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THE CATEGORY OF GROUPOID GRADED MODULES

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Abstract. We introduce the abelian category R-gr of groupoid graded modules and give an answer to the following general question: If $U : R\text{-gr} \to R\text{-mod}$ denotes the functor which associates to any graded left R-module M the underlying ungraded structure $U(M)$, when does either of the following two implications hold: (I) M has property $X \Rightarrow U(M)$ has property X; (II) $U(M)$ has property $X \Rightarrow M$ has property X? We treat the cases when X is one of the properties: direct summand, free, finitely generated, finitely presented, projective, injective, essential, small, and flat. We also investigate when exact sequences are pure in R-gr. Some relevant counterexamples are indicated.

1. Introduction. The notion of group graded rings and modules occurs frequently in the literature (see e.g. [2]–[7] and [9]). In this article, we introduce the category of *groupoid* graded modules. By examining various properties (see below) of this category, we generalize several results from the category of group graded modules to the groupoid graded case.

Recall that a *groupoid* is a small category with the property that all morphisms are isomorphisms. Equivalently, it can be defined as a non-empty set Γ equipped with a unary operation $\Gamma \ni \sigma \mapsto \sigma^{-1} \in \Gamma$ and a partial binary operation $\Gamma \times \Gamma \ni (\sigma,\tau) \mapsto \sigma\tau \in \Gamma$ satisfying the following four axioms:

(i) $d(\sigma) := \sigma^{-1}\sigma$ and $r(\sigma) := \sigma\sigma^{-1}$ are always defined ($d =$ "domain" and $r =$ "range");
(ii) $\sigma\tau$ is defined if and only if $d(\sigma) = r(\tau)$;
(iii) if $\sigma\tau$ and $\tau\varrho$ are defined, then $(\sigma\tau)\varrho$ and $\sigma(\tau\varrho)$ are defined and equal;
(iv) each of $d(\sigma)\tau$, $\tau d(\sigma)$, $r(\sigma)\tau$, and $\tau r(\sigma)$ is equal to τ if it is defined.

For the rest of the article, we fix a groupoid Γ. We say that a ring R is *graded* if there is a family R_σ, $\sigma \in \Gamma$, of additive subgroups of R such that $R = \bigoplus_{\sigma\in\Gamma} R_\sigma$, and for all $\sigma,\tau \in \Gamma$, we have $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ if $d(\sigma) = r(\tau)$, and $R_\sigma R_\tau = \{0\}$ otherwise. Natural examples of such rings are e.g. given by group rings or matrix rings (see Ex. 2.1.2).

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Furthermore, if R is a graded ring, then we say that a left R-module M is *graded* if there is a family M_σ, $\sigma \in \Gamma$, of additive subgroups of M such that $M = \bigoplus_{\sigma\in\Gamma} M_\sigma$, and for all $\sigma, \tau \in \Gamma$, we have $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ if $d(\sigma) = r(\tau)$, and $R_\sigma M_\tau = \{0\}$ otherwise. Let R-mod (resp. R-gr) denote the category of left R-modules (resp. graded left R-modules). The morphisms in the graded case are taken to be R-linear maps $f : M \to M'$ with the property $f(M_\sigma) \subseteq M'_\sigma$, $\sigma \in \Gamma$.

The main objective of this article is to study the following general question:

> *Is it possible to derive information about classical objects over a graded ring making use of graded data?*

More precisely, if $U : R\text{-gr} \to R\text{-mod}$ denotes the functor which associates to any graded left R-module M the underlying ungraded structure $U(M)$, when does either of the following two implications hold:

(I) M has property $X \Rightarrow U(M)$ has property X;
(II) $U(M)$ has property $X \Rightarrow M$ has property X?

In Section 3, we give an answer to this question in the cases when X is one of the properties: direct summand, free, finitely generated, finitely presented, projective, injective, essential, small, and flat. We also investigate when exact sequences are pure in R-gr.

Since some of the proofs of our results resemble their ungraded counterparts, we have sometimes taken the liberty of omitting the details.

2. Basic results. In this section, we prove some results that are needed in Section 3 to give an answer to the general question raised in the introduction.

2.1. *Notation.* For a set X, let $|X|$ denote the cardinality of X, and $\mathcal{P}(X)$ the power set of X.

We assume that all rings R are associative and equipped with a multiplicative identity 1_R, and that ring homomorphisms $R \to S$ map 1_R to 1_S. By abuse of notation, we will write 1 instead of 1_R.

For the rest of the article, we fix a graded ring R. If R' is another graded ring, $R' \subseteq R$, then we say that R' is a *graded subring* of R if $1_{R'} = 1_R$ and $R'_\sigma \subseteq R_\sigma$, $\sigma \in \Gamma$. Note that if Γ' is a *subgroupoid* of Γ, that is, a subset of Γ containing $\Gamma_0 := \{d(\sigma) \mid \sigma \in \Gamma\}$ $(= \{r(\sigma) \mid \sigma \in \Gamma\})$ closed under multiplication and the inverse, then $R' := \bigoplus_{\sigma\in\Gamma'} R_\sigma$, with the grading induced from R, is a graded subring of R.

Let M be a graded left R-module. Elements of $\bigcup_{\sigma\in\Gamma} M_\sigma$ are called *homogeneous elements* of M. If $m \in M_\sigma \setminus \{0\}$ for some $\sigma \in \Gamma$, then m is called *homogeneous of degree* σ and we write $\deg(m) = \sigma$. Any non-zero $m \in M$

has a unique decomposition $m = \sum_{\sigma\in\Gamma} m_\sigma$, where $m_\sigma \in M_\sigma$, $\sigma \in \Gamma$, and all but a finite number of the m_σ are non-zero. The non-zero elements m_σ in the decomposition of m are called the *homogeneous components* of m.

If N is an R-submodule of M, then it is called a *graded submodule* if $N = \bigoplus_{\sigma\in\Gamma}(N \cap M_\sigma)$. In that case, the quotient module M/N can be graded in a natural way. A (left or right) ideal of R is called *graded* if it is graded as a (left or right) submodule of R.

It is easy to see that R-gr is an abelian category with enough projective objects (that is, every module in R-gr can be written as a quotient of a projective module; see Prop. 3.3.4(b) and Lemma 3.4.2). It is even a Grothendieck category (see e.g. [11] for a definition of this concept). Direct sums and direct limits exist in R-gr. Note however that direct products do not always exist in R-gr.

By the next proposition, we can always assume that Γ_0 is *finite.*

2.1.1. Proposition. *With the above notations, we get*

(a) $1 \in \bigoplus_{\sigma\in\Gamma_0} R_\sigma$.

If we put $\Gamma' = \{\sigma \in \Gamma \mid 1_{d(\sigma)}, 1_{r(\sigma)} \neq 0\}$, *then*

(b) *The set* Γ', *with the operations induced from* Γ, *is a groupoid.*
(c) $|\Gamma'_0| < \infty$.
(d) $R = \bigoplus_{\sigma\in\Gamma'} R_\sigma$.

Proof. (a) Let $1 = \sum_{\sigma\in\Gamma} 1_\sigma$ be the homogeneous decomposition of 1 in R. Thus, for $\tau \in \Gamma$, we get $1_\tau = 11_\tau = \sum_{\sigma\in\Gamma} 1_\sigma 1_\tau$. But since $1_\sigma 1_\tau \subseteq R_{\sigma\tau}$, we get $1_\sigma 1_\tau = 0$ if $\sigma \notin \Gamma_0$. Hence, $\sigma \notin \Gamma_0 \Rightarrow 1_\sigma = 1_\sigma 1 = 1_\sigma \sum_{\tau\in\Gamma} 1_\tau = \sum_{\tau\in\Gamma} 1_\sigma 1_\tau = 0$.

(b) follows immediately from the fact that if $\sigma, \tau \in \Gamma$ are chosen so that $d(\sigma) = r(\tau)$, then $d(\sigma\tau) = d(\tau)$ and $r(\sigma\tau) = r(\sigma)$.

(c) follows from (a).

(d) Take $\sigma \in \Gamma$. If $1_{r(\sigma)} = 0$, then $R_\sigma = 1R_\sigma = 1_{r(\sigma)} R_\sigma = \{0\}$. The case when $1_{d(\sigma)} = 0$ is treated similarly. ∎

For future use, we now recall a well known example of graded rings.

2.1.2. Example. Let T be a ring. The *groupoid ring* $T[\Gamma]$, of T over Γ, is defined to be the set of all formal sums $\sum_{\sigma\in\Gamma} t_\sigma \sigma$, with $t_\sigma \in T$, $\sigma \in \Gamma$, and $t_\sigma = 0$ for almost all $\sigma \in \Gamma$. Addition is defined pointwise and multiplication is defined by the T-linear extension of the rule

$$\sigma \cdot \tau = \begin{cases} \sigma\tau & \text{if } d(\sigma) = r(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

The grading is, of course, defined by $T[\Gamma]_\sigma = T\sigma$, $\sigma \in \Gamma$.

If Γ is a group, then $T[\Gamma]$ is the usual group ring of T over Γ. On the other hand, if $\Gamma = I \times I$, where I is a finite set, and Γ is equipped with the operation $(i,j)\cdot(k,l)=(i,l)$ if $j=k$, then $T[\Gamma]$ is the ring of $|I|\times|I|$ matrices over T.

2.2. *The monoid $\mathcal{P}(\Gamma)$.* Recall that a *monoid* is a non-empty set $\mathcal{M}$ equipped with an associative binary operation $*$ and a neutral element e. An element $x\in\mathcal{M}$ is called *invertible* if there is $y\in\mathcal{M}$ such that $x*y=y*x=e$.

2.2.1. Proposition. *If for $\Sigma,\Sigma'\in\mathcal{P}(\Gamma)$ we define*

$$\Sigma*\Sigma'=\{\sigma\tau\mid\sigma\in\Sigma,\,\tau\in\Sigma',\,d(\sigma)=r(\tau)\},$$

then:

(a) *$(\mathcal{P}(\Gamma),*)$ is a monoid with neutral element Γ_0.*
(b) *The element $\Sigma\in\mathcal{P}(\Gamma)$ is invertible if and only if the following two properties hold*:
   (i) $|\Sigma|=|\Gamma_0|$,
   (ii) $\sigma,\tau\in\Sigma,\ \sigma\neq\tau\Rightarrow d(\sigma)\neq d(\tau),\ r(\sigma)\neq r(\tau)$.
(c) *For $\sigma\in\Gamma$, let $\Sigma_\sigma\in\mathcal{P}(\Gamma)$ be defined by*

$$\Sigma_\sigma=\begin{cases}\{\sigma,\sigma^{-1}\}\cup(\Gamma_0\setminus\{d(\sigma),r(\sigma)\}) & \text{if } d(\sigma)\neq r(\sigma),\\ \{\sigma\}\cup(\Gamma_0\setminus\{d(\sigma)\}) & \text{otherwise.}\end{cases}$$

Then Σ_σ is invertible.

Proof. (a) This is clear.

(b) Put $\Sigma^{-1}=\{\sigma^{-1}\mid\sigma\in\Sigma\}$. Note first that Σ is invertible if and only if $\Sigma^{-1}*\Sigma=\Sigma*\Sigma^{-1}=\Gamma_0$.

Assume that Σ is invertible. If $|\Sigma|>|\Gamma_0|$, then there are $\sigma,\tau\in\Sigma$, $\sigma\neq\tau$, such that $r(\sigma)=r(\tau)$ (or $d(\sigma)=d(\tau)$). Thus, we get a contradiction: $\Sigma^{-1}*\Sigma\ni\sigma^{-1}\tau\notin\Gamma_0$ (or $\Sigma*\Sigma^{-1}\ni\sigma\tau^{-1}\notin\Gamma_0$). If $|\Sigma|<|\Gamma_0|$, then we get a contradiction: $|\Gamma_0|=|(\Sigma^{-1}*\Sigma)\cap\Gamma_0|\le|\Sigma|<|\Gamma_0|$. Hence, (i) holds. If (ii) does not hold, then there are $\sigma,\tau\in\Sigma$, $\sigma\neq\tau$, such that $d(\sigma)=d(\tau)$ or $r(\sigma)=r(\tau)$, and we again get a contradiction as above.

On the other hand, if we assume that (i) and (ii) are satisfied, then clearly $\Sigma*\Sigma^{-1}=\Sigma^{-1}*\Sigma=\Gamma_0$.

(c) follows directly from (b). ∎

2.2.2. Remark. If Γ is a group, then the operation $*$ coincides with the usual multiplication of subsets of Γ, that is,

$$\Sigma*\Sigma'=\Sigma\Sigma'=\{\sigma\tau\mid\sigma\in\Sigma,\,\tau\in\Sigma'\}$$

for all $\Sigma,\Sigma'\in\mathcal{P}(\Gamma)$. Furthermore, $\Sigma\in\mathcal{P}(\Gamma)$ is invertible precisely when $\Sigma=\Sigma_\sigma=\{\sigma\}$ for some $\sigma\in\Gamma$.

For a graded left R-module M, let $M(\sigma)$, the *σ-suspension* of M, be M as a left R-module but with the new grading

$$M(\sigma)_\tau = \begin{cases} M_{\tau\sigma} & \text{if } d(\tau) = r(\sigma), \\ \{0\} & \text{otherwise,} \end{cases}$$

for all $\tau \in \varGamma$. It follows immediately that if $\sigma, \tau \in \varGamma$, then

$$M(\sigma)(\tau) = \begin{cases} M(\tau\sigma) & \text{if } d(\tau) = r(\sigma), \\ \{0\} & \text{otherwise.} \end{cases} \tag{1}$$

For $\varSigma \in \mathcal{P}(\varGamma)$, define the functor

$$T_\varSigma : R\text{-gr} \to R\text{-gr}$$

by $T_\varSigma(M) = \bigoplus_{\sigma\in\varSigma} M(\sigma)$ for all graded left R-modules M. This functor enjoys some nice properties (which will come in handy later):

2.2.3. Proposition. *With the above notations, we get*:

(a) *If $\varSigma, \varSigma' \in \mathcal{P}(\varGamma)$, then $T_\varSigma T_{\varSigma'} = T_{\varSigma * \varSigma'}$.*
(b) *If $\varSigma \in \mathcal{P}(\varGamma)$ is invertible, then $T_\varSigma$ is an autoequivalence of R-gr.*

Proof. (a) is a consequence of (1), and (b) follows from (a) if we put $\varSigma' = \varSigma^{-1}$. ∎

2.3. *Graded homomorphisms and tensor products.* Let M and N be graded left R-modules. If $f : M \to N$ is R-linear and $\varSigma \in \mathcal{P}(\varGamma)$, then we say that f is a *map of degree $\varSigma$* if for all $\sigma \in \varGamma$ we have

$$f(M_\sigma) \subseteq \bigoplus_{\tau\in\varSigma,\, r(\tau)=d(\sigma)} N_{\sigma\tau}.$$

The collection of maps of degree $\varSigma$ is denoted $\mathrm{HOM}_R(M,N)_\varSigma$. If $\varSigma = \{\sigma\}$ for some $\sigma \in \varGamma$, then we write $\mathrm{HOM}_R(M,N)_\sigma$ instead of $\mathrm{HOM}_R(M,N)_\varSigma$. Note that the maps of degree $\varGamma_0$ are precisely the morphisms in R-gr (as defined in the introduction). In what follows, we will refer to them simply as *graded maps.*

Let $\mathrm{Ab}_\varGamma$ denote the category of $\varGamma$-graded abelian groups. Groups of this type can always, in a natural way, be viewed as graded left $\mathbb{Z}[\varGamma_0]$-modules (note that $\mathbb{Z}[\varGamma_0]$, being a graded subring of $\mathbb{Z}[\varGamma]$, is a graded ring). We call this the *trivial grading* of the objects in $\mathrm{Ab}_\varGamma$.

Define the functor

$$\mathrm{HOM}_R : R\text{-gr} \times R\text{-gr} \to \mathrm{Ab}_\varGamma$$

by $\mathrm{HOM}_R(M,N) = \bigoplus_{\sigma\in\varGamma} \mathrm{HOM}_R(M,N)_\sigma$. The elements of $\mathrm{HOM}_R(M,N)$ will from now on be called *semi-graded maps.*

2.3.1. Remark. If M and N are graded left R-modules, then

$$\mathrm{HOM}_R(M,N) \subseteq \mathrm{Hom}_R(M,N). \tag{2}$$

It is easy to see that equality holds in (2) e.g. when Γ is finite or M is finitely generated. However, equality does not hold in general (for a counterexample in the case when Γ is a group, see p. 11 in [9]).

We gather some elementary properties of HOM_R that we need later.

2.3.2. PROPOSITION. *Let M and N_i, $i \in I$, be graded left R-modules. Then the following isomorphisms in* Ab_Γ *hold*:

(a) $\mathrm{HOM}_R(R, M) \cong M$.
(b) $\mathrm{HOM}_R(\bigoplus_{i\in I} N_i, M) \cong \bigoplus_{i\in I} \mathrm{HOM}_R(N_i, M)$.

2.3.3. PROPOSITION. *Given an exact sequence $M \to N \to P \to 0$ of graded left R-modules and graded maps, the induced sequence in* Ab_Γ:

$$0 \to \mathrm{HOM}_R(M, Q) \to \mathrm{HOM}_R(N, Q) \to \mathrm{HOM}_R(P, Q)$$

is exact.

The proofs of the last two propositions are analogous to the proofs in the ungraded case (found e.g. in [10]).

2.3.4. REMARK. Let R and S be graded rings. A right S-module (resp. an R-S-bimodule) M is called *graded* if there is a family M_σ, $\sigma \in \Gamma$, of additive subgroups of M such that $M = \bigoplus_{\sigma\in\Gamma} M_\sigma$, and for all $\sigma, \tau \in \Gamma$, we have $M_\sigma S_\tau \subseteq M_{\sigma\tau}$ (resp. $R_\sigma M_\tau S_\varrho \subseteq M_{\sigma\tau\varrho}$) if $d(\sigma) = r(\tau)$ (resp. $d(\sigma) = r(\tau)$ and $d(\tau) = r(\varrho)$), and $M_\sigma S_\tau = \{0\}$ (resp. $R_\sigma M_\tau S_\varrho = \{0\}$) otherwise. Let gr-$S$ (resp. R-gr-S) denote the category of graded left R-modules (resp. graded R-S-bimodules). The morphisms $f : M \to N$ are taken to be right R-module (resp. R-S-bimodule) maps such that $f(M_\sigma) \subseteq N_\sigma$ for all $\sigma \in \Gamma$. The obvious change in the definition of the suspension for graded right modules is left to the reader.

If M is a graded right R-module and N is a graded left R-module, then we may consider $M \otimes_R N$ as an object in Ab_Γ, where the grading is defined by letting $(M \otimes_R N)_\sigma$, $\sigma \in \Gamma$, be the $\mathbb{Z}$-module generated by all $m_\tau \otimes n_\varrho$, $d(\tau) = r(\varrho)$, $\tau\varrho = \sigma$, $m_\tau \in M_\tau$, $n_\varrho \in N_\varrho$. To see that this is well defined, note that $M \otimes_R N = M \otimes_{\mathbb{Z}} N/L$ where L is the graded subgroup of $M \otimes_{\mathbb{Z}} N$ generated by elements of the form $mr \otimes n - m \otimes rn$. The grading on $M \otimes_R N$ is therefore induced by the grading on $M \otimes_{\mathbb{Z}} N$.

For the rest of the article, we fix another graded ring S. Now we state some elementary properties concerning HOM and $\otimes$.

2.3.5. PROPOSITION. *Let M be a graded right R-module*, *N a graded R-S-bimodule and P a graded right S-module. Then*:

(a) *$M \otimes_R N$ is a graded right S-module.*
(b) $\mathrm{HOM}_S(N, P)$ *is a graded right R-module.*

(c) *There is an isomorphism in* Ab_Γ:

$$\mathrm{HOM}_S(M \otimes_R N, P) \cong \mathrm{HOM}_R(M, \mathrm{HOM}_S(N, P)).$$

Proof. Analogous to the ungraded case (see [10]). ∎

We end this section by remarking that the functor U has a right adjoint

$$G : R\text{-mod} \to R\text{-gr}$$

which to a left R-module M associates $G(M) = \bigoplus_{\sigma\in\Gamma} {}^{\sigma}M$, where ${}^{\sigma}M = \{{}^{\sigma}x \mid x \in M\}$, with an R-module structure defined by

$$\begin{cases} {}^{\tau}x + {}^{\tau}y = {}^{\tau}(x+y), \\ r \cdot {}^{\tau}x = \sum_{\sigma\in\Gamma,\, d(\sigma)=r(\tau)} {}^{\sigma\tau}(r_\sigma x), \end{cases}$$

$\tau \in \Gamma$, $x, y \in M$, $r \in R$. If $f : M \to N$ is R-linear, then $G(f) : G(M) \to G(N)$ is defined by $G(f)({}^{\sigma}x) = {}^{\sigma}f(x)$, $\sigma \in \Gamma$, $x \in M$. It is easy to check that G is exact.

3. Further results. In this section, we give an answer to the general question raised in the introduction.

3.1. *Direct summands.* Let A and B be objects in an abelian category. Recall that B is called a *direct summand* of A if there is an object C in the category such that $A \cong B \oplus C$.

The following lemma will be used frequently in what follows.

3.1.1. Lemma. *Let M, N and P be graded left R-modules and suppose that $f : M \to P$, $g : N \to P$ and $h : M \to N$ are R-linear maps such that $f = g \circ h$. If f and g (resp. f and h) are graded maps, then there is a graded map $h' : M \to N$ (resp. $g' : N \to P$) such that $f = g \circ h'$ (resp. $f = g' \circ h$).*

Proof. Let f and g be graded maps. It is enough to define h' on each M_σ, $\sigma \in \Gamma$. For $x_\sigma \in M_\sigma$, $\sigma \in \Gamma$, let $h'(x_\sigma) = h(x_\sigma)_\sigma$. Then $g(h'(x_\sigma)) = g(h(x_\sigma)_\sigma) = (g(h(x_\sigma)))_\sigma = f(x_\sigma)_\sigma = f(x_\sigma)$. The second part is proved in the same way. ∎

We immediately get the following:

3.1.2. Corollary. *Let M and N be graded left R-modules. If N is a graded submodule of M, then N is a direct summand of M if and only if $U(N)$ is a direct summand of $U(M)$.*

3.2. *Free modules.* We say that a graded left R-module M is *free* (*of finite type*) if there are $\sigma_i \in \Gamma$, $i \in I$ (I finite), such that $M \cong \bigoplus_{i\in I} R(\sigma_i)$.

It turns out that neither (I) nor (II) holds for the property of being free (of finite type):

3.2.1. Example. (i) Let T be a ring. Suppose that T and Γ are chosen so that every finitely generated projective left module (without grading) over $A := T[\Gamma]$ can be written uniquely, up to permutation of the factors, as a finite direct sum of indecomposable A-modules (e.g. if T is a field and Γ is finite). Fix $\sigma \in \Gamma$. Then $A(\sigma)$ is, by definition, a free graded left A-module. But $U(A(\sigma))$ is free as a left A-module if and only if Γ is a group. In fact, this follows directly from the direct sum decomposition $A = A(\sigma) \oplus \bigoplus_{\tau \in \Gamma_0,\, \tau \neq d(\sigma)} A(\tau)$ and the assumptions on A. Hence, (I) does not hold in general.

(ii) The implication (II) does not hold in general. For a counterexample in the case when Γ is a group, see p. 8 in [9].

In spite of the above example, we can always prove the following:

3.2.2. Proposition. *Let M be a free graded left R-module* (*of finite type*). *Then there is a free graded left R-module M'* (*of finite type*) *such that $U(M \oplus M')$ is free* (*of finite type*).

Proof. It is enough to prove the result in the case when $M = R(\sigma)$ for some $\sigma \in \Gamma$. Put $M' = \bigoplus_{\tau \in \Gamma_0,\, \tau \neq d(\sigma)} R(\tau)$. Then $U(M \oplus M') \cong U(R)$. ∎

3.3. *Presentation of modules.* Let M be a graded left R-module. If n is a non-negative integer, then we say that M has a (*finite*) *presentation of length n* if there is an exact sequence

$$F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

of free graded left R-modules (of finite type) and graded maps. If M has a (finite) presentation of length 0, then we say that M is (*finitely*) *generated*. If M has a (finite) presentation of length 1, then we say that M is (*finitely*) *presented*.

3.3.1. Proposition. *Let M be a graded left R-module. Then*:

(a) *If M has a* (*finite*) *presentation of length n, then $U(M)$ has a* (*finite*) *presentation of length n.*
(b) *The module M is* (*finitely*) *generated if and only if $U(M)$ is* (*finitely*) *generated.*

Proof. (a) Let

$$A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_0} M \to 0 \tag{3}$$

be an exact sequence of free graded left R-modules (of finite type) and graded maps. In $n+1$ steps we will now transform (3) into an exact sequence

$$B_n \xrightarrow{g_n} B_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_0} M \to 0 \tag{4}$$

of free graded left R-modules (of finite type) and graded maps.

Step 0. By Proposition 3.2.2, there is a free graded left R-module A_0' (of finite type) such that $U(A_0 \oplus A_0')$ is free (of finite type). If $0 \le i \le n$, put $A_i^0 = A_i \oplus A_0'$ and define $f_i^0 : A_i^0 \to A_{i-1}^0$ by

$$f_i^0(a_i \oplus a_0') = \begin{cases} f_0(a_0) & \text{if } i = 0, \\ f_i(a_i) & \text{if } 0 < i \le n,\ i \text{ even}, \\ f_i(a_i) \oplus a_0' & \text{otherwise}, \end{cases}$$

for all $a_i \in A_i$ and all $a_0 \in A_0'$. It is easy to check that the sequence

$$A_n^0 \xrightarrow{f_n^0} A_{n-1}^0 \xrightarrow{f_{n-1}^0} \cdots \xrightarrow{f_0^0} M \to 0 \tag{5}$$

is exact.

Step 1. Repeat the above procedure for the first n modules in (5). This gives us another exact sequence

$$A_n^1 \xrightarrow{f_n^1} A_{n-1}^1 \xrightarrow{f_{n-1}^1} \cdots \xrightarrow{f_0^1} M \to 0$$

where $A_0^1 = A_0^0$, $f_0^1 = f_0^0$, $\mathrm{Im}(f_1^1) = \mathrm{Im}(f_1^0)$ and $U(A_0^1)$ is free (of finite type).

Continuing like this in $n-1$ more steps, we can put $B_i = A_i^i$ and $g_i = f_i^n$ for $i = 0, \ldots, n$.

(b) If M is (finitely) generated, then, by (a), $U(M)$ is (finitely) generated.

Assume that $U(M)$ is (finitely) generated. Let X be a (finite) generating subset of $U(M)$ consisting of non-zero homogeneous elements. For $x \in X$, define a graded map $f_x : R(\deg(x)^{-1}) \to M$ by $f_x(r) = rx$, $r \in R(\deg(x)^{-1})$. If we put $F = \bigoplus_{x\in X} R(\deg(x)^{-1})$, then the maps f_x, $x \in X$, induce, in a canonical way, a surjective graded map $f_X : F \to M$. ∎

3.3.2. Remark. It is not clear if the converse to Proposition 3.3.1(a) holds in general.

To prove the next proposition, we need a lemma.

3.3.3. Lemma. *Let M_1 and M_2 be graded left R-modules. If $f : M_1 \to M_2$ is a graded map, then there is a free graded left R-module F and a graded map $g : F \to M_1$ such that the sequence $F \xrightarrow{g} M_1 \xrightarrow{f} M_2$ is exact.*

Proof. Put $K = \ker(f)$. By the proof of Proposition 3.3.1(b), there is a free graded left R-module F and a surjective graded map $h : F \to K$. If $i : K \to M$ denotes the inclusion, then we can put $g = i \circ h$. ∎

3.3.4. Proposition. *If M is a graded left R-module and n is a non-negative integer, then*:

(a) *The module M admits a presentation of length n.*
(b) *There is a free graded left R-module F and a graded submodule K of F such that $F/K \cong M$.*
(c) *The module M is presented.*

Proof. To prove (a), apply Lemma 3.3.3 repeatedly, starting with $M_1 = M$, $M_2 = 0$ and $f = 0$.

(b) and (c) follow directly from (a) with $n = 0$ and $n = 1$ respectively. ∎

3.3.5. Proposition. *Every graded left R-module is the direct limit of a direct system of finitely presented graded left R-modules and graded maps.*

Proof. Our proof is analogous to that in the ungraded case, given in [1].

Fix a graded left R-module M. By Proposition 3.3.4(c), there is a presentation of M:

$$\bigoplus_{i\in X} R(\sigma_i) \xrightarrow{u} \bigoplus_{j\in Y} R(\tau_j) \xrightarrow{v} M \to 0,$$

where $\sigma_i, \tau_j \in \Gamma$. For $X' \subseteq X$ and $Y' \subseteq Y$, put $M_{X'} = \bigoplus_{i\in X'} R(\sigma_i)$ and $M^{Y'} = \bigoplus_{j\in Y'} R(\tau_j)$ and let

$$I = \{(X', Y') \mid X' \subseteq X,\ Y' \subseteq Y,\ |X'|, |Y'| < \infty,\ u(M_{X'}) \subseteq M^{Y'}\}.$$

For $\alpha = (X', Y') \in I$, let $u_\alpha : M_{X'} \to M^{Y'}$ denote the graded map induced by u. If we put $M_\alpha = \operatorname{coker}(u_\alpha)$, and we let $v_\alpha : M^{Y'} \to M_\alpha$ denote the canonical graded map, then we get the following commutative diagram of graded left R-modules and graded maps, with exact rows:

$$\begin{tikzcd}
M_{X'} \arrow[r, "u"] \arrow[d, "i_\alpha"'] & M^{Y'} \arrow[r, "v"] \arrow[d, "j_\alpha"'] & M_\alpha \arrow[r] \arrow[d, "f_\alpha"'] & 0 \\
M_X \arrow[r, "u_\alpha"] & M^Y \arrow[r, "v_\alpha"] & M \arrow[r] & 0
\end{tikzcd}$$

where i_α and j_α are the canonical injections and f_α is induced from j by passage to quotients. For $\alpha = (X', Y')$ and $\beta = (X'', Y'')$ in I, put $\alpha \le \beta$ if $X' \subseteq X''$ and $Y' \subseteq Y''$. In that case, let $\varphi_{\beta\alpha} : M_\alpha \to M_\beta$ be defined in the canonical way. Since $f_\beta \circ \varphi_{\beta\alpha} = f_\alpha$, $\alpha, \beta \in I$, $\alpha \le \beta$, we can pass to the direct limits and still get a commutative diagram of graded left R-modules and graded maps, with exact rows:

$$\begin{tikzcd}
\varinjlim M_{X'} \arrow[r, "u"] \arrow[d, "i"'] & \varinjlim M^{Y'} \arrow[r, "v"] \arrow[d, "j"'] & \varinjlim M_\alpha \arrow[r] \arrow[d, "f"'] & 0 \\
M_X \arrow[r, "u_\alpha"] & M^Y \arrow[r, "v_\alpha"] & M \arrow[r] & 0
\end{tikzcd}$$

Since i and j are isomorphisms, also f is an isomorphism (e.g. by the five lemma). ∎

3.4. *Projective modules.* Recall that an object A in an abelian category $\mathcal{A}$ is called *projective* if the functor $\operatorname{Hom}(A, \cdot) : \mathcal{A} \to \mathrm{Ab}$ is exact.

To prove our next result, we need a well known proposition and a lemma:

3.4.1. Proposition. *Let $\mathcal{A}$ be an abelian category. Then*:

(a) *If* $(A_i)_{i\in I}$ *is a family of objects in* $\mathcal{A}$, *then* $\bigoplus_{i\in I} A_i$ *is projective if and only if each* A_i *is projective.*
(b) *If* $0 \to A \to B \xrightarrow{\alpha} C \to 0$ *is an exact sequence in* $\mathcal{A}$, *then the sequence splits if and only if there is* $\beta : C \to B$ *such that* $\alpha\circ\beta = \mathrm{id}_C$.

Proof. Both (a) and (b) are standard facts which can be found e.g. in [11]. ∎

3.4.2. Lemma. *If a graded left* R*-module is free*, *then it is projective.*

Proof. By Proposition 3.4.1(a), it is enough to prove the result for $R(\sigma)$, $\sigma \in \Gamma$. Fix $\sigma \in \Gamma$. Take graded left R-modules M_1 and M_2 and assume that there are graded maps $f : R(\sigma) \to M_2$ and $g : M_1 \to M_2$ such that g is surjective. Take $x \in M_1$ such that $g(x) = f(1_{d(\sigma)})$ and define an R-linear map $h : R(\sigma) \to M_1$ by $h(r) = rx$, $r \in R(\sigma)$. Since $R(\sigma)$ is the left principal ideal of R generated by $1_{d(\sigma)}$, and $1_{d(\sigma)}$ is an idempotent, we get $f = g \circ h$. By Lemma 3.1.1, we can assume that h is a graded map. ∎

3.4.3. Proposition. *Let* M *be a graded left* R*-module. Then* M *is projective if and only if* $U(M)$ *is projective.*

Proof. By Proposition 3.3.4(b), there are graded left R-modules F and K such that F is free, K is a graded submodule of F and $M \cong F/K$. Consider the canonical exact sequence

$$0 \to K \to F \xrightarrow{p} F/K \to 0. \tag{6}$$

If M is projective, then, by Proposition 3.4.1(b), (6) splits, which in turn, by Proposition 3.2.2, implies that M is a direct summand of some graded left R-module F' with the property that $U(F')$ is free. Hence, by Corollary 3.1.2 and Proposition 3.4.1(a), $U(M)$ is projective.

Now assume that $U(M)$ is projective. Then there is an R-linear map $f : F/K \to F$ such that $p \circ f = \mathrm{id}_{F/K}$. By Lemma 3.1.1, we can assume that f is a graded map. Therefore, by Proposition 3.4.1(b), (6) splits, and so M is a direct summand of the free graded left R-module F. But by Lemma 3.4.2, F is projective, which, by Proposition 3.4.1(a), implies that M is projective. ∎

As a direct consequence of Propositions 3.4.3 and 3.3.1(b), we get:

3.4.4. Corollary. *Let* M *be a graded left* R*-module. Then* M *is finitely generated and projective if and only if* $U(M)$ *is finitely generated and projective.*

3.5. *Injective modules.* Recall that an object A in an abelian category $\mathcal{A}$ is called *injective* if the functor $\mathrm{Hom}(\cdot, A) : \mathcal{A} \to \mathrm{Ab}$ is exact.

We need the following well known result about injective objects in abelian categories:

3.5.1. Proposition. *Let $(A_i)_{i\in I}$ be a family of objects in an abelian category. Then $\prod_{i\in I} A_i$ is injective if and only if each A_i is injective.*

Proof. This is a standard fact which can be found e.g. in [11]. ∎

Now we give a description of the injective objects in R-gr analogous to Baer's criterion (see e.g. [10]).

3.5.2. Proposition. *Let M be a graded left R-module. Then the following three statements are equivalent*:

(i) *The module M is injective.*
(ii) *The functor* $\mathrm{HOM}_R(\cdot, M) : R\text{-mod} \to \mathrm{Ab}_\Gamma$ *is exact.*
(iii) *For every graded left ideal I of R, the canonical map*

$$\mathrm{HOM}_R(R, M) \to \mathrm{HOM}_R(I, M)$$

is surjective.

Proof. We first show that (i) implies (ii). Since Σ_σ is a finite set for all $\sigma \in \Gamma$, we get, by Proposition 2.2.1(c), Propositions 2.2.3(b) and 3.5.1:

$$\begin{aligned} M \text{ is injective } &\Rightarrow\ \forall \sigma\in\Gamma,\ T_{\Sigma_\sigma}(M) \text{ is injective}\\ &\Rightarrow\ \forall\sigma\in\Gamma,\ M(\sigma) \text{ is injective}\\ &\Rightarrow\ \forall\sigma\in\Gamma,\ \mathrm{HOM}_R(\cdot, M(\sigma))_{\Gamma_0} \text{ is exact}\\ &\Rightarrow\ \forall\sigma\in\Gamma,\ \mathrm{HOM}_R(\cdot, M)_\sigma \text{ is exact}\\ &\Rightarrow\ \mathrm{HOM}_R(\cdot, M) \text{ is exact.}\end{aligned}$$

The implication (ii)⇒(iii) is evident.

Now suppose that (iii) holds. We show (i). Let N and P be graded left R-modules and suppose that there are graded maps $f : N \to M$ and $i : N \to P$ such that i is injective. We want to construct a graded map $\overline{f} : P \to M$ such that $f = \overline{f} \circ i$. Let $\mathcal{F}$ denote the collection of graded maps $f' : P' \to M$ such that $i(N) \subseteq P' \subseteq P$ (where P' is a graded left R-module) and $f|'_{i(N)} \circ i = f$. For $f', f'' \in \mathcal{F}$, put $f' \le f''$ if f'' extends f'. By Zorn's lemma, we can find a maximal $\overline{f} \in \mathcal{F}$, $\overline{f} : \overline{P} \to M$. Seeking a contradiction, assume that $\overline{P} \subsetneq P$. Then we can pick a homogeneous $x \in P \setminus \overline{P}$ of degree, say, $\sigma \in \Gamma$. Put $I = \{r \in R \mid rx \in \overline{P}\}$. Then I is a graded left ideal of R. If we define $\alpha : I \to M$ by $\alpha(r) = \overline{f}(rx)$, $r \in I$, then the degree of α is σ, and hence, by (iii), there is $y \in M_\sigma$ such that $\alpha(r) = ry$, $r \in I$. If we now put $\widetilde{P} = \overline{P} + Rx$ and define $\widetilde{f} : \widetilde{P} \to M$ by $\widetilde{f}(p + rx) = \overline{f}(p) + ry$, $p \in \overline{P}$, $r \in R$, then $\widetilde{f}$ is a well defined graded map that extends $\overline{f}$ non-trivially, which gives a contradiction. ∎

By the above result, we immediately get:

3.5.3. Corollary. *Let M be a graded left R-module. If $U(M)$ is injective, then M is injective.*

3.5.4. Remark. The converse to Corollary 3.5.3 does not hold in general. For a counterexample in the case when Γ is a group, see p. 8 of [9].

3.6. *Essential and small subobjects.* Let A be an object in an abelian category. Recall that a subobject B of A is called *essential* (resp. *small*) in A if $B \cap C \neq 0$ (resp. $B + C \neq A$) for every non-zero subobject C of A.

3.6.1. Proposition. *Let M and N be graded left R-modules, where N is a graded submodule of M. Then:*

(a) *The module N is essential in M if and only if $U(N)$ is essential in $U(M)$.*
(b) *If $U(N)$ is small in $U(M)$, then N is small in M.*

Proof. (a) If $U(N)$ is essential in $U(M)$, then, trivially, N is essential in M.

Assume now that N is essential in M. Take a non-zero submodule P of $U(M)$. We show that $N \cap P \neq \{0\}$. Pick $x \in P \setminus \{0\}$ and let $x = \sum_{i=1}^{n} x_{\sigma_i}$, $\sigma_i \in \Gamma$, $x_{\sigma_i} \in M_{\sigma_i} \setminus \{0\}$, $i = 1, \ldots, n$. By induction over n, we show that $N \cap Rx \neq \{0\}$. If $n = 1$, then $x \in M_{\sigma_1}$, which implies that Rx is a non-zero graded submodule of M. Hence, since N is essential in M, $N \cap Rx \neq \{0\}$. Assume now that $n > 1$. Since Rx_{σ_1} is a non-zero graded submodule of M, there is (again since N is essential in M) $a \in R$ (which we can assume to be homogeneous) such that $ax_{\sigma_1} \in N \setminus \{0\}$. Put $y = x - x_{\sigma_1}$. Then ay has at most $n-1$ non-zero homogeneous components. Therefore, by the inductive hypothesis, there is $b \in R$ (which we can also assume to be homogeneous) such that $bay \in N \setminus \{0\}$. Thus, $Rx \ni bax = bax_{\sigma_1} + bay \in N \setminus \{0\}$.

(b) is immediate. ∎

3.6.2. Remark. The converse to Proposition 3.6.1(b) does not hold in general. For a counterexample in the case when Γ is a group, see p. 10 of [9].

3.7. *Flat modules.* We say that a graded left R-module M is *flat* if the functor $- \otimes_R M : \text{gr-}R \to \mathrm{Ab}_\Gamma$ is exact.

Before we prove the next proposition, we need another lemma.

3.7.1. Lemma. *Let P be a graded right R-module, M a graded S-R-bimodule and N a graded left S-module. Then there is a graded canonical map*

$$P \otimes_R \mathrm{HOM}_S(M, N) \to \mathrm{HOM}_S(\mathrm{HOM}_R(P, M), N).$$

If P is finitely generated and projective, then this map is an isomorphism.

Proof. If we use Propositions 2.3.2(a),(b) and 3.2.2, then we can proceed exactly as in the ungraded case. For the details, see e.g. [10]. ∎

Now we give a description of the flat modules in R-gr analogous to the corresponding classical ungraded result (see [1] or [8]).

3.7.2. Proposition. *Let M be a graded left R-module. Then the following five statements are equivalent*:

(i) *The module $U(M)$ is flat.*
(ii) *The module M is flat.*
(iii) *For every finitely presented graded left R-module P, the canonical graded map $\mathrm{HOM}_R(P,R) \otimes_R M \to \mathrm{HOM}_R(P,M)$ is surjective.*
(iv) *For every finitely presented graded left R-module P and each semi-graded map $u : P \to M$, there is a graded left R-module F, free of finite type, such that $U(F)$ is free of finite type, and there are semi-graded maps $v : P \to F$ and $w : F \to M$ such that $u = w \circ v$.*
(v) *The module M is the direct limit of free graded left R-modules F_i, $i \in I$, of finite type, such that each $U(F_i)$ is free of finite type.*

Proof. The implication (i)⇒(ii) is trivial.

Now suppose that (ii) holds. We show (iii). Since P is finitely presented, there are graded left R-modules F_0 and F_1, free of finite type, and an exact sequence of graded maps

$$F_1 \xrightarrow{v} F_0 \xrightarrow{w} P \to 0.$$

By the proof of Proposition 3.3.1(a), we can assume that $U(F_0)$ and $U(F_1)$ are also free of finite type. If we use, for a graded right R-module A, the notation $A_M := A \otimes_R M$, then the above sequence induces a commutative diagram of graded modules and graded maps:

$$\begin{array}{ccccc}
\mathrm{HOM}_R(P,R)_M & \xrightarrow{i} & \mathrm{HOM}_R(F_0,R)_M & \longrightarrow & \mathrm{HOM}_R(F_1,R)_M \\
\downarrow{\scriptstyle v_P} & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v_1} \\
\mathrm{HOM}_R(P,M) & \xrightarrow{j} & \mathrm{HOM}_R(F_0,M) & \longrightarrow & \mathrm{HOM}_R(F_1,M)
\end{array}$$

By Proposition 2.3.3, the bottom row is exact and j is injective. By the same proposition and the fact that M is flat, the top row is also exact and i is injective. By Proposition 2.3.2(a),(b), v_0 and v_1 are isomorphisms. Hence, by a standard diagram chase, v_P is surjective.

Suppose that (iii) holds. We show (iv). Let P be a finitely presented graded left R-module and take a semi-graded map $u : P \to M$. Suppose that $u = u_1 + \cdots + u_n$ is a decomposition of u into homogeneous components. By (iii), there are semi-graded maps $f_i : P \to R$ and $m_i \in M$, $i = 1, \ldots, n$, such that all f_i and all m_i are homogeneous and $u_i(x) = f_i(x)m_i$ for all $i = 1, \ldots, n$. Define semi-graded maps $v : P \to R^n$ and $w : R^n \to M$ by $v(x) = (f_i(x))_{i=1}^n$, $x \in P$, and $w((r_i)_{i=1}^n) = \sum_{i=1}^n r_i m_i$, $r_i \in R$, $i = 1, \ldots, n$. Then $u = w \circ v$.

The implication (iv)⇒(v) can be proved in exactly the same way as in the ungraded case (see e.g. [1]).

Now assume that (v) holds. Since all the $U(F_i)$ are free, they are flat. But a direct limit of flat modules is again flat (see e.g. [11]), so (i) holds. ■

3.8. *Pure sequences.* Let M, M' and M'' be graded left R-modules. We call an exact sequence of graded maps

$$0 \to M' \xrightarrow{u} M \xrightarrow{u'} M'' \to 0 \tag{7}$$

pure if for every graded right R-module N, the induced sequence

$$0 \to N \otimes_R M' \to N \otimes_R M \to N \otimes_R M'' \to 0$$

is also exact.

The last result of this article gives a characterization of pure sequences in R-gr analogous to the corresponding ungraded result obtained in [8].

3.8.1. Proposition. *With the above notations, the following five statements are equivalent:*

(i) *The sequence* $0 \to U(M') \xrightarrow{u} U(M) \xrightarrow{u'} U(M'') \to 0$ *is pure.*
(ii) *The sequence* (7) *is pure.*
(iii) *Consider a commutative diagram of graded left R-modules*

$$\begin{tikzcd} F' \arrow[r, "v"] \arrow[d, "i"'] & F \arrow[d, "j"'] \\ M' \arrow[r, "u"] & M \end{tikzcd}$$

where i, j and v are semi-graded maps. If F', $U(F')$, F and $U(F)$ are free of finite type, then there is a semi-graded map $w : F \to M'$ such that $i = w \circ v$.
(iv) *For every finitely presented graded left R-module P, the map*

$$\mathrm{HOM}_R(P, M) \to \mathrm{HOM}_R(P, M'')$$

induced by u' is surjective.
(v) *The sequence* (7) *is the direct limit of split sequences*

$$0 \to M' \to M' \oplus P_i \to P_i \to 0$$

of graded left R-modules, where each P_i is finitely presented.

Proof. The implication (i)⇒(ii) is trivial and the implication (ii)⇒(iii) can be proved in exactly the same way as in the ungraded case (see e.g. [8] or [11]).

Now assume that (iii) holds. Take a semi-graded map $f : P \to M''$, where P is a finitely presented graded left R-module. We construct a semi-graded map $h : P \to M$ such that $u' \circ h = f$. There is an exact sequence of graded left R-modules and graded maps $F' \xrightarrow{v} F \xrightarrow{v'} P \to 0$, where F' and F are free of finite type. By the proof of Proposition 3.3.1(a), we can assume that

$U(F')$ and $U(F)$ are free of finite type. There is an induced commutative diagram

$$\begin{tikzcd} F' \arrow[r, "v"] \arrow[d, "i"'] & F \arrow[r, "v'"] \arrow[d, "j"'] & P \arrow[d, "f"'] \\ M' \arrow[r, "u"] & M \arrow[r, "u'"] & M'' \end{tikzcd}$$

By (iii), there is a semi-graded map $w : F \to M'$ such that $i = w \circ v$. If we put $g = j - u \circ w$, then, since $g \circ v = j \circ v - u \circ w \circ v = j \circ v - u \circ i = 0$, we can define a semi-graded map $h : P \to M$ such that $h \circ v' = g$. Then $u' \circ h \circ v' = u' \circ g = u' \circ j - u' \circ u \circ w = f \circ v'$, which, since v' is surjective, implies that $u' \circ h = f$.

Now assume that (iv) holds. We show (v). By Proposition 3.3.5, M'' is the direct limit of finitely presented graded left R-modules P_i, $i \in I$. Let M_i be the fiber product of P_i and M mapping to M'', that is, $M_i = \{(m, p) \in M \times P_i \mid u'(m) = p\}$. Then M_i is a graded left R-module in a natural way. Let Q_i denote the kernel of the canonical surjection $M_i \to P_i$. This gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Q_i \arrow[r, "u_i"] \arrow[d, "f_i'"'] & M_i \arrow[r, "u_i'"] \arrow[d, "f_i"'] & P_i \arrow[r] \arrow[d, "f_i''"'] & 0 \\ 0 \arrow[r] & M' \arrow[r, "u"] & M \arrow[r, "u'"] & M'' \arrow[r] & 0 \end{tikzcd}$$

where $u_i, u_i', f_i, f_i', f_i''$ are defined in the natural way. Then the rows are exact. By (iv) there is a semi-graded map $g_i : P_i \to M$ such that $u' \circ g_i = f_i''$. By the universal property of the fiber product, there is a semi-graded map $u_i'' : P_i \to M_i$ such that $u_i' \circ u_i'' = \mathrm{id}_{P_i}$. Hence, the top horizontal sequence splits in R-gr (see Corollary 3.1.2). If we now pass to the direct limit, we can, since the f_i' are isomorphisms, use the five lemma to get the desired result.

The implication (v)$\Rightarrow$(i) follows directly since the direct limit is an exact functor (see e.g. [10]). ∎

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