoting by $\{s_n(T)\}$ the sequence of eigenvalues of $|T|$ (the singular numbers of T) arranged in non-increasing order and taking into account multiplicities, the nuclearity of T reduces to the single condition $\sum_{n=1}^{\infty} s_n(T) < \infty$. We refer the reader to [C, GK] for more details and examples regarding this concept. We will write $\mathcal{B}_1(\mathcal{H})$ for the subspace of $\mathcal{B}(\mathcal{H})$ formed by the trace-class elements of $\mathcal{B}(\mathcal{H})$. The formula

$$\|T\|_1 := \sum_{n=1}^{\infty} s_n(T), \qquad T \in \mathcal{B}_1(\mathcal{H}), \tag{4.1}$$

defines a norm in $\mathcal{B}_1(\mathcal{H})$. If $\{f_n\}$ is an orthonormal basis of $\mathcal{H}$, the formula

$$\operatorname{tr}(T) := \sum_{n=1}^{\infty} \langle T(f_n), f_n\rangle_{\mathcal{H}}, \quad T \in \mathcal{B}_1(\mathcal{H}), \tag{4.2}$$

defines a linear functional $T \in \mathcal{B}_1(\mathcal{H}) \to \operatorname{tr}(T)$ of norm 1, the *trace* of T. The trace generalizes the concept of trace for matrices in the following sense: if $T \in \mathcal{B}_1(\mathcal{H})$ is normal and $\{a_n\}$ is the sequence of eigenvalues of T, repeated according to their multiplicities, then $\operatorname{tr}(T) = \sum_{n=1}^{\infty} a_n$.

If we assume that X is a Hausdorff and first countable space and σ is a non-degenerate, Borel and locally finite measure on X, there is no guarantee that $L^2(X,\sigma)$ will be separable. Since we will deal with the trace-class concept, we will assume from now on that additional assumptions have been made on $\mathcal{M}$ and σ in order to make $L^2(X,\sigma)$ separable. For example, to achieve this one can assume that $\mathcal{M}$ is countably generated (up to σ-null sets) and X is σ-finite ([D, p. 92]).

THEOREM 4.1. *Let X be a first countable and locally compact topological space endowed with a non-degenerate locally finite measure σ. Let K be a continuous $L^2(X,\sigma)$-positive definite kernel with $x \in X \mapsto K(x,x)$ integrable. Then $\mathcal{K}$ is trace-class and*

$$\operatorname{tr}(\mathcal{K}) = \int_X K(x,x)\,d\sigma(x). \tag{4.3}$$

Proof. Theorem 3.5 implies that K is smooth. The paragraph preceding Lemma 3.1 and the lemma itself imply the existence of an orthonormal complete sequence $\{f_n\} \subset L^2(X,\sigma)$ and a non-increasing sequence $\{a_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} a_n\langle f, f_n\rangle f_n$ is $L^2(X,\sigma)$-convergent to $\mathcal{K}(f)$ whenever $f \in L^2(X,\sigma)$. $\mathcal{K}$ being self-adjoint, after arranging the singular values of $\mathcal{K}$ in decreasing order and counting multiplicities, say, $s_1(\mathcal{K}) \ge s_2(\mathcal{K}) \ge \cdots$, we can deduce that $a_n = s_n(\mathcal{K})$ ([Yo, p. 204]). Theorem 2.7 yields

$$\sum_{n=1}^{\infty} s_n(\mathcal{K}) = \sum_{n=1}^{\infty} a_n\|f_n\|_2^2 = \int_X K(x,x)\,d\sigma(x),$$

which ends the proof. ∎

THEOREM 4.2. *Under the assumptions of Theorem* 4.1,

$$\int_X\int_X |K(x,y)|^2\,d\sigma(x)\,d\sigma(y) = \sum_{n=1}^{\infty} a_n^2.$$

Proof. Theorem 3.5 implies that K is smooth. In particular, $K(x,\cdot) \in L^2(X,\sigma)$, $x \in X$. Applying Parseval's identity, we deduce that

$$\int_X |K(x,y)|^2\,d\sigma(y) = \sum_{n=1}^{\infty} |\mathcal{K}(f_n)(x)|^2 = \sum_{n=1}^{\infty} |a_n f_n(x)|^2, \qquad x \in X.$$

Integration and an application of the Monotone Convergence Theorem leads to

$$\int_X\int_X |K(x,y)|^2\,d\sigma(y)\,d\sigma(x) = \sum_{n=1}^{\infty} a_n^2 \int_X |f_n(x)|^2\,d\sigma(x) = \sum_{n=1}^{\infty} a_n^2.$$

This completes the proof. ∎

COROLLARY 4.3. *Under the assumptions of Theorem* 4.1,

$$\sum_{n=1}^{\infty} a_n f_n(x)\overline{f_n(y)} = K(x,y)$$

in $L^2(X\times X,\sigma\times\sigma)$.

Proof. Direct computation yields

$$\int_X \int_X K(x,y)\overline{f_n(x)}f_n(y)\,d\sigma(y)\,d\sigma(x) = a_n, \qquad n = 1,2,\ldots. \tag{4.4}$$

Another calculation plus the use of (4.4) implies that

$$\int_X \int_X |K(x,y) - \sum_{n=1}^{m} a_n f_n(x)\overline{f_n(y)}|^2\,d\sigma(x)\,d\sigma(y)$$
$$= \int_X \int_X |K(x,y)|^2\,d\sigma(x)\,d\sigma(y) - \sum_{n=1}^{m} a_n^2$$

whenever $m \geq 1$. Theorem 4.2 completes the proof. ■

The reader can find versions of the previous two theorems in the case $X = [0,1]$ endowed with the usual Lebesgue measure in Chapter 4 of [PS]. The establishment of versions of the previous theorems in the case in which the measure σ is finite can be obtained in a similar manner. The details are left to the reader.

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