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AN L^1-STABILITY AND UNIQUENESS RESULT FOR BALANCE LAWS WITH MULTIFUNCTIONS: A MODEL FROM THE THEORY OF GRANULAR MEDIA

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Abstract. We study the uniqueness and L^1-stability of the Cauchy problem for a 2×2 system coming from the theory of granular media [9, 10]. We work in a class of weak entropy solutions. The appearance of a multifunction in a source term, given by the Coulomb–Mohr friction law, requires a modification of definition of the weak entropy solution [5, 6].

1. Introduction. We consider a system describing the motion of an avalanche down a slope, which will be described by the following values:

- the height $h : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ of the avalanche,
- the density $\varrho : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ of the avalanche,
- the velocity $v : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ of the avalanche.

The system consists of a differential inclusion

$$\begin{gathered}\frac{\partial}{\partial t}(\varrho h) + \frac{\partial}{\partial x}(\varrho h v) = 0, \\ \frac{\partial}{\partial t}(\varrho h v) + \frac{\partial}{\partial x}\left(\varrho h v^2 + \frac{1}{2}\beta \varrho h^2\right) \in \varrho h \widetilde{g},\end{gathered} \tag{1.1}$$

where $\beta := \beta(x)$ is a given function and $\widetilde{g} := \widetilde{g}(x, v)$ is a given multifunction. The equation in (1.1) describes the conservation of mass, whereas the differential inclusion describes the balance of linear momentum. For simplicity the dependence of $\widetilde{g}$ and β on x will be ignored. The constant β and the multifunction $\widetilde{g}(v)$ are defined by

$$\beta = k\cos(\gamma),$$
$$\widetilde{g}(v) = \begin{cases} \sin(\gamma) + [-\cos(\gamma), +\cos(\gamma)] & \text{for } v = 0, \\ \sin(\gamma) - \frac{v}{|v|}\cos(\gamma) & \text{for } v \neq 0, \end{cases}$$

where $-\pi/2 < \gamma < \pi/2$ is an angle between the gravitational force and a constant slope ground, and k is a positive constant. The evolution of three

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variables (ϱ, h, v) cannot be determined uniquely by these two balance laws, therefore some additional constitutive relation has to be added. We can assume that ϱ is a function of h and v, namely $\varrho = h^{-1/2}$. This leads to a system of two differential inclusions for two independent variables (h, v). Following the nonlinear transformations of the above system in [6] we obtain the new system

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}F(u) \in \widetilde{G}(u), \tag{1.2}$$

where

$$F(u) = \begin{pmatrix} u_1u_2 \\ u_2^2/2 + u_1^2/2 \end{pmatrix}, \qquad \widetilde{G}(u) = \begin{pmatrix} 0 \\ \widetilde{g}(u_2) \end{pmatrix}. \tag{1.3}$$

We will introduce a class of weak entropy solutions which are appropriate for the above system.

Definition 1.1. Let $\eta = \eta(u_1, u_2)$, $q = q(u_1, u_2)$ be scalar C^1-functions and $F(u_1, u_2)$ be a C^1-function satisfying

$$\nabla_{(u_1,u_2)}\eta(u_1, u_2) \cdot \nabla_{(u_1,u_2)}F(u_1, u_2) = \nabla_{(u_1,u_2)}q(u_1, u_2).$$

Then (η, q) is called an *entropy-entropy flux pair* for the system (1.2). If η is convex, then (η, q) is called a *convex entropy-entropy flux pair*.

Definition 1.2. We call $u \in \mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ a *weak entropy solution* to the system

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}F(u) = G(t, x)$$

with the initial data $u^0 \in \mathbb{L}^\infty(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ and a source term $G \in \mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R}^2)$ if:

(i) u is a *weak solution*, i.e.

$$\int_{[0,T)\times\mathbb{R}} \left[u(t,x) \cdot \frac{\partial}{\partial t}\psi(t,x) + F(u(t,x)) \cdot \frac{\partial}{\partial x}\psi(t,x) + G(t,x) \cdot \psi(t,x)\right] dt\,dx + \int_{\mathbb{R}} u^0(x) \cdot \psi(0,x)\,dx = 0$$

for all test functions $\psi \in C_{\mathrm{c}}^1([0,T) \times \mathbb{R}; \mathbb{R}^2)$.

(ii) The *entropy inequality*

$$\int_{[0,T)\times\mathbb{R}} \left[\eta(u(t,x))\,\frac{\partial}{\partial t}\phi(t,x) + q(u(t,x))\,\frac{\partial}{\partial x}\phi(t,x) + \nabla_u\eta(u(t,x)) \cdot G(t,x)\phi(t,x)\right] dt\,dx + \int_{\mathbb{R}} \eta(u^0(x))\phi(0,x)\,dx \geq 0$$

holds for all nonnegative test functions $\phi \in C_{\mathrm{c}}^1([0,T) \times \mathbb{R}; \mathbb{R})$ and all convex entropy-entropy flux pairs (η, q).

Remark. The above definition is standard in the theory of conservation laws. Nevertheless, it cannot be used for system (1.2) because of the multifunction in the source term. Thus we need the following extension.

Definition 1.3. We call $u \in \mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ a *weak entropy solution* to the system (1.2) with initial data $u^0 \in \mathbb{L}^\infty(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ if:

(i) $\exists\, G(t,x) \in \widetilde{G}(u(t,x))$ for a.a. $(t,x) \in [0,T) \times \mathbb{R}$.

(ii) u is a weak entropy solution according to Definition 1.2 to a system

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} F(u) = G(t,x).$$

The existence of solutions to the Cauchy problem for the system (1.2), (1.3) was shown in [6]. We recall the corresponding theorem below (see Theorem 1.1).

Notation. By $C^0_{1+|\cdot|}(\Omega; X)$ we denote the Banach space of continuous functions $u : \Omega \to X$ with the norm weighted by $1 + |\cdot|$, i.e. $\|u\|_{C^0_{1+|\cdot|}} = \sup_{x\in\Omega} \|(1+|x|)u(x)\|$. The notation $C^r_{\mathrm{b}}(\Omega; X)$ is used for the Banach space of r-times differentiable functions with the usual norm.

Theorem 1.1. *Assume that the initial data satisfies*

$$u^0 = (u^0_1, u^0_2) \in C^3_{\mathrm{b}}(\mathbb{R}; \mathbb{R}^2), \quad \inf_{x\in\mathbb{R}} u^0_1(x) \geq 0, \quad (u^0_1 - \overline{u}, u^0_2) \in C^0_{1+|\cdot|}(\mathbb{R}; \mathbb{R}^2)$$

for some positive constant $\overline{u}$. *Then the system* (1.2), (1.3) *with the above initial data has a weak entropy solution in the sense of Definition* 1.3 *for all positive* T, *with* $\inf_{x\in\mathbb{R}} u_1(t,x) \geq 0$ *for a.a.* $t \in [0,T)$.

For further considerations it is useful to observe that the system (1.2), (1.3) is equivalent to the following system of two independent inclusions coupled only by their right-hand sides:

$$\begin{aligned} &\frac{\partial}{\partial t}(u_1 - u_2) - \frac{1}{2}\frac{\partial}{\partial x}(u_1 - u_2)^2 \in -\widetilde{g}(u_2),\\ &\frac{\partial}{\partial t}(u_1 + u_2) + \frac{1}{2}\frac{\partial}{\partial x}(u_1 + u_2)^2 \in \widetilde{g}(u_2). \end{aligned} \tag{1.4}$$

Introducing new variables $w_1 = u_1 - u_2$, $w_2 = u_1 + u_2$, we can restate this system in the form

$$\begin{aligned} &\frac{\partial}{\partial t} w_1 - \frac{1}{2}\frac{\partial}{\partial x} w_1^2 \in -\widetilde{g}\Big(\frac{w_2 - w_1}{2}\Big),\\ &\frac{\partial}{\partial t} w_2 + \frac{1}{2}\frac{\partial}{\partial x} w_2^2 \in \widetilde{g}\Big(\frac{w_2 - w_1}{2}\Big). \end{aligned} \tag{1.5}$$

The last system can be expressed in the form (1.2) for $w = (w_1, w_2)$, namely

$$\frac{\partial}{\partial t} w + \frac{\partial}{\partial x} F(w) \in \widetilde{G}(w) \tag{1.6}$$

with

$$F(w)=\frac{1}{2}\begin{pmatrix}-w_1^2\\ w_2^2\end{pmatrix},\qquad \widetilde{G}(w)=\begin{pmatrix}-\widetilde{g}\Big(\dfrac{w_2-w_1}{2}\Big)\\ \widetilde{g}\Big(\dfrac{w_2-w_1}{2}\Big)\end{pmatrix}. \tag{1.7}$$

Remark. The transformation of variables $(u_1,u_2)\mapsto(w_1,w_2)$ is linear, hence it preserves convexity of the entropy function. Consequently, each weak entropy solution in the sense of Definition 1.3 to the system (1.2), (1.3) with initial data $u^0=(u_1^0,u_1^0)\in\mathbb{L}^\infty(\mathbb{R};\mathbb{R}^2)$ coincides with a weak entropy solution to the system (1.6), (1.7) with initial data $w^0=(u_1^0-u_2^0,u_1^0+u_2^0)\in\mathbb{L}^\infty(\mathbb{R};\mathbb{R}^2)$.

One could expect that the Cauchy problem for the system consisting of differential inclusions instead of equations should produce a large number of solutions. The problem of nonuniqueness can be observed both for ordinary differential inclusions and for stationary solutions to our problem (see [5] for details), as opposed to the system of two differential equations, where only one stationary solution has been obtained.

For a solution $u\in\mathbb{L}^\infty([0,T)\times\mathbb{R};\mathbb{R}_+\times\mathbb{R})$ in the sense of Definition 1.3 uniqueness for the Cauchy problem cannot be expected because of the possible occurrence of an initial layer to such a solution. Thus it is natural to look for a possible class of solutions (and initial data) for the Cauchy problem in which we have global-in-time existence and uniqueness together. These are the weak entropy solutions (in the sense of Definition 1.3) with the additional condition of time regularity $C^0([0,T);\mathbb{L}^1_{\mathrm{loc}}(\mathbb{R}))$. Note that this is the typical time regularity for uniqueness results for scalar conservation laws (cf. [7], [8], [4]; see also [12]).

Section 3 establishes the global-in-time existence of weak entropy solutions in $C^0([0,T);\mathbb{L}^1_{\mathrm{loc}}(\mathbb{R}))$ (for all positive T) under some additional assumption on the initial data (cf. assumption on ω in Lemma 3.3).

Section 4 contains the proof of L^1-stability (Thm. 1.2), implying the uniqueness of solutions, which is our main result.

Theorem 1.2. *Let (w_1,w_2) and $(\overline{w}_1,\overline{w}_2)$ be two weak entropy solutions in $C^0([0,T);\mathbb{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^2))\cap\mathbb{L}^\infty([0,T)\times\mathbb{R};\mathbb{R}^2)$ to the system* (1.6), (1.7) *with $(w_1-\overline{w}_1,w_2-\overline{w}_2)\in\mathbb{L}^\infty([0,T);\mathbb{L}^1(\mathbb{R};\mathbb{R}^2))$ and initial data (w_1^0,w_2^0), $(\overline{w}_1^0,\overline{w}_2^0)\in\mathbb{L}^\infty(\mathbb{R};\mathbb{R}^2)$. Then for any $0<t<T$,*

$$\|w_1(t)-\overline{w}_1(t)\|_{L^1(\mathbb{R})}+\|w_2(t)-\overline{w}_2(t)\|_{L^1(\mathbb{R})}\le\|w_1^0-\overline{w}_1^0\|_{L^1(\mathbb{R})}+\|w_2^0-\overline{w}_2^0\|_{L^1(\mathbb{R})}.$$

Remark. Similar results for strongly coupled 2×2 systems are not straightforward. Even for a homogeneous system we need an additional assumption on BV norm ($\|u^0\|_{BV(\mathbb{R};\mathbb{R}^2)}\ll 1$) to show the uniqueness (cf. [1]–[3]). This yields a global-in-time estimate on the BV norm of solutions.

However in the case of nonhomogeneous equations some "dissipative properties" of the right-hand side in the sense of a proper BV norm are meaningful.

2. Technical lemma. For technical reasons it will be convenient to formulate the following lemma:

LEMMA 2.1. *Let* $\widetilde{g}:\mathbb{R}\to 2^{\mathbb{R}}$ *be a monotone multifunction, where* $\widetilde{g}(b)\subset[-1,1]$ *is multivalued only if* $b=0$, *and*

$$I(a,b,\overline{a},\overline{b})=[g-\overline{g}][\eta_\delta'(a+b-(\overline{a}+\overline{b}))-\eta_\delta'(a-b-(\overline{a}-\overline{b})],$$

where $g\in\widetilde{g}(b)$, $\overline{g}\in\widetilde{g}(\overline{b})$ *and* $\eta_\delta:\mathbb{R}\to\mathbb{R}$ *for* $\delta>0$ *is a function defined by*

$$\eta_\delta(y)=\begin{cases}0, & y\in(-\infty,0],\\ y^2/4\delta, & y\in(0,2\delta],\\ y-\delta, & y\in(2\delta,\infty).\end{cases}\tag{2.1}$$

Then $I\le 0$ *for every* $a,\overline{a},b,\overline{b}\in\mathbb{R}$.

REMARK. By *monotone multifunction* we mean that $\widetilde{g}$ has the following property:

$$\forall b,\overline{b}\in\mathbb{R}\ \forall g\in\widetilde{g}(b),\ \overline{g}\in\widetilde{g}(\overline{b})\qquad b<\overline{b}\ \Rightarrow\ g\ge\overline{g}.$$

Proof. Observe that η_δ' is nondecreasing and

$$\eta_\delta'(a+b-(\overline{a}+\overline{b}))-\eta_\delta'(a-b-(\overline{a}-\overline{b}))=\eta_\delta'((a-\overline{a})+(b-\overline{b}))-\eta_\delta'((a-\overline{a})-(b-\overline{b})).$$

Consider three cases:

1. If $b<\overline{b}$ then $\eta_\delta'(a+b-(\overline{a}+\overline{b}))-\eta_\delta'(a-b-(\overline{a}-\overline{b}))\le 0$.
2. If $b=\overline{b}$ then $\eta_\delta'(a+b-(\overline{a}+\overline{b}))-\eta_\delta'(a-b-(\overline{a}-\overline{b}))=0$.
3. If $b>\overline{b}$ then $\eta_\delta'(a+b-(\overline{a}+\overline{b}))-\eta_\delta'(a-b-(\overline{a}-\overline{b}))\ge 0$.

The assertion of the lemma is now straightforward. ∎

3. Additional estimates for vanishing viscosity solutions. The main purpose of this section is to show that higher time regularity of the limit solutions, i.e. $C^0([0,T);\mathbb{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^2))$, follows from the properties of the approximate sequence of vanishing viscosity solutions defined in [6]. We begin by proving the stability of solutions to the parabolic system

$$\begin{aligned}\frac{\partial}{\partial t}w_1-\frac{1}{2}\frac{\partial}{\partial x}w_1^2&=\varepsilon\frac{\partial^2}{\partial x^2}w_1-g^\varepsilon\left(\frac{w_2-w_1}{2}\right),\\ \frac{\partial}{\partial t}w_2+\frac{1}{2}\frac{\partial}{\partial x}w_2^2&=\varepsilon\frac{\partial^2}{\partial x^2}w_2+g^\varepsilon\left(\frac{w_2-w_1}{2}\right),\end{aligned}\tag{3.1}$$

with initial data $(w_1^0,w_2^0)=(w_1(0),w_2(0))$ and $\varepsilon\searrow 0$. The multifunction $\widetilde{g}$ has been replaced by a smooth bounded function g^ε, which is constructed by mollifying $\widetilde{g}$ with some smooth function with compact support. The above problem has a classical solution. We only recall the corresponding theorem from [6]:

THEOREM 3.1. *Assume that the initial data satisfies*

$$w^0 = (w_1^0, w_2^0) \in C_{\mathrm{b}}^3(\mathbb{R};\mathbb{R}^2), \qquad (w_1^0 + w_2^0 - 2\overline{u}, w_1^0 - w_2^0) \in C_{1+|\cdot|}^0(\mathbb{R};\mathbb{R}^2)$$

for some positive constant $\overline{u}$. *Then the problem* (3.1) *has a classical global-in-time solution*, *i.e.* $w \in C^0([0,T); C_{\mathrm{b}}^2(\mathbb{R};\mathbb{R}^2))$, $\frac{\partial}{\partial t} w \in C_{\mathrm{b}}^0([0,T) \times \mathbb{R};\mathbb{R}^2)$, *where T is arbitrary. Moreover*:

(i) *if* $\inf_{x\in\mathbb{R}}(w_1^0(x) + w_2^0(x)) \geq 0$, *then* $\inf_{x\in\mathbb{R}}(w_1(t,x) + w_2(t,x)) \geq 0$;

(ii) *for all* $t \in [0,T)$,

$$\|w_1(t)\|_{\mathbb{L}^\infty(\mathbb{R})} + \|w_2(t)\|_{\mathbb{L}^\infty(\mathbb{R})} \leq (|\sin(\gamma)| + |\cos(\gamma)|)t + \sqrt{2}\,(\|w_1^0\|_{\mathbb{L}^\infty(\mathbb{R})} + \|w_2^0\|_{\mathbb{L}^\infty(\mathbb{R})}).$$

For further considerations we need a new, independent of ε, estimate for the solution to the system (3.1).

LEMMA 3.2. *Let* (w_1, w_2) *and* $(\overline{w}_1, \overline{w}_2)$ *be two different solutions to the system* (3.1) *with initial data* (w_1^0, w_2^0) *and* $(\overline{w}_1^0, \overline{w}_2^0)$ *as in Theorem* 3.1, *and moreover* $(w_1^0 - \overline{w}_1^0, w_2^0 - \overline{w}_2^0) \in \mathbb{L}^1(\mathbb{R};\mathbb{R}^2)$. *Then for any* $0 < t < T$,

$$\|w_1(t)-\overline{w}_1(t)\|_{L^1(\mathbb{R})}+\|w_2(t)-\overline{w}_2(t)\|_{L^1(\mathbb{R})} \leq \|w_1^0-\overline{w}_1^0\|_{L^1(\mathbb{R})}+\|w_2^0-\overline{w}_2^0\|_{L^1(\mathbb{R})}.$$

Proof. Let η_δ be defined by (2.1). Simple calculations yield

$$\begin{aligned}
(3.2)\qquad & \frac{\partial}{\partial t}[\eta_\delta(w_1 - \overline{w}_1) + \eta_\delta(w_2 - \overline{w}_2)] - \frac{1}{2}\frac{\partial}{\partial x}\{\eta_\delta'(w_1 - \overline{w}_1)(w_1^2 - \overline{w}_1^2) \\
& - \eta_\delta'(w_2 - \overline{w}_2)(w_2^2 - \overline{w}_2^2)\} + \frac{1}{2}\eta_\delta''(w_1 - \overline{w}_1)(w_1^2 - \overline{w}_1^2)\frac{\partial}{\partial x}(w_1 - \overline{w}_1) \\
& - \frac{1}{2}\eta_\delta''(w_2 - \overline{w}_2)(w_2^2 - \overline{w}_2^2)\frac{\partial}{\partial x}(w_2 - \overline{w}_2) \\
= {} & \varepsilon\frac{\partial^2}{\partial x^2}[\eta_\delta(w_1 - \overline{w}_1) + \eta_\delta(w_2 - \overline{w}_2)] \\
& - \varepsilon\eta_\delta''(w_1 - \overline{w}_1)\left[\frac{\partial}{\partial x}(w_1 - \overline{w}_1)\right]^2 - \varepsilon\eta_\delta''(w_2 - \overline{w}_2)\left[\frac{\partial}{\partial x}(w_2 - \overline{w}_2)\right]^2 \\
& + \left[g^\varepsilon\left(\frac{w_2 - w_1}{2}\right) - g^\varepsilon\left(\frac{\overline{w}_2 - \overline{w}_1}{2}\right)\right][\eta_\delta'(w_2 - \overline{w}_2) - \eta_\delta'(w_1 - \overline{w}_1)].
\end{aligned}$$

Since the last term on the right-hand side is nonpositive by Lemma 2.1, integrating the above equation over $\mathbb{R} \times (0,t)$ with $0 < t < T$ fixed leads to

$$\begin{aligned}(3.3)\qquad &\int_{\mathbb{R}} [\eta_\delta(w_1(t,x)-\overline{w}_1(t,x))+\eta_\delta(w_2(t,x)-\overline{w}_2(t,x))]\,dx\\
&\quad -\int_{\mathbb{R}} [\eta_\delta(w_1^0(x)-\overline{w}_1^0(x))+\eta_\delta(w_2^0(x)-\overline{w}_2^0(x))]\,dx\\
&\le \int_{\mathbb{R}\times(0,t)} \frac{1}{2}\Big\{\eta_\delta''(w_2-\overline{w}_2)(w_2^2-\overline{w}_2^2)\frac{\partial}{\partial x}(w_2-\overline{w}_2)\\
&\quad -\eta_\delta''(w_1-\overline{w}_1)(w_1^2-\overline{w}_1^2)\frac{\partial}{\partial x}(w_1-\overline{w}_1)\Big\}\,dt\,dx\\
&\quad -\int_{\mathbb{R}\times(0,t)} \varepsilon\eta_\delta''(w_1-\overline{w}_1)\Big[\frac{\partial}{\partial x}(w_1-\overline{w}_1)\Big]^2\,dt\,dx\\
&\quad -\int_{\mathbb{R}\times(0,t)} \varepsilon\eta_\delta''(w_2-\overline{w}_2)\Big[\frac{\partial}{\partial x}(w_2-\overline{w}_2)\Big]^2\,dt\,dx,\end{aligned}$$

which is due to the fact that information on initial data implies that also $w_1(t)+w_2(t)-2\overline{u}$ and $\overline{w}_1(t)+\overline{w}_2(t)-2\overline{u}$ are bounded in $C^0_{1+|\cdot|}(\mathbb{R};\mathbb{R})$ for all $t\in[0,T)$ (for details see [6]). Consequently, for a fixed $t\in[0,T)$, the functions $w_i(t,x)-\overline{w}_i(t,x)$ vanish at infinity, which together with boundedness of solutions in $C^0_{\mathrm{b}}([0,T);C^2_{\mathrm{b}}(\mathbb{R};\mathbb{R}^2))$ implies that the integrals

$$\int_{\mathbb{R}\times(0,t)} \frac{\partial}{\partial x}\{\eta_\delta'(w_i-\overline{w}_i)(w_i^2-\overline{w}_i^2)\}\,dx$$

and

$$\int_{\mathbb{R}\times(0,t)} \frac{\partial^2}{\partial x^2}\eta_\delta(w_i-\overline{w}_i)\,dx=\int_{\mathbb{R}}\frac{\partial}{\partial x}\Big\{\eta_\delta'(w_i-\overline{w}_i)\frac{\partial}{\partial x}(w_i-\overline{w}_i)\Big\}\,dx$$

vanish for $i=1,2$. Note additionally that

$$\begin{aligned}(3.4)\qquad &\int_{\mathbb{R}\times(0,t)} \frac{1}{2}\Big|\eta_\delta''(w_2-\overline{w}_2)(w_2^2-\overline{w}_2^2)\frac{\partial}{\partial x}(w_2-\overline{w}_2)\\
&\qquad -\eta_\delta''(w_1-\overline{w}_1)(w_1^2-\overline{w}_1^2)\frac{\partial}{\partial x}(w_1-\overline{w}_1)\Big|\,dt\,dx\\
&\le \int_{\mathbb{R}\times(0,t)} \varepsilon\eta_\delta''(w_1-\overline{w}_1)\Big[\frac{\partial}{\partial x}(w_1-\overline{w}_1)\Big]^2\\
&\quad +\varepsilon\eta_\delta''(w_2-\overline{w}_2)\Big[\frac{\partial}{\partial x}(w_2-\overline{w}_2)\Big]^2\,dt\,dx\\
&\quad +\frac{\|w_1+\overline{w}_1\|^2_{C^0_{\mathrm{b}}}}{\varepsilon}\int_{A^1_\delta}\eta_\delta(w_1-\overline{w}_1)\,dt\,dx+\frac{\|w_2+\overline{w}_2\|^2_{C^0_{\mathrm{b}}}}{\varepsilon}\int_{A^2_\delta}\eta_\delta(w_2-\overline{w}_2)\,dt\,dx,\end{aligned}$$

where $A_\delta^i = \{(t,x) \in [0,T) \times \mathbb{R} \mid 0 \le w_i(t,x) - \overline{w}_i(t,x) \le 2\delta\}$ for $i = 1,2$. Using (3.3) and (3.4) together with the Gronwall lemma applied to the inequality

$$\int_{\mathbb{R}} [\eta_\delta(w_1 - \overline{w}_1) + \eta_\delta(w_2 - \overline{w}_2)]\,dx \le \int_{\mathbb{R}} [\eta_\delta(w_1^0 - \overline{w}_1^0) + \eta_\delta(w_2^0 - \overline{w}_2^0)]\,dx \tag{3.5}$$

$$+ \max\left\{\frac{\|w_1 + \overline{w}_1\|^2_{C^0_{\mathrm{b}}}}{\varepsilon}, \frac{\|w_2 + \overline{w}_2\|^2_{C^0_{\mathrm{b}}}}{\varepsilon}\right\} \int_{\mathbb{R}\times(0,t)} [\eta_\delta(w_1 - \overline{w}_1) + \eta_\delta(w_2 - \overline{w}_2)]\,dt\,dx$$

implies uniform boundedness (w.r.t. δ) of $\eta_\delta(w_1 - \overline{w}_1)$ and $\eta_\delta(w_2 - \overline{w}_2)$ in the space $\mathbb{L}^\infty([0,T);\mathbb{L}^1(\mathbb{R};\mathbb{R}))$. Note that $\eta_\delta(w_i(t,x) - \overline{w}_i(t,x))$ converges monotonically pointwise to $[w_i(t,x) - \overline{w}_i(t,x)]^+$ as $\delta \to 0$, for $i = 1,2$. Thus estimates (3.3) and (3.4) yield

$$\int_{\mathbb{R}} [w_1(t,x) - \overline{w}_1(t,x)]^+\,dx + \int_{\mathbb{R}} [w_2(t,x) - \overline{w}_2(t,x)]^+\,dx$$
$$\le \int_{\mathbb{R}} [w_1^0(x) - \overline{w}_1^0(x)]^+\,dx + \int_{\mathbb{R}} [w_2^0(x) - \overline{w}_2^0(x)]^+\,dx.$$

Interchanging w_i with $\overline{w}_i$ leads to an analogous inequality. Adding both inequalities yields the assertion of the lemma. ∎

LEMMA 3.3. *Let* (w_1, w_2) *be a classical solution to system* (3.1) *with initial data* (w_1^0, w_2^0) *as in Theorem* 3.1. *Let* $\omega \in C^0(\mathbb{R};\mathbb{R})$ *be such that* $\omega(0) = 0$ *and*

$$\int_{\mathbb{R}} |w_1^0(x) - w_1^0(x+h)|\,dx + \int_{\mathbb{R}} |w_2^0(x) - w_2^0(x+h)|\,dx \le \omega(h)$$

for all $h \in \mathbb{R}_+$. *Then for any* $0 < t < T - h$ *and* $r > 0$,

$$\int_{-r}^{r} \{|w_1(t+h,x) - w_1(t,x)| + |w_2(t+h,x) - w_2(t,x)|\}\,dx$$
$$\le c_1(h + h^{2/3} + h^{1/3})(r+1)(\|w_1^2\|_{\mathbb{L}^\infty([t,t+h)\times\mathbb{R})} + \|w_2^2\|_{\mathbb{L}^\infty([t,t+h)\times\mathbb{R})} + 1)$$
$$+ c_2\omega(h^{1/3}),$$

where the constants c_1, c_2 *do not depend on* ε.

Proof. Observe that by Lemma 3.2,

$$\int_{\mathbb{R}} \{|w_1(t,x) - w_1(t,x+y)| + |w_2(t,x) - w_2(t,x+y)|\}\,dx$$
$$\le \int_{\mathbb{R}} \{|w_1^0(x) - w_1^0(x+y)| + |w_2^0(x) - w_2^0(x+y)|\}\,dx \le \omega(|y|).$$

We begin by analyzing the equations of system (3.1) separately. Multiplying each by a bounded test function $\phi_i(x)K(x)$, where ϕ_i and K are in $W^{2,\infty}(\mathbb{R})$ with

$$K(x)=\begin{cases}1 & \text{for } |x|\le r,\\ 0 & \text{for } |x|>r+1,\\ (x+r+1)^2(x+r-1) & \text{for } -(r+1)\le x<-r,\\ (x-r-1)^2(x-r+1) & \text{for } r<x\le r+1,\end{cases}$$

then integrating the equations over $\mathbb{R}\times(t,t+h)$ yields

$$\begin{aligned}\int_{\mathbb{R}}&\phi_i(x)K(x)[w_i(t+h,x)-w_i(t,x)]\,dx\\ &=\int_{(t,t+h)\times\mathbb{R}}\frac{(-1)^i}{2}\left[\frac{\partial}{\partial x}\phi_i(x)K(x)+\phi_i(x)\frac{\partial}{\partial x}K(x)\right]w_i^2(t,x)\,dt\,dx\\ &\quad+\int_{(t,t+h)\times\mathbb{R}}\left\{\varepsilon\left[\frac{\partial^2}{\partial x^2}\phi_i(x)K(x)+\frac{\partial}{\partial x}\phi_i(x)\frac{\partial}{\partial x}K(x)\right.\right.\\ &\qquad\qquad\left.\left.+\phi_i(x)\frac{\partial^2}{\partial x^2}K(x)\right]w_i(t,x)\right\}dt\,dx\\ &\quad+\int_{(t,t+h)\times\mathbb{R}}\phi_i(x)K(x)g(t,x)\,dt\,dx\end{aligned}$$

for $i=1,2$. A way to obtain the assertion of the lemma would be to take $\operatorname{sgn} v_i(x)$ with $v_i(x)=w_i(t+h,x)-w_i(t,x)$ as a test function. But since $\operatorname{sgn} v_i(x)$ is discontinuous, we have to mollify it first. Hence, we define the test function

$$\phi_i(x)=(\xi_h*\operatorname{sgn} v_i)(x),$$

where $\xi_h(x)=h^{-1/3}\xi(x/h^{1/3})$, with some smooth and nonnegative function ξ of compact support and total mass one. Note that $|K|,\left|\frac{\partial}{\partial x}K\right|,\left|\frac{\partial^2}{\partial x^2}K\right|$, $|\phi_i|,\left|h^{1/3}\frac{\partial}{\partial x}\phi_i\right|,\left|h^{2/3}\frac{\partial^2}{\partial x^2}\phi_i\right|$ are bounded and $|g^\varepsilon|$ is uniformly bounded (w.r.t. to ε) in $\mathbb{L}^\infty([0,T)\times\mathbb{R};\mathbb{R})$. Thus we conclude that

$$\int_{-r}^{r}\phi_i(x)v_i(x)\,dx\le c_1(h+h^{2/3}+h^{1/3})(r+1)(\|w_i^2\|_{\mathbb{L}^\infty([t,t+h)\times\mathbb{R})}+1).$$

Since $|v_i(x)|-v_i(x)\operatorname{sgn} v_i(z)\le 2|v_i(x)-v_i(z)|$, we have

$$\begin{aligned}|v_i(x)|&-v_i(x)\operatorname{sgn} v_i(z)\\ &\le 2\int_{\mathbb{R}}\xi_h(s)\{|w_1(t,x)-w_1(t,x-h^{1/3}s)|+|w_2(t,x)-w_2(t,x-h^{1/3}s)|\}\,ds.\end{aligned}$$

This yields the assertion of the lemma. ∎

Proof of the higher time regularity

STEP 1. Note that the sequence of the approximate solutions to (3.1) has a subsequence strongly convergent in $\mathbb{L}^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R};\mathbb{R}^2)$ (cf. [6, pp. 77, item 4]) denoted by (w_1^k,w_2^k).

STEP 2. This subsequence is also strongly convergent in $\mathbb{L}^1([0,T);\mathbb{L}^1([-r,r];\mathbb{R}^2))$ (for all positive T,r) and then $(w_1^k(t),w_2^k(t))$ converges for a.a. $t\in[0,T)$ in the strong topology of $\mathbb{L}^1([-r,r];\mathbb{R}^2)$.

STEP 3. By Lemma 3.3 and the additional information on solutions from Theorem 3.1, the family of functions (w_1^k,w_2^k) is uniformly equicontinuous in $C^0([0,T);\mathbb{L}^1([-r,r];\mathbb{R}^2))$ (for all positive T,r), which also implies that the convergence from Step 2 holds for all $t\in[0,T)$.

STEP 4. That $w\in C^0([0,T);X)$ follows from the following claim (which is a consequence of the Ascoli–Arzelà Theorem, cf. [11, pp. 71]):

CLAIM. *Let X be a Banach space and (w^k) be a uniformly equicontinuous family of functions in $C^0([0,T);X)$ such that $(w^k(t))$ is relatively compact in X for all $t\in[0,T)$. Then (w^k) is relatively compact in $C^0([0,T);X)$.*

4. Stability of solutions. The entropy-entropy flux pair (Definition 1.1) has to satisfy the condition

$$\nabla_{(w_1,w_2)}\eta(w_1,w_2)\cdot\nabla_{(w_1,w_2)}F(w_1,w_2)=\nabla_{(w_1,w_2)}q(w_1,w_2).$$

In our case the matrix $\nabla_{(w_1,w_2)}F(w_1,w_2)$ is diagonal, hence the above vector equation takes the form

$$(\partial_{w_1}\eta(w_1,w_2),\partial_{w_2}\eta(w_1,w_2))\cdot\begin{pmatrix}-w_1 & 0\\ 0 & w_2\end{pmatrix}=(\partial_{w_1}q(w_1,w_2),\partial_{w_2}q(w_1,w_2)).$$

Hence there are entropy-entropy flux pairs (η^1,q^1) dependent only on w_1 and (η^2,q^2) on w_2. Thus the vector equation can be decoupled into scalar equations

$$\begin{aligned}(\eta^1)'(w_1)\cdot(-w_1)&=(q^1)'(w_1),\\ (\eta^2)'(w_2)\cdot w_2&=(q^2)'(w_2).\end{aligned}$$

Following the notation from Section 1, g will denote some measurable selection from $\widetilde{g}$, namely $g(t,x)\in\widetilde{g}\big(\frac{w_2(t,x)-w_1(t,x)}{2}\big)$ for a.a. $(t,x)\in[0,T)\times\mathbb{R}$. Then the entropy inequality (Definitions 1.2(ii) and 1.3(i)) takes the form

$$\begin{aligned}(4.1)\qquad \int\limits_{[0,T)\times\mathbb{R}}\Big\{&[\eta^1(w_1(t,x))+\eta^2(w_2(t,x))]\,\frac{\partial}{\partial t}\phi(t,x)\\ &+[q^1(w_1(t,x))+q^2(w_2(t,x))]\,\frac{\partial}{\partial x}\phi(t,x)\\ &+g(t,x)[(\eta^2)'(w_2(t,x))-(\eta^1)'(w_1(t,x))]\phi(t,x)\Big\}\,dt\,dx\\ &+\int\limits_{\mathbb{R}}[\eta^1(w_1^0(x))+\eta^2(w_2^0(x))]\phi(0,x)\,dx\ge 0\end{aligned}$$

for all nonnegative functions $\phi \in C^1_{\mathrm{c}}([0,T)\times\mathbb{R};\mathbb{R})$. Hence $\eta^i_\delta(w_i,\overline{w}_i)$ defined below becomes an entropy function; moreover, $q^i_\delta(w_i,\overline{w}_i)$ denotes a corresponding entropy flux, where $\overline{w}_i$ is a parameter taking values in $\mathbb{R}$, $i=1,2$. We set

$$\eta^i_\delta(w_i,\overline{w}_i)=\begin{cases}0 & \text{for } w_i\le\overline{w}_i\\ \dfrac{(w_i-\overline{w}_i)^2}{4\delta} & \text{for } \overline{w}_i<w_i\le\overline{w}_i+2\delta,\\ w_i-\overline{w}_i-\delta & \text{for } w_i>\overline{w}_i+2\delta.\end{cases}$$

Note that

$$\partial_{w_i}\eta^i_\delta(w_i,\overline{w}_i)=\begin{cases}0 & \text{for } w_i\le\overline{w}_i,\\ \dfrac{w_i-\overline{w}_i}{2\delta} & \text{for } \overline{w}_i<w_i\le\overline{w}_i+2\delta,\\ 1 & \text{for } w_i>\overline{w}_i+2\delta,\end{cases}$$

$$\partial_{\overline{w}_i}\eta^i_\delta(w_i,\overline{w}_i)=\begin{cases}0 & \text{for } w_i\le\overline{w}_i,\\ -\dfrac{w_i-\overline{w}_i}{2\delta} & \text{for } \overline{w}_i<w_i\le\overline{w}_i+2\delta,\\ -1 & \text{for } w_i>\overline{w}_i+2\delta.\end{cases}$$

Proof of Theorem 1.2. In the above entropy inequality, we use a nonnegative function $\phi(t,x,\overline{t},\overline{x})\in C^1_{\mathrm{c}}(((0,T)\times\mathbb{R})^2;\mathbb{R})$ as a test function. Then for some fixed $(\overline{t},\overline{x})$ the inequality takes the form

$$\begin{aligned}\int\limits_{[0,T)\times\mathbb{R}}\bigg\{&\frac{\partial}{\partial t}\phi(t,x,\overline{t},\overline{x})[\eta^1_\delta(w_1,\overline{w}_1)+\eta^2_\delta(w_2,\overline{w}_2)]\\ &+\partial_x\phi(t,x,\overline{t},\overline{x})[q^1_\delta(w_1,\overline{w}_1)+q^2_\delta(w_2,\overline{w}_2)]\\ &+\phi(t,x,\overline{t},\overline{x})g(t,x)[\partial_{w_2}\eta^2_\delta(w_2,\overline{w}_2)-\partial_{w_1}\eta^1_\delta(w_1,\overline{w}_1)]\bigg\}\,dt\,dx\ge 0.\end{aligned}$$

In the same manner, for (t,x) fixed, we obtain

$$\begin{aligned}\int\limits_{[0,T)\times\mathbb{R}}\bigg\{&\frac{\partial}{\partial\overline{t}}\phi(t,x,\overline{t},\overline{x})[\eta^1_\delta(w_1,\overline{w}_1)+\eta^2_\delta(w_2,\overline{w}_2)]\\ &+\frac{\partial}{\partial\overline{x}}\phi(t,x,\overline{t},\overline{x})[q^1_\delta(w_1,\overline{w}_1)+q^2_\delta(w_2,\overline{w}_2)]\\ &-\phi(t,x,\overline{t},\overline{x})\overline{g}(\overline{t},\overline{x})[\partial_{w_2}\eta^2_\delta(w_2,\overline{w}_2)-\partial_{w_1}\eta^1_\delta(w_1,\overline{w}_1)]\bigg\}\,d\overline{t}\,d\overline{x}\ge 0.\end{aligned}$$

Here $\overline{g}(\overline{t},\overline{x})\in\widetilde{g}\big(\frac{\overline{w}_2(\overline{t},\overline{x})-\overline{w}_1(\overline{t},\overline{x})}{2}\big)$ for a.a. $(t,x)\in[0,T)\times\mathbb{R}$. Integrating the first inequality with respect to $(\overline{t},\overline{x})$ and the second with respect to (t,x), then adding them leads to the following result:

$$\int_{([0,T)\times\mathbb{R})^2} \bigg\{ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \overline{t}}\right)\phi(t,x,\overline{t},\overline{x})[\eta^1_\delta(w_1,\overline{w}_1)+\eta^2_\delta(w_2,\overline{w}_2)] + \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial \overline{x}}\right)\phi(t,x,\overline{t},\overline{x})[q^1_\delta(w_1,\overline{w}_1)+q^2_\delta(w_2,\overline{w}_2)] + \phi(t,x,\overline{t},\overline{x})[g(t,x) - \overline{g}(\overline{t},\overline{x})][\partial_{w_2}\eta^2_\delta(w_2,\overline{w}_2)-\partial_{w_1}\eta^1_\delta(w_1,\overline{w}_1)]\bigg\}\,dt\,dx\,d\overline{t}\,d\overline{x} \ge 0. \tag{4.2}$$

We fix a smooth, compactly supported function $\xi:\mathbb{R}\to\mathbb{R}_+$ satisfying $\int_{\mathbb{R}}\xi(x)\,dx=1$ and we test (4.2) against

$$\phi(t,x,\overline{t},\overline{x})=\frac{1}{\varepsilon^2}\,\psi\left(\frac{t+\overline{t}}{2},\frac{x+\overline{x}}{2}\right)\xi\left(\frac{t-\overline{t}}{2\varepsilon}\right)\xi\left(\frac{x-\overline{x}}{2\varepsilon}\right),$$

where $\psi:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}_+$ is a C^1-function with compact support ($\mathrm{supp}(\psi)\subset((0,T)\times\mathbb{R})^2$). Note that

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \overline{t}}\right)\phi(t,x,\overline{t},\overline{x})=\frac{1}{\varepsilon^2}\,\frac{\partial}{\partial t}\psi\left(\frac{t+\overline{t}}{2},\frac{x+\overline{x}}{2}\right)\xi\left(\frac{t-\overline{t}}{2\varepsilon}\right)\xi\left(\frac{x-\overline{x}}{2\varepsilon}\right),$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial \overline{x}}\right)\phi(t,x,\overline{t},\overline{x})=\frac{1}{\varepsilon^2}\,\frac{\partial}{\partial x}\psi\left(\frac{t+\overline{t}}{2},\frac{x+\overline{x}}{2}\right)\xi\left(\frac{t-\overline{t}}{2\varepsilon}\right)\xi\left(\frac{x-\overline{x}}{2\varepsilon}\right).$$

Letting $\varepsilon\searrow 0$ we find that (4.2) leads to (for more details on this step we refer the reader to [7])

$$\int_{[0,T)\times\mathbb{R}} \bigg\{\frac{\partial}{\partial t}\psi(t,x)[\eta^1_\delta(w_1,\overline{w}_1)+\eta^2_\delta(w_2,\overline{w}_2)] + \frac{\partial}{\partial x}\psi(t,x)[q^1_\delta(w_1,\overline{w}_1)+q^2_\delta(w_2,\overline{w}_2)] + \psi(t,x)[g(t,x)-\overline{g}(\overline{t},\overline{x})][\partial_{w_2}\eta^2_\delta(w_2,\overline{w}_2)-\partial_{w_1}\eta^1_\delta(w_1,\overline{w}_1)]\bigg\}\,dt\,dx\ge 0$$

for all nonnegative C^1-functions ψ with compact support in $(0,T)\times\mathbb{R}$. Density of $C^1_c((0,T)\times\mathbb{R};\mathbb{R})$ in $\mathbb{W}^{1,1}_0([0,T)\times\mathbb{R};\mathbb{R})$ together with the fact that $(w_1,w_2),(\overline{w}_1,\overline{w}_2)\in\mathbb{L}^\infty([0,T)\times\mathbb{R};\mathbb{R}^2)$ implies that the above inequality also holds for $\psi\in\mathbb{W}^{1,1}_0([0,T)\times\mathbb{R};\mathbb{R})$. Therefore we use a test function $\psi(t,x)=\zeta_r(x)\theta_{\varepsilon,s}(t)$, where

$$\zeta_r(x)=\begin{cases}0, & |x|>r+1,\\ r+1-|x|, & r<|x|<r+1,\\ 1, & |x|<r,\end{cases}$$

$$\theta_{\varepsilon,s}(t)=\begin{cases}0, & 0<t\le s \text{ or } t>\tau+\varepsilon,\\ 1, & s+\varepsilon<t\le\tau,\\ \dfrac{1}{\varepsilon}t-\dfrac{s}{\varepsilon}, & s<t\le s+\varepsilon,\\ -\dfrac{1}{\varepsilon}t+1+\dfrac{\tau}{\varepsilon}, & \tau<t\le\tau+\varepsilon,\end{cases}$$

for $r>0$, $0<s<\tau$, $0<\varepsilon<\tau-\varepsilon$. According to Lemma 2.1 the nonpositive term containing g can be omitted. We have

$$\frac{1}{\varepsilon}\int_{-r}^{r}\int_{\tau}^{\tau+\varepsilon}[\eta_\delta^1(w_1,\overline{w}_1)+\eta_\delta^2(w_2,\overline{w}_2)]\,dt\,dx - \frac{1}{\varepsilon}\int_{\{r<|x|<r+1\}}\int_{s}^{\tau+\varepsilon}\theta_{\varepsilon,s}(t)[q_\delta^1(w_1,\overline{w}_1)+q_\delta^2(w_2,\overline{w}_2)]\,dt\,dx$$
$$\le\frac{1}{\varepsilon}\int_{-r}^{r}\int_{s}^{s+\varepsilon}[\eta_\delta^1(w_1,\overline{w}_1)+\eta_\delta^2(w_2,\overline{w}_2)]\,dt\,dx.$$

Let first $s\searrow 0$, and then $\varepsilon\searrow 0$. Using in both cases continuity with respect to t (i.e. $w,\overline{w}\in C^0([0,T);\mathbb{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^2))$) we conclude that

$$\int_{-r}^{r}[\eta_\delta^1(w_1,\overline{w}_1)+\eta_\delta^2(w_2,\overline{w}_2)]\,dx-\int_0^{\tau}\int_{\{r<|x|<r+1\}}[q_\delta^1(w_1,\overline{w}_1)+q_\delta^2(w_2,\overline{w}_2)]\,dx\,dt$$
$$\le\int_{-(r+1)}^{r+1}[\eta_\delta^1(w_1^0,\overline{w}_1^0)+\eta_\delta^2(w_2^0,\overline{w}_2^0)]\,dx$$

for all $t\in[0,T)$. Letting first $\delta\searrow 0$, and then $r\to\infty$, we conclude with a standard dominated convergence theorem argument that

$$\int_{\mathbb{R}}\{[w_1(\tau,x)-\overline{w}_1(\tau,x)]^+ + [w_2(\tau,x)-\overline{w}_2(\tau,x)]^+\}\,dx$$
$$\le\int_{\mathbb{R}}\{[w_1^0(x)-\overline{w}_1^0(x)]^+ + [w_2^0(x)-\overline{w}_2^0(x)]^+\}\,dx.$$

Interchanging w_i with $\overline{w}_i$ leads to the analogous inequality. Adding both inequalities yields the assertion of the theorem. ∎

REMARK. The proof of Theorem 1.2 is a modification of one from [7].

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