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PARTLY DISSIPATIVE SYSTEMS IN UNIFORMLY LOCAL SPACES

BY

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Abstract. We study the existence of attractors for partly dissipative systems in $\mathbb{R}^n$. For these systems we prove the existence of global attractors with attraction properties and compactness in a slightly weaker topology than the topology of the phase space. We obtain abstract results extending the usual theory to encompass such two-topologies attractors. These results are applied to the FitzHugh–Nagumo equations in $\mathbb{R}^n$ and to Field–Noyes equations in $\mathbb{R}$. Some embeddings between uniformly local spaces are also proved.

1. Introduction. The prototype for the problems considered in this paper is the FitzHugh–Nagumo system in $\mathbb{R}^n$,

$$(1.1)\qquad \begin{cases} u_t = \Delta u - \alpha v + f(u), \\ v_t = -\delta v + \beta u + h(x), \qquad t > 0,\ x \in \mathbb{R}^n, \\ u(0) = u_0, \quad v(0) = v_0, \end{cases}$$

where α, β, δ are positive constants and the assumptions on f and h will be specified later.

It is known that the above system, for $n = 1$, may exhibit a relaxation wave solution (see [15]). We aim to give a result on existence of a global attractor for (1.1) in such a way that these relaxation wave solutions are included in it. Also, by setting $h(x) \equiv 0$ one may have constant equilibria for (1.1) and our attractor should include these equilibria as well. There have been some efforts to obtain the existence of an attractor for (1.1) in $\mathbb{R}^n$ (see, for example, [21]) but, in this case, none of the above described special solutions are in the attractor. This is due to the fact that problem (1.1) has been set in usual Sobolev spaces (say $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$), which require that the elements in the attractor "vanish" at infinity and therefore do not include constant functions or relaxation wave solutions.

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In order to include these special solutions (equilibria and relaxation waves) in the attractor we must choose sufficiently "large" spaces to work in. One attempt is to consider weighted L^p spaces as in [1, 3]. These spaces are defined in the usual way by replacing the Lebesgue measure dx with $\varrho(x)\,dx$, where the weight ϱ is a positive integrable $C^2(\mathbb{R}^n)$ function satisfying some additional conditions that will be specified later. The disadvantage of such spaces is that they ignore the behavior of the solutions for large spatial values, and moreover, the usual Sobolev type embeddings are not available for them. In [9] locally compact attractors are considered for damped wave equations and this idea led many people to work in uniformly local Sobolev spaces which are the completions of $C_B^\infty(\mathbb{R}^n)$ (C^∞ functions bounded with all partial derivatives in $\mathbb{R}^n$) in the norm $\|u\|_{W^{m,p}_{\mathrm{lu}}(\mathbb{R}^n)} = \sup_{y\in\mathbb{R}^n} \|u\|_{W^{m,p}_{\varrho_y}(\mathbb{R}^n)}$ (here $\varrho_y(x) = \varrho(x-y)$). In [19] the existence of global attractors for problems like the Ginzburg–Landau equation is established in the framework of uniformly local Sobolev spaces. The existence of attractors in unbounded domains has been studied in many other works, e.g. [1, 10, 18, 19, 4].

The theory of attractors introduced in [3] includes the possibility that the attractor is not compact in the phase space. This is also the case in [9, 19]. In [19] an abstract definition of the so called (Z-Z_ϱ)-attractors is given. Here Z and Z_ϱ are Banach spaces with $\|\cdot\|_{Z_\varrho} \le c\|\cdot\|_Z$. In this case the attractor must attract bounded subsets of Z in the norm of Z_ϱ, and the attractor is compact in Z_ϱ. This approach differs from the theory in [11] by the fact that in the latter Z and Z_ϱ are the same space.

Here we give abstract conditions for the existence of (Z-Z_ϱ)-attractors and then apply them to partly dissipative parabolic partial differential equations like (1.1).

2. Abstract results. One of the basic questions in the discussion of asymptotics of semigroups is what are the weakest conditions necessary for the existence of a (nonempty) compact invariant set. Evidently, dissipativity itself is not enough to obtain this strong conclusion. Namely, a bounded dissipative system may be constructed which does not have a compact invariant set (see [7]; also [6]). It is well known that if the dynamical system acting on a metric space $\mathcal{X}$ is S-dissipative and asymptotically smooth, then there is a compact invariant set which attracts S-sets (S-sets may be points, neighborhood of points, compact sets or bounded sets). The aim of this section is to formulate a similar result but under a weaker hypothesis concerning asymptotic smoothness. This result may be applied to systems which (generically) are not compact at infinity.

In this section we recall the notion of an (E-E_ϱ)-*attractor* introduced in [19] and give a condition for the existence of such attractors. That result can be applied to many problems in unbounded domains. We follow closely the developments in [11] and in [16] making the necessary changes to accommodate the lack of compactness in such problems.

A typical result on existence of global attractors states that *a point dissipative, bounded and asymptotically smooth continuous semigroup* $\{T(t): t \geq 0\}$ *on a complete metric space* E *has a compact global attractor*. This attractor is invariant and attracts bounded sets under the semigroup. We try to obtain similar conditions for the case when asymptotic compactness of the semigroup is not available in the usual phase space E but is available in a larger space E_ϱ in which E is continuously embedded.

Assumptions. We shall assume that:

(I) E and E_ϱ are metric spaces with metrics d and d_ϱ respectively (not necessarily complete),

(II) $E \subset E_\varrho$ algebraically and topologically,

(III) there exists $B_0 \subset E$ with $\operatorname{diam}_d(B_0) < \infty$ such that

$$\forall_{B \subset E,\, \operatorname{diam}_d(B) < \infty}\ \exists_{t_B \geq 0} \qquad \bigcup_{t \geq t_B} T(t)(B) \subset B_0,$$

(IV) restriction of the semigroup $T(t) : E \to E$, $t \geq 0$, to the set B_0 is continuous in the following sense: for each $t > 0$, if $\{v_n\} \subset B_0$ converges to $v \in E$ in E_ϱ then $T(t)v_n \to T(t)v$ in E_ϱ,

(V) if $B \subset E$ is nonempty, $\operatorname{diam}_d(\bigcup_{t \geq t_B} T(t)(B)) < \infty$ for some $t_B \geq 0$, $\{u_n\} \subset B$, and $t_n \to \infty$, then $\{T(t_n)u_n\}$ has a subsequence convergent in the metric d_ϱ to some element $a \in E$.

Theorem 1. *Under the above assumptions there exists a nonempty set* $A \subset E$ *with the following properties*:

(i) $T(t)(A) = A,\ t \geq 0$,

(ii) $\operatorname{cl}_E(A) = A$,

(iii) *A is compact in the metric d_ϱ (in particular*, $\operatorname{cl}_{E_\varrho}(A) = \operatorname{cl}_E(A) = A$),

(iv) $\forall_{B \subset E,\, \operatorname{diam}_d(B) < \infty} \forall_{\mathcal{O}_{d_\varrho}(A) \textit{ neighborhood of } A \textit{ in metric } d_\varrho}\ \exists_{\tau_B \geq 0}$

$$\bigcup_{t \geq \tau_B} T(t)(B) \subset \mathcal{O}_{d_\varrho}(A).$$

Proof. Define

$$A := \{a \in E : T(t_n)u_n \xrightarrow{d_\varrho} a \text{ for some } \{u_n\} \subset B_0 \text{ and } t_n \to \infty\},$$

and note that $A \neq \emptyset$ as a consequence of assumptions (V) and (III).

To obtain (i) we first prove that $T(t)(A) \subset A$. For this we take $a \in A$, $\{u_n\} \subset B_0$, $t_n \to \infty$ such that $d_\varrho(T(t_n)u_n, a) \to 0$ and observe from (IV) that

$$d_\varrho(T(t + t_n)u_n, T(t)a) \to 0.$$

From the definition of A it is then clear that $T(t)a \in A$.

The inclusion $A \subset T(t)(A)$ is proved in a similar way. We take $a \in A$, $\{u_n\} \subset B_0$, $t_n \to \infty$ such that $d_\varrho(T(t_n)u_n, a) \to 0$ and observe (via (V)) that the sequence $\{T(t_n - t)u_n\}$ has a subsequence $\{T(t_{n_k} - t)u_{n_k}\}$ such that $d_\varrho(T(t_{n_k} - t)u_{n_k}, b) \to 0$ for some $b \in E$. From the definition of A we see that $b \in A$. Finally from (IV) we infer that $d_\varrho(T(t_{n_k})u_{n_k}, T(t)b) \to 0$. Recalling that $d_\varrho(T(t_{n_k})u_{n_k}, a) \to 0$ we obtain the equality $a = T(t)b$.

To prove (ii) take $a \in \mathrm{cl}_E(A)$ and $\{a_n\} \subset A$ such that $d(a_n, a) \to 0$. From (II) we then have $d_\varrho(a_n, a) \to 0$ whereas from the definition of A for $n \in \mathbb{N}$ there exist $u_n \in B_0$ and $t_n > 0$ satisfying $d_\varrho(a_n, T(t_n)u_n) < 1/n$. We thus have $d_\varrho(T(t_n)u_n, a) \le 1/n + d_\varrho(a_n, a) \to 0$, which shows that $a \in A$.

Condition (iii) is a consequence of (V). Indeed, if $\{a_n\} \subset A$ then (from the definition of A) there exist sequences $\{u_n\} \subset B_0$ and $t_n \to \infty$ such that $d_\varrho(a_n, T(t_n)u_n) < 1/n$. As a result of (V) (note that $\mathrm{diam}_E(B_0) < \infty$) there is a subsequence $\{T(t_{n_k})u_{n_k}\}$ convergent in the metric d_ϱ to some $a \in E$. By the definition of A we deduce that $a \in A$, and by the triangle inequality we have $d_\varrho(a, a_{n_k}) \le d_\varrho(a, T(t_{n_k})u_{n_k}) + d_\varrho(a_{n_k}, T(t_{n_k})u_{n_k}) \to 0$.

Suppose finally that (iv) is not true. Then we may choose $B \subset E$ with $\mathrm{diam}_d(B) < \infty$, $\varepsilon > 0$ and sequences $\{u_n\} \subset B$, $t_n \to \infty$ such that

$$\inf_{n \in \mathbb{N}} d_\varrho(T(t_n)u_n, A) > \varepsilon. \tag{2.1}$$

However, from (V), there exists a subsequence $\{T(t_{n_k})u_{n_k}\}$ and $a \in E$ such that $d_\varrho(T(t_{n_k})u_{n_k}, a) \to 0$. Then a must belong to A, which contradicts (2.1). ∎

Definition 1. A set A satisfying conditions (i)–(iv) of Theorem 1 is called an $(E\text{-}E_\varrho)$-*attractor* for the semigroup $\{T(t)\}$.

Definition 2. The property described in assumption (V) is called the $(E\text{-}E_\varrho)$-*asymptotic compactness* of $\{T(t)\}$.

In the language of the well known monograph [2] and recent articles [19], [18], the results of Theorem 1 may be expressed briefly as follows.

Corollary 1. *If $\{T(t)\}$ is a dissipative semigroup on a metric space E which satisfies the continuity assumption* (IV) *and is $(E\text{-}E_\varrho)$-asymptotically compact, then $\{T(t)\}$ has an $(E\text{-}E_\varrho)$-global attractor.*

Remark 1. Often in examples, where the spaces E and E_ϱ are specified, we immediately obtain boundedness of the set A in E. In that case A may be considered as a bounded global $(E\text{-}E_\varrho)$-attractor for $\{T(t)\}$ in E.

3. Application to FitzHugh–Nagumo equations. In this section we introduce a functional framework for problem (1.1) to put it in the abstract setting of Section 2. We start with a description of the function spaces that will be used throughout. Let $\varrho : \mathbb{R}^n \to (0, \infty)$ be a $C^2(\mathbb{R}^n)$ integrable weight function and denote by $L^p_\varrho(\mathbb{R}^n)$, $p>1$, the set of all functions φ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|\varphi\|_{L^p_\varrho(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |\varphi(x)|^p \varrho(x)\,dx\Big)^{1/p} < \infty.$$

In a similar way one defines the spaces $W^{m,p}_\varrho(\mathbb{R}^n)$.

In order to deal with energy estimates involving the application of the Divergence Theorem these weights should enjoy the following additional property:

$$|\nabla\varrho| \le \varrho_0\varrho, \qquad |\Delta\varrho| \le c\varrho, \tag{3.1}$$

where ϱ can be chosen such that the constant $\varrho_0>0$ is as small as needed. Such weights are well known in the literature (see [8]); an example is $\varrho_\varepsilon(x) = (1+|\varepsilon x|^2)^{-n}$. As mentioned previously, these weighted spaces are not appropriate (from our point of view) to describe the dynamics of evolution equations in $\mathbb{R}^n$; instead we will work in the spaces $W^{m,p}_{\mathrm{lu}}(\mathbb{R}^n)$ which are given as the completion of $C^\infty_B(\mathbb{R}^n)$ in the norm $\|\varphi\|_{W^{m,p}_{\mathrm{lu}}(\mathbb{R}^n)} = \sup_{y\in\mathbb{R}^n} \|\varphi\|_{W^{m,p}_{\varrho_y}(\mathbb{R}^n)}$, where $\varrho_y(x) = \varrho(x-y)$. Another definition is recalled in the Appendix.

Assumption (V) of the abstract formulation is the hard point in our examples. It will be clear from the considerations below that this assumption is connected both with special properties of the spaces involved, and with the smoothing action of the semigroup. We need the following considerations to verify assumption (V).

First we specify an extra assumption on the weight function ϱ:

$$\forall_{y\in\mathbb{R}^n}\ \exists_{c(y)}\ \sup_{x\in\mathbb{R}^n} \frac{\varrho(x-y)}{\varrho(x)} = c(y) < \infty; \tag{VI}$$

moreover, let $c(y)$ be bounded on bounded subsets of $\mathbb{R}^n$. It will be shown that assumption (VI) follows from condition (3.1).

Next we need to generalize the considerations of [19, p. 748]. We have the following properties of the closure in $H^1_\varrho(\mathbb{R}^n)$ of bounded sets in $H^2_{\mathrm{lu}}(\mathbb{R}^n)$.

Property 1. *Under assumption* (VI), *translation of argument*

$$T_y v(x) = v(x+y), \qquad y\in\mathbb{R}^n \text{ fixed},$$

is a continuous linear operator in $H^1_\varrho(\mathbb{R}^n)$.

Proof. We need to show that

$$v_m \to v \text{ in } H^1_\varrho(\mathbb{R}^n) \ \Rightarrow\ T_y v_m \to T_y v \text{ in } H^1_\varrho(\mathbb{R}^n).$$

It is sufficient to study one component appearing in the norm (the others are estimated in a similar way):

$$\Big(\int_{\mathbb{R}^n}|T_y v_m(x)-T_y v(x)|^2\varrho(x)\,dx\Big)^{1/2}=\Big(\int_{\mathbb{R}^n}|v_m(z)-v(z)|^2\varrho(z-y)\,dz\Big)^{1/2}$$
$$\le\Big(\sup_{z\in\mathbb{R}^n}\frac{\varrho(z-y)}{\varrho(z)}\Big)^{1/2}\Big(\int_{\mathbb{R}^n}|v_m(z)-v(z)|^2\varrho(z)\,dz\Big)^{1/2}\to 0$$

so the convergence is proved. ∎

Property 2. *The closure, in $H^1_\varrho(\mathbb{R}^n)$, of a ball $B_{H^1_{\rm lu}(\mathbb{R}^n)}(0,r)$ consists of elements* (*not necessarily "translation continuous"*; *see* (5.1)) *with*

$$\|v\|_{H^1_{\rm lu}(\mathbb{R}^n)}\le r.$$

Proof. Let $\{v_m\}\subset B_{H^1_{\rm lu}(\mathbb{R}^n)}(0,r)$ and $v_m\to v$ in $H^1_\varrho(\mathbb{R}^n)$. Then

$$\|T_y v\|_{H^1_\varrho(\mathbb{R}^n)}\le\|T_y v_m\|_{H^1_\varrho(\mathbb{R}^n)}+\|T_y v-T_y v_m\|_{H^1_\varrho(\mathbb{R}^n)}$$
$$\le\|v_m\|_{H^1_{\rm lu}(\mathbb{R}^n)}+c(y)\|v-v_m\|_{H^1_\varrho(\mathbb{R}^n)}=r+\varepsilon_m.$$

Letting first $m\to\infty$ we can then take the supremum over $y\in\mathbb{R}^n$ on the left hand side to get the result. ∎

Property 3. *If v belongs to the $H^1_\varrho(\mathbb{R}^n)$-closure of a ball $B_{H^2_{\rm lu}(\mathbb{R}^n)}(0,r)$, then $v\in H^1_{\rm lu}(\mathbb{R}^n)$, i.e. $\|v\|_{H^1_{\rm lu}(\mathbb{R}^n)}\le r$ and v is translation continuous in the $H^1_{\rm lu}(\mathbb{R}^n)$ norm.*

Proof. Because of Property 2 we need only show the translation continuity of v. We claim the following property of bounded subsets of $H^2_{\rm lu}(\mathbb{R}^n)$;

$$\text{(3.2)}\qquad \text{if}\quad v\in B_{H^2_{\rm lu}(\mathbb{R}^n)}(0,r)\quad\text{then}\quad \|T_z v-v\|_{H^1_\varrho(\mathbb{R}^n)}\le \mathrm{const}(|z|),$$

with a continuous function const : $[0,\infty)\to[0,\infty)$, independent of v and satisfying $\mathrm{const}(0)=0$.

Property (3.2) expresses the connection between generalized derivatives and difference quotients. The simplest way to show (3.2) is to use local characterizations of the spaces $H^1_{\rm lu}(\mathbb{R}^n)$ and $H^2_{\rm lu}(\mathbb{R}^n)$ (as in the Appendix) and compactness of the embedding $H^2(\Omega)\hookrightarrow H^1(\Omega)$ together with the M. Riesz criterion of compactness in $L^p(\Omega)$ for a bounded domain Ω.

Let now $v_m\to v$ in $H^1_\varrho(\mathbb{R}^n)$. Then

$$\|T_z(T_y v-v)\|_{H^1_\varrho(\mathbb{R}^n)}\le\|T_z(T_y v-T_y v_m)\|_{H^1_\varrho(\mathbb{R}^n)}+\|T_z(T_y v_m-v_m)\|_{H^1_\varrho(\mathbb{R}^n)}$$
$$+\|T_z(v_m-v)\|_{H^1_\varrho(\mathbb{R}^n)}.$$

By Property 1 applied to the first and third components and the condition (3.2) applied to the middle component, we get

$$\|T_z(T_y v-v)\|_{H^1_\varrho(\mathbb{R}^n)}$$
$$\le\Big(\sup_{x\in\mathbb{R}^n}\frac{\varrho(x-z)}{\varrho(x)}\Big)^{1/2}\Big(\Big(\sup_{x\in\mathbb{R}^n}\frac{\varrho(x-y)}{\varrho(x)}\Big)^{1/2}+1\Big)\|v-v_m\|_{H^1_\varrho(\mathbb{R}^n)}+\mathrm{const}(|y|).$$

Letting $m \to \infty$, we find that

$$\|T_z(T_y v - v)\|_{H^1_\varrho(\mathbb{R}^n)} \le \mathrm{const}(|y|),$$

which, after taking the supremum over $z \in \mathbb{R}^n$ on the left hand side, proves Property 3. ■

Observation 1. As claimed in (3.3) below, the embeddings $H^2_{\mathrm{lu}}(\mathbb{R}^n) \hookrightarrow H^1_\varrho(\mathbb{R}^n)$ and $H^2_{\mathrm{lu}}(\mathbb{R}^n) \hookrightarrow L^{2n/(n-2)}_\varrho(\mathbb{R}^n)$ are compact. Property 3 allows us to sharpen this compactness result to the following observation: any sequence $\{v_m\}$ bounded in $H^2_{\mathrm{lu}}(\mathbb{R}^n)$ has a subsequence convergent in $H^1_\varrho(\mathbb{R}^n) \cap L^{2n/(n-2)}_\varrho(\mathbb{R}^n)$ to some $v \in H^1_{\mathrm{lu}}(\mathbb{R}^n)$.

It is known that $-\Delta$ defines a sectorial operator in $L^p_{\mathrm{lu}}(\mathbb{R}^n)$, $p \in (1,\infty)$, with domain $W^{2,p}_{\mathrm{lu}}(\mathbb{R}^n)$ (see [18, 4]). Because of the translation invariance property these spaces do not enjoy compact embeddings as do Sobolev spaces in bounded domains; they are, however, compactly embedded in weighted spaces (provided that ϱ is decreasing with respect to the absolute value of each variable). More precisely, the following embeddings are *compact* (see Lemma 2):

$$W^{m,p}_{\mathrm{lu}}(\mathbb{R}^n) \hookrightarrow W^{j,q}_\varrho(\mathbb{R}^n), \qquad j - \frac{n}{q} < m - \frac{n}{p},\ 1 < p \le q < \infty. \tag{3.3}$$

Besides these we have the following *continuous* embeddings:

$$W^{m,p}_{\mathrm{lu}}(\mathbb{R}^n) \hookrightarrow W^{j,q}_{\mathrm{lu}}(\mathbb{R}^n), \quad j - \frac{n}{q} \le m - \frac{n}{p},\ 1 < p \le q < \infty \tag{3.4}$$

(see [19, Theorem 3.2] for a proof of the case $p = q = 2$ and $n = j = 1$).

We also quote the Nirenberg–Gagliardo type inequality

$$\|\phi\|_{L^r_{\mathrm{lu}}(\mathbb{R}^n)} \le C\|\phi\|^{1-\theta}_{L^q_{\mathrm{lu}}(\mathbb{R}^n)}\|\phi\|^\theta_{W^{m,p}_{\mathrm{lu}}(\mathbb{R}^n)}, \qquad \phi \in L^q_{\mathrm{lu}}(\mathbb{R}^n) \cap W^{m,p}_{\mathrm{lu}}(\mathbb{R}^n), \tag{3.5}$$

where

$$\frac{1}{r} = \theta\left(\frac{1}{p} - \frac{m}{n}\right) + \frac{1-\theta}{q}, \qquad 1 < p,q,r < \infty, \quad 0 < \theta < 1,$$

m is a positive integer and C depends on the weight function ϱ only through the value of its integral.

Let

$$E = H^1_{\mathrm{lu}}(\mathbb{R}^n) \times L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n),$$
$$E_\varrho = (H^1_\varrho(\mathbb{R}^n) \cap L^{2n/(n-2)}_\varrho(\mathbb{R}^n)) \times L^{2n/(n-2)}_\varrho(\mathbb{R}^n)$$

with the usual topology given by the norm (the intersection is normed by the sum of norms). Assume, for $n \ge 3$, that $h \in L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)$ and that $f : \mathbb{R} \to \mathbb{R}$ in (1.1) is differentiable and satisfies the following conditions:

$$|f'(s)| \le a(1 + |s|^{2/(n-2)}), \qquad s \in \mathbb{R}, \tag{3.6}$$

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} = -2c < 0. \tag{3.7}$$

In space dimensions $n = 1, 2$, we need (3.7) and, instead of (3.6), only the assumption that f grows like a polynomial. For space dimensions $n > 3$ we need to increase the regularity requirements on the data f, g, u_0, v_0, as described in Remark 7 below.

Under these assumptions we will show that problem (1.1) is globally well posed for $(u_0, v_0) \in E$ and that the semigroup generated by this problem is asymptotically (E-E_ϱ)-compact and bounded. This implies, from the results in Section 2, the existence of a global (E-E_ϱ)-attractor for (1.1).

REMARK 2. A simple example of a nonlinearity f satisfying all the above assumptions for $n = 3$ is the function $f(u) = u(1 - |u|^{p-1})$, $1 < p \leq 3$.

Before proceeding let us rewrite (1.1) in the following matrix form:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & -\delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u) - \alpha v \\ \beta u + h(x) \end{pmatrix}. \tag{3.8}$$

The above problem can be seen as an abstract semilinear parabolic problem, in $X = L^2_{\mathrm{lu}}(\mathbb{R}^n) \times L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)$, having the form

$$\frac{d}{dt} e = Ae + F(e), \tag{3.9}$$

where $e = \binom{u}{v}$, $A : D(A) \subset X \to X$ is the minus sectorial operator given by $D(A) = H^2_{\mathrm{lu}}(\mathbb{R}^n) \times L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)$, $Ae = \binom{\Delta u}{-\delta v}$ and $F(e) = \binom{f(u)-\alpha v}{\beta u + h(x)}$. It is known that $X^{1/2} = E$ and using the local existence results in [13] we find that (3.9) is locally well posed in E. More precisely, for any initial data $e_0 \in X^{1/2}$ there is a maximal positive time τ and a function $e \in C([0, \tau), X^{1/2})$ such that $e(0) = e_0$, $e \in C^1((0, \tau), X^{1/2})$ and (3.9) is satisfied. The above smoothness properties of the solutions justify the computations in the a priori estimates below.

To obtain global existence of a solution in E we show its E-norm does not blow up in a finite time. This is accomplished through the following a priori estimates.

Firstly, we obtain a priori estimate of the solutions in X.

3.1. *First a priori estimate.* Multiplying the first equation in (1.1) by $\beta u \varrho_y$, the second by $\alpha v \varrho_y$, integrating over $\mathbb{R}^n$ and adding the results we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big[\beta \int_{\mathbb{R}^n} u^2 \varrho_y(x)\, dx + \alpha \int_{\mathbb{R}^n} v^2 \varrho_y(x)\, dx\Big] &= \beta \int_{\mathbb{R}^n} \Delta u u \varrho_y(x)\, dx \\ - \alpha\delta \int_{\mathbb{R}^n} v^2 \varrho_y(x)\, dx + \beta \int_{\mathbb{R}^n} f(u) u \varrho_y(x)\, dx &+ \alpha \int_{\mathbb{R}^n} h(x) v \varrho_y(x)\, dx. \end{aligned} \tag{3.10}$$

Note that

$$\begin{aligned}(3.11)\qquad &\int_{\mathbb{R}^n} \Delta u u \varrho_y(x)\,dx \\ &\leq -\int_{\mathbb{R}^n} |\nabla u|^2 \varrho_y(x)\,dx + \frac{\varrho_0}{2}\Big[\int_{\mathbb{R}^n} |\nabla u|^2 \varrho_y(x)\,dx + \int_{\mathbb{R}^n} u^2 \varrho_y(x)\,dx\Big];\end{aligned}$$

moreover, due to (3.7), for some $d > 0$ we have

$$(3.12)\qquad \int_{\mathbb{R}^n} f(u)u\varrho_y(x)\,dx \leq -c\int_{\mathbb{R}^n} u^2\varrho_y(x)\,dx + d\int_{\mathbb{R}^n} \varrho_y(x)\,dx$$

and

$$(3.13)\qquad \Big|\int_{\mathbb{R}^n} h(x)v\varrho_y(x)\,dx\Big| \leq \frac{\eta}{2}\int_{\mathbb{R}^n} v^2\varrho_y(x)\,dx + \frac{1}{2\eta}\int_{\mathbb{R}^n} h(x)^2\varrho_y(x)\,dx.$$

Choosing η and ϱ_0 suitably small we have

$$\begin{aligned}(3.14)\qquad &\frac{d}{dt}\Big[\beta\int_{\mathbb{R}^n} u^2\varrho_y(x)\,dx + \alpha\int_{\mathbb{R}^n} v^2\varrho_y(x)\,dx\Big] + \frac{\beta}{2}\int_{\mathbb{R}^n} |\nabla u|^2\varrho_y(x)\,dx \\ &\leq -\min\{c,\delta\}\Big(\beta\int_{\mathbb{R}^n} u^2\varrho_y(x)\,dx + \alpha\int_{\mathbb{R}^n} v^2\varrho_y(x)\,dx\Big) \\ &\quad + 2\beta d\int_{\mathbb{R}^n} \varrho_y(x)\,dx + \frac{\alpha}{\eta}\int_{\mathbb{R}^n} h(x)^2\varrho_y(x)\,dx.\end{aligned}$$

This estimate implies that

$$(3.15)\qquad \beta\int_{\mathbb{R}^n} u(t)^2\varrho_y(x)\,dx + \alpha\int_{\mathbb{R}^n} v(t)^2\varrho_y(x)\,dx \leq C(\|u_0\|_{L^2_{\varrho_y}(\mathbb{R}^n)}, \|v_0\|_{L^2_{\varrho_y}(\mathbb{R}^n)}),$$

for all $t \geq 0$, as long as the local $X^{1/2}$-solution of (1) exists. Here $C : \mathbb{R}^2 \to [0,\infty)$ is increasing in each argument and locally bounded. Since $(u_0, v_0) \in X^{1/2}$, by taking the supremum over all y in $\mathbb{R}^n$ the estimate in (3.15) can be extended to

$$(3.16)\qquad \|(u,v)\|^2_{L^2_{\mathrm{lu}}(\mathbb{R}^n)\times L^2_{\mathrm{lu}}(\mathbb{R}^n)} \leq \Big(\frac{1}{\alpha} + \frac{1}{\beta}\Big)C(\|u_0\|_{L^2_{\mathrm{lu}}(\mathbb{R}^n)}, \|v_0\|_{L^2_{\mathrm{lu}}(\mathbb{R}^n)}).$$

Also, (3.14) provides the asymptotic estimate

$$\begin{aligned}(3.17)\qquad &\limsup_{t\to\infty}(\|u(t)\|^2_{L^2_{\varrho_y}(\mathbb{R}^n)} + \|v(t)\|^2_{L^2_{\varrho_y}(\mathbb{R}^n)}) \\ &\leq \frac{1}{\min\{c,\delta\}}\Big(2\beta d\int_{\mathbb{R}^n} \varrho_y(x)\,dx + \frac{\alpha}{\eta}\int_{\mathbb{R}^n} h(x)^2\varrho_y(x)\,dx\Big),\end{aligned}$$

the convergence being uniform for (u_0, v_0) in a bounded set $B \subset X^{1/2}$.

Moreover, returning to (3.14) one can see that

$$\int_t^{t+r}\int_{\mathbb{R}^n} |\nabla u(\tau)|^2 \varrho_y(x)\,dx\,d\tau \le R, \tag{3.18}$$

where R depends only on r and the $L^2_{\varrho_y}(\mathbb{R}^n)$ norms of u_0, v_0, but is independent of $t \ge 0$. Due to (3.14) and (3.17) we also have an asymptotic estimate

$$\limsup_{t\to\infty} \int_t^{t+r}\int_{\mathbb{R}^n} |\nabla u(\tau)|^2 \varrho_y(x)\,dx\,d\tau \le R_1(\alpha,\beta,c,d,h), \tag{3.19}$$

with R_1 independent of u_0, v_0, the convergence being uniform for (u_0, v_0) varying in bounded $B \subset X^{1/2}$.

3.2. *Second a priori estimate.* Now we will estimate the expression

$$\int_{\mathbb{R}^n} |\nabla u|^2 \varrho_y(x)\,dx,$$

which appears in the $H^1_{\varrho_y}(\mathbb{R}^n)$ norm of u. Multiplying the first equation in (1.1) by $u_t\varrho_y$ and integrating over $\mathbb{R}^n$ we obtain

$$\int_{\mathbb{R}^n} u_t^2 \varrho_y(x)\,dx = \int_{\mathbb{R}^n} \Delta u u_t \varrho_y(x)\,dx + \int_{\mathbb{R}^n} f(u) u_t \varrho_y(x)\,dx - \alpha \int_{\mathbb{R}^n} v u_t \varrho_y(x)\,dx.$$

Integrating by parts we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Delta u u_t \varrho_y(x)\,dx \le{}& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla u|^2 \varrho_y(x)\,dx \\ &+ \frac{\varrho_0}{2}\Big[\int_{\mathbb{R}^n} |\nabla u|^2 \varrho_y(x)\,dx + \int_{\mathbb{R}^n} u_t^2 \varrho_y(x)\,dx\Big],\end{aligned}$$

and moreover,

$$\int_{\mathbb{R}^n} f(u) u_t \varrho_y(x)\,dx = \frac{d}{dt}\int_{\mathbb{R}^n} F(u) \varrho_y(x)\,dx,$$

where $F(s) = \int_0^s f(z)\,dz$. With the above estimates and Young's inequality we get

$$\begin{aligned}&\frac{d}{dt}\bigg(\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(t)|^2 \varrho_y(x)\,dx - \int_{\mathbb{R}^n} F(u(t)) \varrho_y(x)\,dx\bigg) \\ &+ \Big(1 - \frac{\varrho_0}{2} - \frac{\eta}{2}\Big)\int_{\mathbb{R}^n} u_t^2 \varrho_y(x)\,dx \le \frac{\varrho_0}{2}\int_{\mathbb{R}^n} |\nabla u|^2 \varrho_y(x)\,dx + \frac{1}{2\eta}\int_{\mathbb{R}^n} v^2 \varrho_y(x)\,dx.\end{aligned} \tag{3.20}$$

Note that the primitive $F(u(t))$ is well defined when $u(t) \in H^1_{\mathrm{lu}}(\mathbb{R}^n)$, thanks to condition (3.6). Choose now η and ϱ_0 such that $1/4 \le 1 - \varrho_0/2 - \eta/2 \le 1/2$.

Since, by the assumption (3.7),

$$F(s) \le -cs^2 + d \tag{3.21}$$

we have

$$-\int_{\mathbb{R}^n} F(u)\varrho_y(x)\,dx + d\int_{\mathbb{R}^n} \varrho_y(x)\,dx \ge 0, \tag{3.22}$$

and we can add formally the term $d\int_{\mathbb{R}^n} \varrho_y(x)\,dx$ under the time derivative. The estimates (3.15) and (3.18) allow us to apply the Uniform Gronwall Lemma (see [23, p. 89]) to the differential inequality (3.20) to deduce that, for any $t \ge 0$ and $r > 0$ fixed,

$$\begin{aligned}\mathcal{L}(u(t+r)) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(t+r)|^2 \varrho_y(x)\,dx - \int_{\mathbb{R}^n} F(u(t+r))\varrho_y(x)\,dx \\ + d\int_{\mathbb{R}^n} \varrho_y(x)\,dx \le M,\end{aligned} \tag{3.23}$$

for some $M = M(\|u_0\|_{L^2_{\mathrm{lu}}(\mathbb{R}^n)}, \|v_0\|_{L^2_{\mathrm{lu}}(\mathbb{R}^n)}, r)$ independent of t. Thanks to (3.22), (3.23) we obtain a bound on $\|\nabla u(t)\|_{L^2_{\mathrm{lu}}(\mathbb{R}^n)}$, uniform for $t \in [r,\infty)$ and for initial conditions varying in bounded subsets of X.

From (3.17), (3.19) and the Uniform Gronwall Lemma we also obtain the asymptotic estimate (following from (3.20))

$$\limsup_{t\to\infty} \|\nabla u(t)\|_{L^2_{\mathrm{lu}}(\mathbb{R}^n)} \le \text{const}, \tag{3.24}$$

with const independent of the initial data u_0, v_0, the convergence being uniform for (u_0, v_0) varying in bounded sets B.

The bound just obtained for u in $H^1_{\mathrm{lu}}(\mathbb{R}^n)$ implies a bound in $L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)$ uniform for $t \in [r,\infty)$. Then the Uniform Gronwall Lemma applied to the second equation in (1.1) gives an $L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)$ bound for v uniform for $t \in [r,\infty)$:

$$\|v(t)\|_{L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)} \le C(\|u_0\|_{H^1_{\mathrm{lu}}(\mathbb{R}^n)}, \|v_0\|_{L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)}), \tag{3.25}$$

and also, by (3.17), (3.24), an asymptotic estimate

$$\limsup_{t\to\infty} \|v(t)\|_{L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)} \le \text{const}, \tag{3.26}$$

with const independent of the initial data, the convergence being uniform for (u_0, v_0) varying in bounded sets $B \subset X^{1/2}$.

Now that we have an estimate of (u, v) uniform on bounded subsets of $X^{1/2}$, and for $t \in [r,\infty)$, global in time solvability of (1.1) is well known [13], [5]. Next, the abstract smoothness Lemma 3.2.1 of [5] allows us to sharpen these estimates and conclude:

Remark 3. As a consequence of [5, p. 76], the $X^{1/2}$ estimate of (u, v), uniform on bounded subsets of $X^{1/2}$ and for $t \in [r,\infty)$, extends to a $D(A)$

estimate

$$(3.27)\quad \|u(t)\|_{H^2_{\mathrm{lu}}(\mathbb{R}^n)} + \|v(t)\|_{L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)} \le C(\|u_0\|_{H^1_{\mathrm{lu}}(\mathbb{R}^n)}, \|v_0\|_{L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)}),$$

valid for $t \in [r+\varepsilon, \infty)$, $\varepsilon > 0$.

REMARK 4. In all the estimates for which the supremum over y must be taken, one can easily see that the bounds involved do not depend on y. Such suprema over y will be added in estimates (3.18), (3.23).

REMARK 5. Estimates (3.17), (3.24) and (3.26) justify the existence of a set $\overline{B}_0$ bounded in $X^{1/2}$, which satisfies all the requirements of condition (III). However, for our further needs define

$$B_0 = T(r+1)(\overline{B}_0).$$

Thanks to (3.27) the set B_0 fulfils condition (III) as well, and moreover it is bounded in $D(A) = H^2_{\mathrm{lu}}(\mathbb{R}^n) \times L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)$.

3.3. *Existence of a global attractor.* We will now check that the semigroup associated to problem (1.1) on $X^{1/2}$ fulfills assumption (V). Let $B \subset X^{1/2}$ be bounded, let $\{(u_{m0}, v_{m0})\} \subset B$ and $t_m \to \infty$. Due to (3.27) and Observation 1 it is clear that we can find a subsequence $\{u_{m_k}(t_{m_k})\}$ convergent in $H^1_\varrho(\mathbb{R}^n) \cap L^{2n/(n-2)}_\varrho(\mathbb{R}^n)$ to some $\overline{u} \in H^1_{\mathrm{lu}}(\mathbb{R}^n)$. For the second coordinate v we use the Variation of Constants Formula for (3.8); that is, if $T(t)(u_0, v_0) = \binom{u(t,u_0,v_0)}{v(t,u_0,v_0)}$ is the semigroup associated to (3.8), then

$$(3.28)\quad \begin{cases} u(t,u_0,v_0) = e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)}[f(u(s,u_0,v_0)) - \alpha v(s,u_0,v_0)]\,ds, \\[2ex] v(t,u_0,v_0) = e^{-\delta t}v_0 + \int_0^t e^{-\delta(t-s)}\beta u(s,u_0,v_0)\,ds + \dfrac{1-e^{-\delta t}}{\delta}\,h. \end{cases}$$

Rearranging the second formula, for $t \ge r+1$, as

$$(3.29)\quad v(t,u_0,v_0) = \left[e^{-\delta t}v_0 + \beta\int_0^{r+1} e^{-\delta(t-s)}u(s,u_0,v_0)\,ds - \frac{e^{-\delta t}}{\delta}\,h\right] + \beta\int_{r+1}^t e^{-\delta(t-s)}u(s,u_0,v_0)\,ds + \frac{1}{\delta}\,h,$$

we observe that the expression in brackets will decay (uniformly in $(u_0, v_0) \in B$) to 0 in $L^{2n/(n-2)}_\varrho(\mathbb{R}^n)$ as $t = t_m \to \infty$. Thanks to (3.27) the integral over $[r+1, t]$ is bounded in $H^2_{\mathrm{lu}}(\mathbb{R}^n)$ uniformly in $t \ge r+1$ and in B. Due to Observation 1, we thus justify assumption (V) for v as well.

REMARK 6. As claimed in Remark 5 there exists a set B_0 bounded in $D(A)$ and fulfilling assumption (III). The above reasoning shows that the invariant set A (an $(X^{1/2}\text{-}E_\varrho)$-global attractor) introduced in Theorem 1 in connection with B_0 is bounded in $X^{1/2}$.

There is one more assumption (IV) of the abstract theory requiring verification in this example. We need to check the continuity of the semigroup $T(t)$ restricted to B_0 in the $(H^1_\varrho(\mathbb{R}^n)\cap L^{2n/(n-2)}_\varrho(\mathbb{R}^n))\times L^{2n/(n-2)}_\varrho(\mathbb{R}^n)$ topology. Let $(u_i(0),v_i(0))$, $i=1,2$, be initial data for (1.1) belonging to B_0. For such smooth data the corresponding solutions $(u_i(t),v_i(t))$, $i=1,2$, will stay in a bounded subset of $D(A)$ for all $t\geq 0$ (compare Remark 3). Let us concentrate on the case $n=3$, and let

$$U=u_1-u_2,\qquad V=v_1-v_2$$

be the difference of such solutions. We want first to get $L^2_\varrho(\mathbb{R}^n)\times L^2_\varrho(\mathbb{R}^n)$ and $L^{2n/(n-2)}_\varrho(\mathbb{R}^n)\times L^{2n/(n-2)}_\varrho(\mathbb{R}^n)$ estimates for (U,V) solving the system

$$\begin{cases} U_t=\Delta U-\alpha V+f(u_1)-f(u_2),\\ V_t=-\delta V+\beta U.\end{cases}\tag{3.30}$$

Let $2\leq q_0\leq 2n/(n-2)$; note also that if $\phi\in L^{2n/(n-2)}_\varrho(\mathbb{R}^n)$, then $\phi\in L^{q_0}_\varrho(\mathbb{R}^n)$ for any such q_0. Multiplying the first equation in (3.30) by $U|U|^{q_0-2}\varrho$, integrating, then multiplying the second equation by $V|V|^{q_0-2}\varrho$, integrating and adding the results, we get

$$\begin{aligned}\frac{1}{q_0}\frac{d}{dt}\int_{\mathbb{R}^n}(|U|^{q_0}+|V|^{q_0})\varrho\,dx &\leq -(q_0-1)\int_{\mathbb{R}^n}|\nabla U|^2|U|^{q_0-2}\varrho\,dx\\ &+\int_{\mathbb{R}^n}|\nabla U|\,|U|^{(q_0-2)/2}|U|^{q_0/2}|\nabla\varrho|\,dx-\alpha\int_{\mathbb{R}^n}VU|U|^{q_0-2}\varrho\,dx\\ &+\int_{\mathbb{R}^n}(f(u_1)-f(u_2))U|U|^{q_0-2}\varrho\,dx-\delta\int_{\mathbb{R}^n}|V|^{q_0}\varrho\,dx+\beta\int_{\mathbb{R}^n}UV|V|^{q_0-2}\varrho\,dx.\end{aligned}$$

Since $|\nabla\varrho|\leq\varrho_0\varrho$ and $u_i(t)$, $i=1,2$, vary in a bounded subset of $H^2_{\mathrm{lu}}(\mathbb{R}^n)\hookrightarrow L^\infty(\mathbb{R}^n)$, $n\leq 3$, it follows that for some $b>0$,

$$\int_{\mathbb{R}^n}(f(u_1)-f(u_2))U|U|^{q_0-2}\varrho\,dx\leq b\int_{\mathbb{R}^n}|U|^{q_0}\varrho\,dx,\tag{3.31}$$

and the standard use of the Cauchy and Young inequalities leads to the estimate

$$\frac{d}{dt}\int_{\mathbb{R}^n}(|U|^{q_0}+|V|^{q_0})\varrho\,dx\leq c(q_0,\varrho_0,b,\alpha,\beta)\int_{\mathbb{R}^n}(|U|^{q_0}+|V|^{q_0})\varrho\,dx,\tag{3.32}$$

providing an exponential bound for the $L^{q_0}_\varrho(\mathbb{R}^n)\times L^{q_0}_\varrho(\mathbb{R}^n)$ norm of (U,V).

Next we proceed to the $H^1_\varrho(\mathbb{R}^n)$ estimate of U. Multiplying the first equation of (3.30) by $U_t\varrho$ and integrating we obtain

$$\int_{\mathbb{R}^n} U_t^2\varrho\,dx = -\int_{\mathbb{R}^n} \nabla U_t\nabla U\varrho\,dx - \int_{\mathbb{R}^n} U_t\nabla U\nabla\varrho\,dx - \alpha\int_{\mathbb{R}^n} VU_t\varrho\,dx + \int_{\mathbb{R}^n}(f(u_1)-f(u_2))U_t\varrho\,dx. \tag{3.33}$$

Thanks to the estimate

$$\int_{\mathbb{R}^n}(f(u_1)-f(u_2))U_t\varrho\,dx \le \int_{\mathbb{R}^n} b|U|\,|U_t|\varrho\,dx \tag{3.34}$$

and the Cauchy inequality, from (3.33) we get

$$\frac{d}{dt}\int_{\mathbb{R}^n}|\nabla U|^2\varrho\,dx + \frac{1}{2}\int_{\mathbb{R}^n}U_t^2\varrho\,dx \le \int_{\mathbb{R}^n}|\nabla U|^2\varrho\,dx + \mathrm{const}\Big(\int_{\mathbb{R}^n}V^2\varrho\,dx + \int_{\mathbb{R}^n}U^2\varrho\,dx\Big).$$

The last estimate together with (3.32) provides a bound for $\|\nabla U\|_{L^2_\varrho(\mathbb{R}^n)}$ in terms of $\|V(0)\|_{L^2_\varrho(\mathbb{R}^n)}$, $\|U(0)\|_{L^2_\varrho(\mathbb{R}^n)}$, $\|U(0)\|_{H^1_\varrho(\mathbb{R}^n)}$, $\|u_1\|_{H^2_{\mathrm{lu}}(\mathbb{R}^n)}$, $\|u_2\|_{H^2_{\mathrm{lu}}(\mathbb{R}^n)}$, uniform on bounded time intervals $[0,T]$.

REMARK 7. For higher dimensions $n = 4,5,\ldots$, to get the estimates (3.31), (3.34), we need to use higher order regularity of the semigroup defined by (1.1). In particular, for $n = 4,5$, we assume that $v_0, h \in H^1_{\mathrm{lu}}(\mathbb{R}^n)$ and the nonlinear term f has locally Lipschitz continuous derivative f'. We are then able to estimate the solution (u,v) in $H^3_{\mathrm{lu}}(\mathbb{R}^n)\times H^1_{\mathrm{lu}}(\mathbb{R}^n)$ uniformly on compact subintervals of $(0,T]$. The absorbing set B_0 will then be bounded in $H^3_{\mathrm{lu}}(\mathbb{R}^n)\times H^1_{\mathrm{lu}}(\mathbb{R}^n)$, and thanks to the embedding $H^3_{\mathrm{lu}}(\mathbb{R}^n)\hookrightarrow L^\infty(\mathbb{R}^n)$, $n = 4,5$, the rest of the considerations will be the same as for $n = 3$.

Another possibility to cover all space dimensions $n \ge 3$ with one unified reasoning is to add one more assumption on the nonlinear term f:

$$\exists_{b>0}\ \forall_{s\in\mathbb{R}}\quad f'(s)\le b.$$

This condition, known as *quasimonotonicity*, simplifies the estimates significantly.

With all the above computations and Theorem 1 we have proved the following result:

THEOREM 2. *If f fulfils* (3.6), (3.7), *$n = 3$ and $h \in L^{2n/(n-2)}_{\mathrm{lu}}(\mathbb{R}^n)$, then problem* (1.1) *defines a bounded dissipative $(X^{1/2}\text{-}E_\varrho)$-asymptotically compact semigroup on $X^{1/2}$ and therefore* (1.1) *has an $(X^{1/2}\text{-}E_\varrho)$-attractor $\mathcal{A}$ bounded in $X^{1/2}$. The same conclusion is true for $n = 4,5$ under the additional regularity assumptions described in Remark* 7.

When $h \in H^1_{\mathrm{lu}}(\mathbb{R}^n)$, it follows from the a priori estimate (3.27), the Variation of Constants Formula (3.28) and the definition of $(X^{1/2}\text{-}E_\varrho)$-attractor that the following holds:

COROLLARY 2. *The attractor for* (1.1) *is a bounded subset of* $H^2_{\mathrm{lu}}(\mathbb{R}^n) \times H^1_{\mathrm{lu}}(\mathbb{R}^n)$ *whenever* $h \in H^1_{\mathrm{lu}}(\mathbb{R}^n)$.

REMARK 8. The calculations described above are much simpler in space dimensions $n_0 = 1, 2$, thanks to the embedding $H^1_{\mathrm{lu}}(\mathbb{R}^{n_0}) \hookrightarrow L^q_{\mathrm{lu}}(\mathbb{R}^{n_0})$ valid for arbitrary $q \in [1, \infty)$.

4. Application to Field–Noyes equations. In this section we consider a slight generalization of the *Field-Noyes system* which is a model for the Belousov–Zhabotinskii reactions in chemical kinetics (see for example [22, 12, 14, 17]). To avoid unnecessary technicalities we restrict the presentation to the case of dimension $n = 1$; that is, we consider the system

$$(4.1)\qquad \begin{cases} u_t = u_{xx} + \alpha(v - uv + f(u)), \\ v_t = \frac{1}{\alpha}(\gamma w - v - uv), \\ w_t = \sigma(u - w), \quad t > 0,\ x \in \mathbb{R}, \end{cases}$$

where α, γ, σ are positive constants.

If f is locally Lipschitz then the above problem is locally well posed in $H^1_{\mathrm{lu}}(\mathbb{R}) \times L^2_{\mathrm{lu}}(\mathbb{R}) \times L^2_{\mathrm{lu}}(\mathbb{R})$.

We will be interested in the solutions of the above system which start at nonnegative initial data. Following [22, Chapter 14], it can be shown that for a dense subset of the cone $E = H^1_{\mathrm{lu}}(\mathbb{R}, \mathbb{R}^+) \times L^2_{\mathrm{lu}}(\mathbb{R}, \mathbb{R}^+) \times L^2_{\mathrm{lu}}(\mathbb{R}, \mathbb{R}^+)$ (say $[C^\infty_{\mathrm{b}}(\mathbb{R}, \mathbb{R}^+)]^3$) the solution will stay in E as long as it exists. To prove that solutions that start in E will stay in E as long as they exist we consider the following auxiliary system:

$$(4.2)\qquad \begin{cases} u_t = u_{xx} + \alpha(v - uv + f(u)) + \eta, \\ v_t = \frac{1}{\alpha}(\gamma w - v - uv) + \eta, \\ w_t = \sigma(u - w) + \eta, \quad t > 0,\ x \in \mathbb{R},\ \eta > 0, \end{cases}$$

with initial data in $[C^\infty_{\mathrm{b}}(\mathbb{R}, \mathbb{R}^+)]^3$ and apply the reasoning used in [22]. Letting η go to zero we obtain the same conclusion for the system (4.1) with initial data in $[C^\infty_{\mathrm{b}}(\mathbb{R}, \mathbb{R}^+)]^3$. To obtain the result for (4.1) in E it is enough to use continuity with respect to initial data.

More simply than in the previous example, let

$$E = H^1_{\mathrm{lu}}(\mathbb{R}, \mathbb{R}^+) \times L^2_{\mathrm{lu}}(\mathbb{R}, \mathbb{R}^+) \times L^2_{\mathrm{lu}}(\mathbb{R}, \mathbb{R}^+),$$
$$E_\varrho = H^1_\varrho(\mathbb{R}, \mathbb{R}^+) \times L^2_\varrho(\mathbb{R}, \mathbb{R}^+) \times L^2_\varrho(\mathbb{R}, \mathbb{R}^+).$$

Assume that f satisfies (3.7) with a suitably large constant c and $\sigma\alpha - \gamma^2 > 0$. Under these assumptions we prove that the semigroup generated by (4.1) in E is bounded dissipative, (E-E_ϱ)-asymptotically compact and therefore has an (E-E_ϱ)-global attractor.

In what follows we sketch the a priori estimates required to obtain boundedness of the semigroup in E and (E-E_ϱ)-asymptotic compactness.

The first a priori estimate gives a bound for solutions in $L^2_{\mathrm{lu}}(\mathbb{R},\mathbb{R}^+)\times L^2_{\mathrm{lu}}(\mathbb{R},\mathbb{R}^+)\times L^2_{\mathrm{lu}}(\mathbb{R},\mathbb{R}^+)$ and it is obtained by multiplying each equation by the corresponding variable times ϱ_y, integrating and adding the results. Here we use the fact that all variables assume only nonnegative values to get rid of the terms $-u^2v$ and $-uv^2$. Using the Young inequality and Divergence Theorem we obtain

$$(4.3)\qquad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(u^2\varrho_y+v^2\varrho_y+w^2\varrho_y)\,dx+\frac{1}{2}\int_{\mathbb{R}}|u_x|^2\varrho_y\,dx\le -\frac{\alpha c}{2}\int_{\mathbb{R}}u^2\varrho_y\,dx \\ -\frac{1}{2\alpha}\int_{\mathbb{R}}v^2\varrho_y\,dx-\frac{\sigma}{2}\int_{\mathbb{R}}w^2\varrho_y\,dx+\frac{\gamma}{\alpha}\int_{\mathbb{R}}vw\varrho_y\,dx+M_0$$

for some constant M_0.

Since $\sigma\alpha-\gamma^2>0$ we find that $\|u\|^2_{L^2_{\mathrm{lu}}(\mathbb{R})}+\|v\|^2_{L^2_{\mathrm{lu}}(\mathbb{R})}+\|w\|^2_{L^2_{\mathrm{lu}}(\mathbb{R})}$ remains bounded uniformly for initial data in bounded subsets of E and for $t\ge 0$. Furthermore, there is a constant $K=K(d,\varrho)>0$ such that

$$(4.4)\qquad \limsup_{t\to\infty}\big(\|u\|^2_{L^2_{\mathrm{lu}}(\mathbb{R})}+\|v\|^2_{L^2_{\mathrm{lu}}(\mathbb{R})}+\|w\|^2_{L^2_{\mathrm{lu}}(\mathbb{R})}\big)\le K$$

uniformly for initial data in bounded subsets of E. Moreover, returning to (4.3) one can see that

$$(4.5)\qquad \int_t^{t+r}\int_{\mathbb{R}}|u_x(\tau)|^2\varrho_y(x)\,dx\,d\tau\le R,$$

where R depends only on r but is independent of t.

In the second a priori estimate we obtain dissipation for the first coordinate u in $H^1_{\mathrm{lu}}(\mathbb{R})$. Since this differs considerably from the estimates in the previous example we include the computations. Multiply the first equation in (4.1) by $u_t\varrho_y$ and integrate over $\mathbb{R}$ to obtain

$$(4.6)\qquad \int_{\mathbb{R}}u_t^2\varrho_y(x)\,dx=\int_{\mathbb{R}}u_{xx}u_t\varrho_y(x)\,dx+\alpha\int_{\mathbb{R}}(w-uv+f(u))u_t\varrho_y(x)\,dx \\ \le -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}u_x^2\varrho_y(x)\,dx+\alpha\frac{d}{dt}\int_{\mathbb{R}}F(u)\varrho_y(x)\,dx+\frac{\varrho_0}{2}\int_{\mathbb{R}}u_x^2\varrho_y(x)\,dx \\ +\frac{\varrho_0}{2}\int_{\mathbb{R}}u_t^2\varrho_y(x)\,dx+\alpha\int_{\mathbb{R}}vu_t\varrho_y(x)\,dx-\alpha\int_{\mathbb{R}}vuu_t\varrho_y(x)\,dx.$$

The last two terms in the inequality above are estimated as follows:

$$\left|\alpha \int_{\mathbb{R}} v u_t \varrho_y(x)\,dx\right| \le \frac{1}{4}\int_{\mathbb{R}} u_t^2 \varrho_y(x)\,dx + k_0 \int_{\mathbb{R}} v^2 \varrho_y(x)\,dx \tag{4.7}$$

for some $k_0 > 0$, and by the Nirenberg–Gagliardo type inequality (3.5) there are constants k_1 and k_2 such that

$$\begin{aligned} -\alpha \int_{\mathbb{R}} v u u_t \varrho_y(x)\,dx &= -\frac{\alpha}{2}\int_{\mathbb{R}} (u^2)_t v \varrho_y(x)\,dx \\ = -\frac{\alpha}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2 v \varrho_y(x)\,dx &+ \frac{1}{2}\int_{\mathbb{R}} u^2(\gamma w - v - uv)\varrho_y(x)\,dx \\ \le -\frac{\alpha}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2 v \varrho_y(x)\,dx &+ k_1 \int_{\mathbb{R}} (u^4 + u^6 + v^2 + w^2)\varrho_y(x)\,dx \\ \le -\frac{\alpha}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2 v \varrho_y(x)\,dx &+ k_2(\|u\|^3_{L^2_{\mathrm{lu}}(\mathbb{R})}\|u\|_{H^1_{\mathrm{lu}}(\mathbb{R})} + \|u\|^4_{L^2_{\mathrm{lu}}(\mathbb{R})}\|u\|^2_{H^1_{\mathrm{lu}}(\mathbb{R})}) \\ &+ k_1 \int_{\mathbb{R}} (v^2 \varrho_y(x) + w^2 \varrho_y(x))\,dx. \end{aligned} \tag{4.8}$$

Rearranging estimate (4.6) and using (4.7), (4.8) we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} u_x^2 \varrho_y(x)\,dx - \alpha \int_{\mathbb{R}} F(u)\varrho_y(x)\,dx + \frac{\alpha}{2}\int_{\mathbb{R}} u^2 v \varrho_y(x)\,dx\right) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} u_t^2 \varrho_y(x)\,dx \le \frac{\varrho_0}{2}\int_{\mathbb{R}} u_x^2 \varrho_y(x)\,dx + (k_0 + k_1)\int_{\mathbb{R}} v^2 \varrho_y(x)\,dx \\ &\quad + k_1 \int_{\mathbb{R}} w^2 \varrho_y(x)\,dx + k_2(\|u\|^3_{L^2_{\mathrm{lu}}(\mathbb{R})}\|u\|_{H^1_{\mathrm{lu}}(\mathbb{R})} + \|u\|^4_{L^2_{\mathrm{lu}}(\mathbb{R})}\|u\|^2_{H^1_{\mathrm{lu}}(\mathbb{R})}). \end{aligned} \tag{4.9}$$

Thanks to the estimates (4.5) and (4.4) we are able to apply the Uniform Gronwall Lemma ([23]) to (4.9) to conclude that the functional

$$\frac{1}{2}\int_{\mathbb{R}} u_x^2 \varrho_y(x)\,dx - \alpha \int_{\mathbb{R}} F(u)\varrho_y(x)\,dx$$

is uniformly bounded for initial data in bounded subsets of E and for $t \in [r,\infty)$, with $r > 0$ fixed. Therefore $u(t) \in H^1_{\mathrm{lu}}(\mathbb{R})$ with the norm uniformly bounded for initial data in bounded subsets of E and $t \in [r,\infty)$. We will also obtain an asymptotic estimate

$$\limsup_{t\to\infty} \|u(t)\|_{H^1_{\mathrm{lu}}(\mathbb{R})} \le k_3, \tag{4.10}$$

with k_3 independent of the initial data. An asymptotic estimate, in $L^2_{\mathrm{lu}}(\mathbb{R})$, of the two components (v, w) is already known (4.4), which proves assumption (III) for the present example.

Thus we observe that the semigroup generated by (4.1) is bounded dissipative. As in the previous example, using Lemma 3.2.1 of [5], we will extend the last $H^1_{\rm lu}(\mathbb{R})$ estimate to the estimate of u in $H^2_{\rm lu}(\mathbb{R})$, uniform for initial data in bounded subsets of E and for $t \in [\varepsilon, \infty)$. This last a priori estimate implies, in the light of Observation 1, that assumption (V) is satisfied for the first coordinate u of the semigroup generated by (4.1). Using an argument with the Variation of Constants Formula as in the previous example we verify assumption (V) for the other two coordinates v, w.

Since the problem is one-dimensional and $H^1_{\rm lu}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, the calculations are much simpler than in the previous example. In particular it is easy to check the continuity assumption (IV) of Theorem 1. We will omit these considerations here, concluding that:

THEOREM 3. *If f has locally bounded first derivative and satisfies* (3.7), *then the problem* (4.1) *defines a bounded dissipative* (E-E_ϱ)*-asymptotically compact semigroup on E and thus* (4.1) *has an* (E-E_ϱ)*-attractor* $\mathcal{A}$.

5. Appendix. In this appendix we recall a characterization of the spaces $L^p_{\rm lu}(\mathbb{R}^n)$ which is suitable for obtaining the compact embedding (3.3), and compare the results in this paper with the previously known results for partly dissipative systems.

5.1. *Characterization of the spaces $L^p_{\rm lu}(\mathbb{R}^n)$.* Recall first an equivalent definition of the space $L^q_{\rm lu}(\mathbb{R}^n)$, $q \in [1, \infty)$:

$$\phi \in L^q_{\rm lu}(\mathbb{R}^n) \ \Leftrightarrow\ \|\phi\|_{L^q_{\rm lu}(\mathbb{R}^n)} < \infty \text{ and } \|T_z\phi - \phi\|_{L^q_{\rm lu}(\mathbb{R}^n)} \to 0 \text{ as } z \to 0. \tag{5.1}$$

We next give an equivalent characterization of the norm of that space.

LEMMA 1. *Assume that ϱ is a strictly positive, integrable weight function, decreasing with respect to the absolute value of each variable. Then*

$$\|\phi\|_{L^q_{\rm lu}(\mathbb{R}^n)} < \infty \ \Leftrightarrow\ \exists_{C,r>0}\ \forall_{y\in\mathbb{R}^n} \int\limits_{B(y,r)} |\phi(x)|^q\,dx \le C, \tag{5.2}$$

where $B(y, r) \subset \mathbb{R}^n$ is the ball of radius r centered at y.

Proof. It is evident that if the above condition holds for one value of r it must hold for any $r > 0$. Assume that $\|\phi\|_{L^q_{\rm lu}(\mathbb{R}^n)} < \infty$. Then, for each ball $B(y, r)$,

$$\int\limits_{B(y,r)} |\phi(x)|^q \varrho_y(x)\,dx \le M.$$

Since ϱ is strictly positive in $B(0, r)$ it is bounded below by a positive constant m, and the same lower bound will be valid for $\varrho_y(x)$ in $B(y, r)$. Therefore

$$m \int\limits_{B(y,r)} |\phi(x)|^q\,dx \le \int\limits_{B(y,r)} |\phi(x)|^q \varrho(x-y)\,dx \le M$$

and consequently

$$\int\limits_{B(y,r)} |\phi(x)|^q\,dx \le \frac{M}{m}.$$

To show the other implication note that the balls in (5.2) can be replaced with cubes. Choose $\tau > 0$ and let $C_{(i_1,\ldots,i_n)} = \prod_{j=1}^n [\tau i_j, \tau(i_j+1)]$. It follows from the integral criterion for convergence of multiple series that

$$S_\tau = \sum_{(i_1,\ldots,i_n)} \sup_{x\in C_{(i_1,\ldots,i_n)}} \varrho(x) < \infty.$$

The proof is completed by noting the estimate

$$\begin{aligned}\int\limits_{\mathbb{R}^n} |\phi(x)|^q \varrho_y(x)\,dx &= \int\limits_{\mathbb{R}^n} |\phi(z+y)|^q \varrho(z)\,dz \\ &\le \sum_{(i_1,\ldots,i_n)} \Big(\int\limits_{C_{(i_1,\ldots,i_n)}} |\phi(z+y)|^q\,dz \sup_{x\in C_{(i_1,\ldots,i_n)}} \varrho(x)\Big) \\ &\le C \sum_{(i_1,\ldots,i_n)} \sup_{x\in C_{(i_1,\ldots,i_n)}} \varrho(x),\end{aligned}$$

whose right hand side is independent of y. ■

Thanks to the above characterization we have the following compactness result:

LEMMA 2. *Let ϱ be a strictly positive integrable function, satisfying* (3.1), *which is decreasing with respect to the absolute value of each variable. Then the embedding*

$$W^{1,p}_{\mathrm{lu}}(\mathbb{R}^n) \hookrightarrow L^q_\varrho(\mathbb{R}^n), \qquad 1 < p \le q < \frac{np}{n-p},$$

is compact, and therefore the general embedding (3.3) *is also compact.*

Proof. As in the previous lemma we can cover $\mathbb{R}^n$ by a countable number of cubes, C_j, $j = 1, 2, \ldots$, having disjoint interiors. For each j we have the compact embedding

$$W^{1,p}(C_j) \hookrightarrow L^q(C_j), \qquad 1 < p \le q < \frac{np}{n-p},$$

and $L^q(C_j) = L^q_\varrho(C_j)$ with equivalent norms. Let $\{\phi_m\}$ be a bounded sequence in $W^{1,p}_{\mathrm{lu}}(\mathbb{R}^n)$. Extract a subsequence $\{\phi_{m_1}\}$ which is convergent in $L^q(C_1)$. Proceeding by induction, once we have constructed a subsequence $\{\phi_{m_k}\}$, let $\{\phi_{m_{k+1}}\}$ be a subsequence of $\{\phi_{m_k}\}$ which is convergent in $L^q(C_k)$. Now the Cantor diagonal process implies that $\{\phi_m\}$ has a subsequence, denoted again by $\{\phi_m\}$, which is convergent in $L^q(C_j)$ for every j. Let ϕ be the limit of this subsequence.

Thanks to the characterization in the previous lemma, to (3.4) and the fact that $\{\phi_m\}$ is a bounded sequence in $L^q_{\mathrm{lu}}(\mathbb{R}^n)$, the convergence of $\{\phi_m\}$ to ϕ in the sense described above implies that ϕ is in $L^q_\varrho(\mathbb{R}^n)$. It remains to prove that $\{\phi_m\}$ converges to ϕ in $L^q_\varrho(\mathbb{R}^n)$. To prove this, note that

$$
\begin{aligned}
(5.3)\qquad \int_{\mathbb{R}^n} |\phi_m(x)-\phi(x)|^q\varrho(x)\,dx &\le \sum_{j=1}^{\infty}\sup_{x\in C_j}\varrho(x)\int_{C_j}|\phi_m(x)-\phi(x)|^q\,dx\\
&\le 2^qC\sum_{j=1}^{\infty}\sup_{x\in C_j}\varrho(x).
\end{aligned}
$$

Since the above series is convergent, given $\varepsilon>0$ there exists a natural number N such that

$$
\begin{aligned}
(5.4)\qquad &\int_{\mathbb{R}^n} |\phi_m(x)-\phi(x)|^q\varrho(x)\,dx\\
&\qquad\le \sum_{j=1}^{N}\sup_{x\in C_j}\varrho(x)\int_{C_j}|\phi_m(x)-\phi(x)|^q\,dx+2^qC\varepsilon
\end{aligned}
$$

and therefore, by the definition of ϕ_m,

$$
\limsup_{m\to\infty}\int_{\mathbb{R}^n}|\phi_m(x)-\phi(x)|^q\varrho(x)\,dx\le 2^qC\varepsilon
$$

for any $\varepsilon>0$. This proves the lemma. ∎

5.2. *Further remarks.* Now we compare the results in this paper with those in [21]. Note first that the assumptions imposed on the nonlinear term f in the present paper are weaker; in particular we do not need any quasi-monotonicity condition $f'(s)\le C$ which was essentially used in [21]. Note also that with the assumptions in [21], following the same a priori estimates as in Section 3, one can state that the global attractor $\mathcal{B}$ obtained in [21] is a bounded subset of $H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ and therefore a bounded subset of $H^2_{\mathrm{lu}}(\mathbb{R}^n)\times H^1_{\mathrm{lu}}(\mathbb{R}^n)$, as long as ϱ is a bounded weight function. This together with the invariance of the attractor and with the remarks stated in the Introduction concerning travelling waves and constant stationary solutions ensures that $\mathcal{B}\subsetneq\mathcal{A}$.

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