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GLOBAL EXISTENCE OF AXIALLY SYMMETRIC SOLUTIONS TO NAVIER–STOKES EQUATIONS WITH LARGE ANGULAR COMPONENT OF VELOCITY

BY

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Abstract. Global existence of axially symmetric solutions to the Navier–Stokes equations in a cylinder with the axis of symmetry removed is proved. The solutions satisfy the ideal slip conditions on the boundary. We underline that there is no restriction on the angular component of velocity. We obtain two kinds of existence results. First, under assumptions necessary for the existence of weak solutions, we prove that the velocity belongs to $W^{2,1}_{4/3}(\Omega\times(0,T))$, so it satisfies the Serrin condition. Next, increasing regularity of the external force and initial data we prove existence of solutions (by the Leray–Schauder fixed point theorem) such that $v\in W^{2,1}_r(\Omega\times(0,T))$ with $r>4/3$, and we prove their uniqueness.

1. Introduction. We consider the motion of a viscous incompressible fluid described by the Navier–Stokes equations in a bounded cylinder, with ideal boundary slip conditions (see [17]):

$$\begin{aligned}
&v_{,t}+v\cdot\nabla v-\operatorname{div}\mathbb{T}(v,p)=f &&\text{in } \Omega^T=\Omega\times(0,T),\\
&\operatorname{div}v=0 &&\text{in } \Omega^T,\\
&v\cdot\overline{n}=0 &&\text{on } S^T=S\times(0,T),\\
&\overline{n}\cdot\mathbb{T}(v,p)\cdot\overline{\tau}_\alpha=0,\quad \alpha=1,2, &&\text{on } S^T,\\
&v|_{t=0}=v(0) &&\text{in } \Omega,
\end{aligned}\tag{1.1}$$

where $v=v(x,t)=(v_1(x,t),v_2(x,t),v_3(x,t))\in\mathbb{R}^3$ is the velocity vector, $p=p(x,t)\in\mathbb{R}$ the pressure, $f=f(x,t)=(f_1(x,t),f_2(x,t),f_3(x,t))\in\mathbb{R}^3$ the external force field, $\overline{n}$ is the unit outward vector normal to S, $\overline{\tau}_\alpha$, $\alpha=1,2$, are tangent vectors to S.

By $\mathbb{T}(v,p)$ we denote the stress tensor of the form

$$\mathbb{T}(v,p)=\nu\mathbb{D}(v)-pI,\tag{1.2}$$

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where ν is the constant viscosity coefficient, $\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}$ is the dilatation tensor and I is the unit matrix.

Finally, dot denotes the scalar product in $\mathbb{R}^3$.

To describe the domain Ω and the motion, we introduce the cylindrical coordinates r, φ, z by the relations $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$, where x_1, x_2, x_3 are the Cartesian coordinates.

We assume that $\Omega = \{x \in \mathbb{R}^3 : R_1 < r < R_2,\ -a < z < a,\ \varphi \in [0, 2\pi]\}$. Then $\partial\Omega = S = S_1 \cup S_2$, where $S_1 = \{x \in \mathbb{R}^3 : r$ is either R_1 or R_2, $-a < z < a,\ \varphi \in [0, 2\pi]\}$ and $S_2 = \{x \in \mathbb{R}^3 : z$ is either $-a$ or a and $R_1 < r < R_2,\ \varphi \in [0, 2\pi]\}$.

Let u be any vector. We introduce the cylindrical coordinates of u in the following way: $u_r = u\cdot\bar{e}_r$, $u_\varphi = u\cdot\bar{e}_\varphi$, $u_z = u\cdot\bar{e}_z$, where $\bar{e}_r = (\cos\varphi, \sin\varphi, 0)$, $\bar{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$, $\bar{e}_z = (0, 0, 1)$.

Definition 1.1. By an *axially symmetric solution* to (1.1) we mean a solution such that the cylindrical components of $v, f, v(0)$ and p do not depend on φ. Then, instead of Ω and S, we consider the intersections of Ω and S with the plane $\varphi = \mathrm{const} \in [0, 2\pi]$.

Following [17] we introduce the following notation. We distinguish the angular component of the velocity, v_φ, by writing $v_\varphi = w$. Let $\alpha = \operatorname{rot} v$ be the vorticity vector. Its cylindrical coordinates in the case of the axially symmetric solution take the form

$$\alpha_r = -w_{,z}, \quad \alpha_\varphi = v_{r,z} - v_{z,r} \equiv \chi, \quad \alpha_z = \frac{w}{r} + w_{,r}. \tag{1.3}$$

To prove the existence of global axially symmetric solutions to (1.1) more regular than weak solutions, we follow the ideas of Ladyzhenskaya [5] and Ukhovskiĭ–Yudovich [16] to obtain an energy estimate for χ. For this purpose we need

Lemma 1.2 (see also [17, 18]). *Let v, w, $F_\varphi = (\operatorname{rot} f)_\varphi$ be given. Then χ is a solution to the problem*

$$\begin{aligned}
&\chi_{,t} + v\cdot\nabla\chi + (v_{r,r} + v_{z,z})\chi \\
&\quad = \nu\bigg[\bigg(r\Big(\frac{\chi}{r}\Big)_{,r}\bigg)_{,r} + \chi_{,zz} + 2\Big(\frac{\chi}{r}\Big)_{,r}\bigg] \\
&\qquad + \frac{2}{r}ww_{,z} + F_\varphi && \text{in } \Omega^T, \\
&\chi = 0 && \text{on } S^T, \\
&\chi|_{t=0} = \chi(0) && \text{in } \Omega.
\end{aligned} \tag{1.4}$$

Proof. First we show $(1.4)_1$. For axially symmetric solutions we express the r and z components of $(1.1)_1$ in the form

$$
\begin{aligned}
& v_{r,t} + v_r v_{r,r} + v_z v_{r,z} - \frac{v_\varphi^2}{r} + p_{,r} = \nu \Delta v_r - \frac{2\nu}{r^2} v_r + f_r, \\
& v_{z,t} + v_r v_{z,r} + v_z v_{z,z} + p_{,z} = \nu \Delta v_z + f_z,
\end{aligned} \tag{1.5}
$$

where

$$
\Delta u = \frac{1}{r}(r u_{,r})_{,r} + u_{,zz}. \tag{1.6}
$$

Differentiating $(1.5)_1$ with respect to z, $(1.5)_2$ with respect to r and subtracting yields $(1.4)_1$.

To find the boundary condition $(1.4)_2$ we express the boundary conditions $(1.1)_{3,4}$ for axially symmetric solutions in cylindrical coordinates. They take the form

$$
\begin{aligned}
& v_r = 0, \\
& v_{\varphi,r} - \frac{1}{r} v_\varphi = 0, \\
& v_{z,r} + v_{r,z} = 0,
\end{aligned} \tag{1.7}
$$

on S_1, and

$$
\begin{aligned}
& v_z = 0, \\
& v_{\varphi,z} = 0, \\
& v_{z,r} + v_{r,z} = 0,
\end{aligned} \tag{1.8}
$$

on S_2. Since $\chi = v_{r,z} - v_{z,r}$, we have $\chi|_{S_1} = 0$ in view of $(1.7)_{1,3}$ and $v_{r,z}|_{S_1} = 0$. Next, in view of $(1.8)_{1,3}$, we see that $\chi|_{S_2} = 0$ because $v_{z,r}|_{S_2} = 0$. Hence $(1.4)_2$ is established, which ends the proof.

LEMMA 1.3 (see also [17, 18]). *Let v and f_φ be given. Then w is a solution to the problem*

$$
\begin{aligned}
& w_{,t} + v \cdot \nabla w + \frac{v_r}{r} w - \nu \Delta w + \nu \frac{w}{r^2} = f_\varphi && \text{in } \Omega^T, \\
& w_{,r}|_{r=R_i} = \frac{1}{R_i} w|_{r=R_i}, \quad i = 1, 2, && \text{on } S_1^T, \\
& w_{,z} = 0 && \text{on } S_2^T, \\
& w|_{t=0} = w(0) && \text{in } \Omega.
\end{aligned} \tag{1.9}
$$

Proof. Taking the φ-component of $(1.1)_1$ we obtain $(1.9)_1$. The conditions $(1.7)_2$, $(1.8)_2$ imply $(1.9)_2$ and $(1.9)_3$, respectively. This ends the proof.

There are numerous results concerning axially symmetric motions with $v_\varphi = 0$. The first results were established by Ladyzhenskaya [5] and Ukhovskiĭ–Yudovich [16]. Ladyzhenskaya proved the global existence of axially symmetric solutions with $v_\varphi = 0$ in a cylinder with the axis of symmetry removed. To prove the global existence, Ladyzhenskaya assumed that

$v \cdot \overline{n}|_S = 0$, $\chi = v_{r,z} - v_{z,r}|_S = 0$, which follows from the ideal boundary slip conditions $(1.1)_{3,4}$ (see Lemma 1.2). However, the boundary condition $(1.4)_2$ is crucial for the proof of global existence in [5] as well as in this paper. Global existence for the Cauchy problem only was proved in [16]. In this case the behaviour of solutions in a neighbourhood of the axis of symmetry was described in weighted Sobolev spaces. An improved version of the proof in [16] was given in [7, 8]. Again, we underline that all the above results were obtained for axially symmetric solutions with $v_\varphi = 0$ (see also [4]).

In this paper we consider the case $v_\varphi \neq 0$. By the abstract technique of semigroups, global existence of regular solutions to the Navier–Stokes equations with Dirichlet boundary conditions was proved by Ströhmer–Zajączkowski [15]. The result was obtained for solutions with some invariance property. In particular, the case of axial symmetry was covered. After small modifications, that result is also valid for problem (1.1). In this paper we remove the axis of symmetry for simplicity only. The existence in a full cylinder will be considered elsewhere. Finally, the slip boundary conditions were also considered in [3, 9, 17, 18].

Now we formulate the main results of this paper.

Theorem 1 (existence and uniqueness). *Let $k_0 \in \mathbb{R}_+$ and $T > 0$ be given. Let $f \in L_\infty(0,\infty; L_{1'}(\Omega))$, $1' > 1$, $v(0) \in W^{1/2}_{4/3}(\Omega)$, $\chi(0) \in L_2(\Omega)$, $f \in L_{4/3}(\Omega \times (k_0, k_0+T))$, $\sup_t |\int_{\Omega^t} f_\eta \, dx\, dt'| < \infty$, $|\int_\Omega v_\eta(0)\, dx| < \infty$, $\eta = re_\varphi$, $f_\eta = f \cdot \eta$, $v_\eta = v \cdot \eta$. Then there exists a unique solution to problem* (1.1) *such that $v \in W^{2,1}_{4/3}(\Omega \times (k_0, k_0+T))$, $\nabla p \in L_{4/3}(\Omega \times (k_0, k_0+T))$ and*

$$
\begin{aligned}
(1.10)\qquad & \|v\|_{2,4/3,\Omega\times(k_0,k_0+T)} + |\nabla p|_{4/3,\Omega\times(k_0,k_0+T)} \\
&\le \varphi_1\Big(T, \|f\|_{L_\infty(0,\infty;L_{1'}(\Omega))}, |f|_{4/3,\Omega\times(k_0,k_0+T)}, \\
&\qquad \|v(0)\|_{1/2,4/3,\Omega}, \Big|\int_\Omega v_\eta(0)\, dx\Big|, \sup_t \Big|\int_{\Omega^t} f_\eta\, dx\, dt'\Big|\Big) \equiv A,
\end{aligned}
$$

where φ_1 is an increasing positive function.

Theorem 2 (existence and regularity). *Let the assumptions of Theorem 1 hold. Let $f' \in L_\infty(0,\infty; L_2(\Omega))$, $f' = (f_r, f_z)$, $\chi(0) \in L_2(\Omega)$, $f \in L_r(\Omega \times (k_0, k_0+T))$, $v(0) \in W^{2-2/r}_r(\Omega)$, $r > 4/3$. Then there exists a solution to problem* (1.1) *such that $v \in W^{2,1}_r(\Omega \times (k_0, k_0+T))$, $\nabla p \in L_r(\Omega \times (k_0, k_0+T))$ and*

$$
\begin{aligned}
(1.11)\qquad & \|v\|_{2,r,\Omega\times(k_0,k_0+T)} + |\nabla p|_{r,\Omega\times(k_0,k_0+T)} \\
&\le \varphi_2(A, \|f'\|_{L_\infty(0,\infty;L_2(\Omega))}, |\chi(0)|_{2,\Omega}, \\
&\qquad |f|_{r,\Omega\times(k_0,k_0+T)}, \|v(0)\|_{2-2/r,r,\Omega}),
\end{aligned}
$$

where φ_2 is an increasing positive function.

THEOREM 3 (uniqueness). *Assume that there exists a solution to problem* (1.1) *such that* $v \in L_2(0,\infty; L_\infty(\Omega))$. *Then uniqueness of solutions to problem* (1.1) *holds in this class.*

Theorem 3 implies that uniqueness holds for $v \in W_r^{2,1}(\Omega^T)$, $r > 4/3$. Hence, Theorem 2 yields uniqueness too. Uniqueness in Theorem 1 follows from the Serrin argument (see [11]). The following lemma corresponds to the results from [12, 13] (see [1]).

LEMMA 1.4. *Assume that* $f \in L_r(\Omega^T)$, $v(0) \in W_r^{2-2/r}(\Omega)$, $S \in C^2$; $r > 1$. *Then there exists a solution to the problem*

$$
\begin{aligned}
& v_{,t} - \operatorname{div} \mathbb{T}(v,p) = f && \text{in } \Omega^T,\\
& \operatorname{div} v = 0 && \text{in } \Omega^T,\\
& v \cdot \overline{n} = 0 && \text{on } S^T,\\
& \overline{n} \cdot \mathbb{T}(v,p) \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T,\\
& v|_{t=0} = v(0) && \text{in } \Omega,
\end{aligned}
\tag{1.12}
$$

such that $v \in W_r^{2,1}(\Omega^T)$, $\nabla p \in L_r(\Omega^T)$ *and*

$$
\|v\|_{2,r,\Omega^T} + |\nabla p|_{r,\Omega^T} \le c(|f|_{r,\Omega^T} + \|v(0)\|_{2-2/r,r,\Omega}). \tag{1.13}
$$

The same result holds for cylindrical domains.

2. Notation and auxiliary results. To simplify notation we introduce

$$
\begin{aligned}
& |u|_{p,Q} = \|u\|_{L_p(Q)}, && Q \in \{\Omega, S, \Omega^T, S^T\},\ p \in [1,\infty],\\
& \|u\|_{s,Q} = \|u\|_{H^s(Q)}, && Q \in \{\Omega, S\}, \quad 0 \le s \in \mathbb{R}_+,\\
& \|u\|_{s,Q} = \|u\|_{W_2^{s,s/2}(Q)}, && Q \in \{\Omega^T, S^T\}, \quad 0 \le s \in \mathbb{R}_+,\\
& \|u\|_{s,p,Q} = \|u\|_{W_p^s(Q)}, && Q \in \{\Omega, S\}, \quad 0 \le s \in \mathbb{R}_+,\ p \in [1,\infty],\\
& \|u\|_{s,p,Q} = \|u\|_{W_p^{s,s/2}(Q)}, && Q \in \{\Omega^T, S^T\}, \quad 0 \le s \in \mathbb{R}_+,\ p \in [1,\infty],\\
& |u|_{p_1,p_2,\Omega^T} = \|u\|_{L_{p_2}(0,T;L_{p_1}(\Omega))}.
\end{aligned}
$$

By c we denote generic constants. To distinguish a certain constant we denote it by c_k, $k \in \mathbb{N}$.

For bounded $\Omega \subset \mathbb{R}^2$ the following interpolation inequality holds (see [2]):

$$
|w|_{4,\Omega} \le c|\nabla w|_{2,\Omega}^{1/2} |w|_{2,\Omega}^{1/2} + c|w|_{2,\Omega}. \tag{2.1}
$$

For functions from $W_s^{2,1}(\Omega^T)$, $\Omega \subset \mathbb{R}^2$, $s > 1$, we have the interpolation inequality

$$
|\nabla^r u|_{q,\Omega^T} \le c|u|_{\sigma,\Omega^T}^{1-\theta} \|u\|_{2,s,\Omega^T}^{\theta}, \tag{2.2}
$$

where r is either 1 or 0, and

$$\theta=\frac{r+4/\sigma-4/q}{2+4/\sigma-4/s}.$$

Moreover, θ satisfies the condition

$$r/2<\theta<1. \tag{2.3}$$

In the case $r=1$, the left inequality in (2.3) takes the form

$$4/q-2/s<2/\sigma. \tag{2.4}$$

The inequality (2.2) is equivalent to

$$|\nabla^r u|_{q,\Omega^T}\le\varepsilon^{1-\theta}\|u\|_{2,s,\Omega^T}+c\varepsilon^{-\theta}|u|_{\sigma,\Omega^T}, \tag{2.5}$$

where ε is any positive parameter.

To examine the existence of weak solutions we need the space

$$V_2^1(\Omega^T)=\{u:\operatorname*{ess\,sup}_{t\le T}|u(t)|_{2,\Omega}+|\nabla u|_{2,\Omega^T}<\infty\},$$

where u is a scalar or vector-valued function.

By l.h.s. (r.h.s.) we denote the left-hand side (right-hand side), respectively.

3. A priori estimates. In this section we find some a priori estimates for axially symmetric solutions to problem (1.1).

LEMMA 3.1. *Assume that $w\in L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;H^1(\Omega))$, $f'=(f_r,f_z)\in L_2(\Omega^T)$ and $\chi(0)\in L_2(\Omega)$. Then*

$$\begin{aligned}\left|\frac{\chi}{r}\right|_{2,\Omega}^2+\nu\int_0^t\left|\nabla\frac{\chi}{r}\right|_{2,\Omega}^2dt'&\le\frac{c}{R_1^4}\sup_t|w|_{2,\Omega}^2\int_0^t|\nabla w(t')|_{2,\Omega}^2\,dt'\\&\quad+\frac{c}{R_1^4}\sup_t|w|_{2,\Omega}^2\int_0^t\|w(t')\|_{1,\Omega}^2\,dt'+\frac{c}{R_1^2}|f'|_{2,\Omega^t}^2+\left|\frac{\chi(0)}{r}\right|_{2,\Omega}^2\end{aligned} \tag{3.1}$$

for all $t\le T$.

Proof. Multiplying $(1.4)_1$ by χ/r^2 and integrating over Ω we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\left|\frac{\chi}{r}\right|^2dx+\nu\int_\Omega\left|\nabla\frac{\chi}{r}\right|^2dx=\int_\Omega\frac{1}{r}(w^2)_{,z}\frac{\chi}{r^2}\,dx+\int_\Omega F_\varphi\frac{\chi}{r^2}\,dx, \tag{3.2}$$

where the last integral equals

$$\begin{aligned}\int_\Omega(f_{r,z}-f_{z,r})\frac{\chi}{r^2}\,dx&=\int(f_{r,z}-f_{z,r})\frac{\chi}{r}\,dr\,dz\\&=\int_\Omega\left(-f_r\left(\frac{\chi}{r}\right)_{,z}+f_z\left(\frac{\chi}{r}\right)_{,r}\right)\frac{1}{r}\,dx.\end{aligned}$$

Integrating by parts in the first term on the r.h.s., using $\chi|_S = 0$ and applying the Hölder and Young inequalities we obtain

$$\frac{1}{2}\frac{d}{dt}\left|\frac{\chi}{r}\right|^2_{2,\Omega} + \frac{3}{4}\nu\left|\nabla\frac{\chi}{r}\right|^2_{2,\Omega} \le \frac{c}{R_1^4}|w|^4_{4,\Omega} + \frac{c}{R_1^2}|f'|^2_{2,\Omega}, \tag{3.3}$$

where the structure of Ω was utilized. Using (2.1) in (3.3) and integrating the result with respect to time, we obtain (3.1), which concludes the proof.

To prove the Korn inequality we introduce

$$E_\Omega(v) = \int_\Omega (v_{i,x_j} + v_{j,x_i})^2\,dx, \tag{3.4}$$

where summation over repeated indices is assumed.

Lemma 3.2. *Suppose* $|\int_\Omega v_\eta\,dx| < \infty$ *and* $E_\Omega(v) < \infty$. *Then*

$$\|v\|^2_{1,\Omega} \le c\Big(E_\Omega(v) + \Big|\int_\Omega v_\eta\,dx\Big|^2\Big), \tag{3.5}$$

where $v_\eta = v\cdot\eta$ *and* $\eta = r\bar{e}_\varphi$.

Proof. Multiplying $(1.1)_1$ by $\eta = (-x_2, x_1, 0)$, integrating the result over Ω^t, using $\eta\cdot\bar{n}|_S = 0$ and $(1.1)_{3,4}$, and the fact that $\nabla\eta$ is an antisymmetric tensor, we obtain

$$\int_\Omega v\cdot\eta\,dx = \int_{\Omega^t} f\cdot\eta\,dx\,dt' + \int_\Omega v(0)\cdot\eta\,dx. \tag{3.6}$$

To show (3.5), we express v in the form

$$v = v' + \frac{\alpha}{|\eta|^2_{2,\Omega}}\eta, \tag{3.7}$$

where

$$v' = v_r\bar{e}_r + \Big(v_\varphi - \frac{\alpha r}{|\eta|^2_{2,\Omega}}\Big)\bar{e}_\varphi + v_z\bar{e}_z \equiv v'' + \Big(v_\varphi - \frac{\alpha}{|\eta|^2_{2,\Omega}}r\Big)\bar{e}_\varphi \equiv v'' + v_*,$$

$$\alpha = \int_\Omega v\cdot\eta\,dx, \qquad \int_\Omega v'\cdot\eta\,dx = 0.$$

First, we shall obtain a relation between $E_\Omega(v)$ and $|\nabla v|_{2,\Omega}$. Write

$$E_\Omega(v) = \int_\Omega (v_{i,x_j} + v_{j,x_i})^2\,dx = 2\int_\Omega v^2_{i,x_j}\,dx + 2\int_\Omega v_{i,x_j}v_{j,x_i}\,dx, \tag{3.8}$$

where the second integral equals

$$\int_\Omega (v_{i,x_j}v_j)_{,x_i}\,dx = \int_S n_i v_{i,x_j}v_j\,dS = \int_S n_i v_{i,x_j}v_{\tau_\alpha}\tau_{\alpha j}\,dS \equiv I,$$

where $v_{\tau_\alpha} = v\cdot\bar{\tau}_\alpha$, and summation over all repeated indices is assumed.

In view of $(1.1)_3$ we have

$$I = -\int\limits_S n_{i,x_j} v_i \tau_{\alpha j} v_{\tau_\alpha}\, dS = -\int\limits_{S_1} n_{i,x_j} v_i \tau_{\alpha j} v_{\tau_\alpha}\, dS_1 - \int\limits_{S_2} n_{i,x_j} v_i \tau_{\alpha j} v_{\tau_\alpha}\, dS_2$$
$$\equiv I_1 + I_2.$$

Since $\overline{n}|_{S_1} = (\cos\varphi, \sin\varphi, 0) = \overline{e}_r$, $\overline{\tau}_1|_{S_1} = (-\sin\varphi, \cos\varphi, 0) = \overline{e}_\varphi$, $\overline{\tau}_2|_{S_1} = (0,0,1) = \overline{e}_z$, $\overline{n}|_{S_2} = (0,0,1) = \overline{e}_z$, $\overline{\tau}_1|_{S_2} = (\cos\varphi, \sin\varphi, 0) = \overline{e}_r$, and $\overline{\tau}_2|_{S_2} = (-\sin\varphi, \cos\varphi, 0) = \overline{e}_\varphi$, we have $I_2 = 0$ and

$$I_1 = \sum_{\sigma=1}^{2} (-1)^{\sigma+1} \int\limits_{S_1} \frac{1}{R_\sigma} e_{ri,\varphi} v_i v_\varphi|_{r=R_\sigma} R_\sigma\, d\varphi\, dz$$
$$= \sum_{i=1}^{2} (-1)^{i+1} \int\limits_0^{2\pi} d\varphi \int\limits_{-a}^{a} dz\, v_\varphi^2|_{r=R_i}.$$

where e_{ri}, $i = 1,2,3$, are the Cartesian coordinates of $\overline{e}_r$. Summing (3.8) implies

$$|\nabla v|_{2,\Omega}^2 = \frac{1}{2} E_\Omega(v) + \sum_{i=1}^{2} (-1)^{i+1} \int\limits_0^{2\pi} d\varphi \int\limits_{-a}^{a} dz\, v_\varphi^2|_{r=R_i}. \tag{3.9}$$

Hence, by the trace theorem we have

$$|\nabla v|_{2,\Omega}^2 \le c(E_\Omega(v) + |v|_{2,\Omega}^2). \tag{3.10}$$

Since $v_r|_{S_1} = 0$ and $v_z|_{S_2} = 0$ we obtain

$$|v_r|_{2,\Omega} \le c|\partial_r v_r|_{2,\Omega} \le c|\partial_r v'|_{2,\Omega} \le c|\nabla v'|_{2,\Omega}, \tag{3.11}$$
$$|v_z|_{2,\Omega} \le c|\partial_z v_z|_{2,\Omega} \le c|\partial_z v'|_{2,\Omega} \le c|\nabla v'|_{2,\Omega}. \tag{3.12}$$

To obtain an estimate for $|v_\varphi|_{2,\Omega}$ we have to prove that there exists constant $M = M(\delta)$ such that

$$|v'|_{2,\Omega}^2 \le \delta|\nabla v'|_{2,\Omega}^2 + ME_\Omega(v'), \tag{3.13}$$

where $\delta > 0$ can be chosen as small as we wish.

Assume that such an M does not exist. Then for any $m \in \mathbb{N}$ there exists $v'^m \in H^1(\Omega)$ such that

$$|v'^m|_{2,\Omega}^2 \ge \delta|\nabla v'^m|_{2,\Omega}^2 + mE_\Omega(v'^m) \equiv G_m(v'^m).$$

For $u^m = v'^m/|v'^m|_{2,\Omega}$ we have

$$|u^m|_{2,\Omega} = 1, \quad G_m(u^m) = \frac{G_m(v'^m)}{|v'^m|_{2,\Omega}} \le 1.$$

Therefore, we can choose a subsequence $\{u^{m_k}\}$ which converges weakly in $H^1(\Omega)$ and strongly in $L_2(\Omega)$ to a limit $u \in H^1(\Omega)$. Moreover, $E_\Omega(u^{m_k}) \le$

$1/m_k \to 0$. Hence $E_\Omega(u) = 0$, so $u = c\eta$. But $c = 0$ because $u \perp \eta$ in $L_2(\Omega)$. This contradicts

$$|u|_{2,\Omega} = \lim_{m_k\to\infty} |u^{m_k}|_{2,\Omega} = 1.$$

Hence, (3.13) holds.

In view of (3.7) inequality (3.13) takes the form

$$|v|^2_{2,\Omega} \le \delta|\nabla v|^2_{2,\Omega} + ME_\Omega(v) + c\Big|\int\limits_\Omega v_\eta\,dx\Big|^2, \tag{3.14}$$

where we used the fact that $E_\Omega(v') = E_\Omega(v)$. Utilizing (3.14) in (3.10) yields

$$|\nabla v|^2_{2,\Omega} \le c\Big(E_\Omega(v) + \Big|\int\limits_\Omega v_\eta\,dx\Big|^2\Big). \tag{3.15}$$

From the form of v' we have

$$\begin{aligned} |v_\varphi|^2_{2,\Omega} &\le c\Big(|v_r|^2_{2,\Omega} + |v_z|^2_{2,\Omega} + |v'|^2_{2,\Omega} + \Big|\int\limits_\Omega v_\eta\,dx\Big|^2\Big) \\ &\le c\Big(E_\Omega(v) + \Big|\int\limits_\Omega v_\eta\,dx\Big|^2\Big), \end{aligned} \tag{3.16}$$

where we used (3.11), (3.12), (3.13) and (3.15) to obtain the last inequality.

Summing up, from (3.11), (3.12), (3.15) and (3.16) we obtain (3.5), which ends the proof.

LEMMA 3.3. *Assume that there exist constants* a_1, a_2 *such that* $\sup_t |f(t)|_{1',\Omega} \equiv a_1 < \infty$, $1' > 1$ *but close to* 1, *and* $\sup_t |\int_{\Omega^t} f_\eta\,dx\,dt'| \equiv a_2 < \infty$. *Then there exist constants*

$$\begin{aligned} d_1^2 &= \frac{c}{\nu_1}\Big(a_1^2 + a_2^2 + \Big|\int\limits_\Omega v_\eta(0)\,dx\Big|^2\Big) + |v(0)|^2_{2,\Omega}, \\ d_2^2(T) &= (e^{\nu_1 T} + 3)d_1^2, \end{aligned} \tag{3.17}$$

independent of $k_0 = kT$ *such that*

$$\begin{aligned} &|v(t)|_{2,\Omega} \le d_1 && \textit{for any } t > 0, \\ &|v(t)|_{2,\Omega} + \Big(\int\limits_{kT}^t \|v(t')\|^2_{1,\Omega}dt'\Big)^{1/2} \le d_2 && \textit{for } t \in (kT, (k+1)T). \end{aligned} \tag{3.18}$$

Proof. Multiplying $(1.1)_1$ by v and integrating over Ω we obtain

$$\frac{1}{2}\frac{d}{dt}|v|^2_{2,\Omega} + \nu E_\Omega(v) = \int\limits_\Omega f\cdot v\,dx. \tag{3.19}$$

Utilizing (3.5) yields

$$\frac{1}{2}\frac{d}{dt}|v|^2_{2,\Omega} + \nu\|v\|^2_{1,\Omega} \le c\Big(\int\limits_\Omega f\cdot v\,dx + \Big|\int\limits_\Omega v_\eta\,dx\Big|^2\Big). \tag{3.20}$$

Expressing (3.6) in the form

$$\int_{\Omega} v_\eta\,dx = \int_{\Omega^t} f_\eta\,dx\,dt' + \int_{\Omega} v_\eta(0)\,dx \tag{3.21}$$

we replace (3.20) by

$$\frac{1}{2}\frac{d}{dt}|v|^2_{2,\Omega}+\nu\|v\|^2_{1,\Omega} \le c\int_{\Omega} f\cdot v\,dx+c\Big|\int_{\Omega^t} f_\eta\,dx\,dt'\Big|^2+c\Big|\int_{\Omega} v_\eta(0)\,dx\Big|^2. \tag{3.22}$$

Applying the Hölder and Young inequalities to the first term on the r.h.s. of (3.22) and using the imbedding $|v|_{q,\Omega} \le c\|v\|_{1,\Omega}$ for any finite q, we rewrite the estimate (3.22) in the form

$$\frac{d}{dt}|v|^2_{2,\Omega} + \nu\|v\|^2_{1,\Omega} \le c|f|^2_{1',\Omega} + c\Big|\int_{\Omega^t} f_\eta\,dx\,dt'\Big|^2 + c\Big|\int_{\Omega} v_\eta(0)\,dx\Big|^2, \tag{3.23}$$

where $1'$ is any number greater than 1 but close to 1.

For $\nu = \nu_1 + \nu_2$, (3.23) takes the form

$$\begin{aligned}\frac{d}{dt}|v|^2_{2,\Omega} + \nu_1|v|^2_{2,\Omega} + \nu_2\|v\|^2_{1,\Omega}\\ \le c|f|^2_{1',\Omega} + c\Big|\int_{\Omega^t} f_\eta\,dx\,dt'\Big|^2 + c\Big|\int_{\Omega} v_\eta(0)\,dx\Big|^2.\end{aligned} \tag{3.24}$$

Now, (3.24) implies

$$\begin{aligned}\frac{d}{dt}(|v|^2_{2,\Omega}e^{\nu_1 t}) + \nu_2\|v\|^2_{1,\Omega}e^{\nu_1 t}\\ \le c|f|^2_{1',\Omega}e^{\nu_1 t} + \Big|\int_{\Omega^t} f_\eta\,dx\,dt'\Big|^2 e^{\nu_1 t} + c\Big|\int_{\Omega} v_\eta(0)\,dx\Big|^2 e^{\nu_1 t}.\end{aligned} \tag{3.25}$$

Integrating (3.25) with respect to time yields

$$\begin{aligned}|v(t)|^2_{2,\Omega} + \nu_2 e^{-\nu_1 t}\int_0^t \|v(t')\|^2_{1,\Omega}e^{\nu_1 t'}\,dt'\\ \le ce^{-\nu_1 t}\int_0^t |f(t')|^2_{1',\Omega}e^{\nu_1 t'}\,dt' + ce^{-\nu_1 t}\int_0^t\Big|\int_{\Omega^{t'}} f_\eta dx dt''\Big|^2 e^{\nu_1 t'}\,dt'\\ + e^{-\nu_1 t}\int_0^t\Big|\int_{\Omega} v_\eta(0)\,dx\Big|^2 e^{\nu_1 t'}\,dt' + e^{-\nu_1 t}|v(0)|^2_{2,\Omega}.\end{aligned} \tag{3.26}$$

Dropping the second term on the l.h.s. of (3.26) and utilizing the assumptions of the lemma we obtain

$$(3.27)\quad |v(t)|_{2,\Omega}^2 \le c\Big(a_1^2+a_2^2+\Big|\int\limits_\Omega v_\eta(0)\,dx\Big|^2\Big)e^{-\nu_1 t}\int\limits_0^t e^{\nu_1 t'}\,dt' + e^{-\nu_1 t}|v(0)|_{2,\Omega}^2$$
$$\le \frac{c}{\nu_1}\Big(a_1^2+a_2^2+\Big|\int\limits_\Omega v_\eta(0)\,dx\Big|^2\Big) + e^{-\nu_1 t}|v(0)|_{2,\Omega}^2$$
$$\le \frac{c}{\nu_1}\Big(a_1^2+a_2^2+\Big|\int\limits_\Omega v_\eta(0)\,dx\Big|^2\Big) + |v(0)|_{2,\Omega}^2 \equiv d_1^2.$$

Taking into account estimate (3.27), we consider (3.26) for $t\in(kT,(k+1)T)$, which yields

$$(3.28)\quad |v(t)|_{2,\Omega}^2 + \nu_2 e^{-\nu_1 t}\int\limits_{kT}^t \|v(t')\|_{1,\Omega}^2 e^{\nu_1 t'}\,dt'$$
$$\le c\Big(a_1^2+a_2^2+\Big|\int\limits_\Omega v_\eta(0)\,dx\Big|^2\Big)e^{-\nu_1 t}\int\limits_{kT}^t e^{\nu_1 t'}\,dt' + e^{-\nu_1(t-kT)}|v(kT)|_{2,\Omega}^2.$$

Continuing, we get

$$(3.29)\quad |v(t)|_{2,\Omega}^2 + \nu_2 e^{-\nu_1(t-kT)}\int\limits_{kT}^t \|v(t')\|_{1,\Omega}^2\,dt'$$
$$\le \frac{c}{\nu_1}\Big(a_1^2+a_2^2+\Big|\int\limits_\Omega v_\eta(0)\,dx\Big|^2\Big) + e^{-\nu_1(t-kT)}|v(kT)|_{2,\Omega}^2$$

for $t\in(kT,(k+1)T))$. Finally, this leads to

$$(3.30)\quad |v(t)|_{2,\Omega}^2 + \nu_2\int\limits_{kT}^t \|v(t')\|_{1,\Omega}^2\,dt'$$
$$\le \frac{c}{\nu_1}(e^{\nu_1 T}+1)\Big(a_1^2+a_2^2+\Big|\int\limits_\Omega v_\eta(0)dx\Big|^2\Big) + 2|v(kT)|_{2,\Omega}^2$$
$$\le (e^{\nu_1 T}+1)d_1^2 + 2d_1^2 \equiv d_2^2(T).$$

This ends the proof.

The fact that d_1 and d_2 do not depend on k_0 and estimates (3.18) suggest that the proof of global existence can be done step by step.

LEMMA 3.4. *Assume that* $v(0),\chi(0)\in L_2(\Omega)$. *Assume that* $\sup_t|f'|_{2,\Omega} \equiv a_3<\infty$. *Then*

$$\Big|\frac{\chi(t)}{r}\Big|_{2,\Omega} \le d_3,$$

$$(3.31)\quad \Big|\frac{\chi(t)}{r}\Big|_{2,\Omega} + \Big(\int\limits_{k_0}^{k_0+t}\Big\|\frac{\chi(t')}{r}\Big\|_{1,\Omega}^2\,dt'\Big)^{1/2} \le d_4(T),$$

where $t \in (k_0, k_0+T)$, $k_0 \in \mathbb{R}_+$,

$$
\begin{aligned}
d_3 &= cd_1(d_1 + |v(0)|_{2,\Omega}) + ca_3 + \left|\frac{\chi(0)}{r}\right|_{2,\Omega},\\
d_4(T) &= c(d_1^2 + a_3)e^{\nu_1 T} + cd_3
\end{aligned}
\tag{3.32}
$$

and c *is a constant.*

Proof. In view of $(1.6)_2$, (2.1), and the Poincaré inequality, we write (3.3) in the form

$$
\frac{d}{dt}\left|\frac{\chi}{r}\right|_{2,\Omega}^2 + \nu_1\left|\frac{\chi}{r}\right|_{2,\Omega}^2 + \left(\frac{\nu}{2}+\nu_2\right)\left\|\frac{\chi}{r}\right\|_{1,\Omega}^2 \le c|w|_{2,\Omega}^2\|w\|_{1,\Omega}^2 + c|f'|_{2,\Omega}^2,
$$

where $\nu = \nu_1 + \nu_2$. Multiplying by $e^{\nu_1 t}$ and integrating with respect to time yields

$$
\begin{aligned}
&\left|\frac{\chi(t)}{r}\right|_{2,\Omega}^2 e^{\nu_1 t} + \nu_3\int_0^t \left\|\frac{\chi(t')}{r}\right\|_{1,\Omega}^2 e^{\nu_1 t'}\,dt'\\
&\le c_1\sup_t |w(t)|_{2,\Omega}^2 \int_0^t \|w(t')\|_{1,\Omega}^2 e^{\nu_1 t'}\,dt' + c_2 a_3^2\int_0^t e^{\nu_1 t'}\,dt' + \left|\frac{\chi(0)}{r}\right|_{2,\Omega}^2,
\end{aligned}
\tag{3.33}
$$

where $\nu_3 = \nu_2 + \nu/2$. Integrating (3.25) with respect to time yields

$$
\begin{aligned}
|v(t)|_{2,\Omega}^2 e^{\nu_1 t} + \nu_2\int_0^t \|v(t')\|_{1,\Omega}^2 e^{\nu_1 t'}\,dt'
&\le \frac{c}{\nu_1}\left(a_1^2 + a_2^2 + \left|\int_\Omega v_\eta(0)\,dx\right|^2\right)e^{\nu_1 t} + |v(0)|_{2,\Omega}^2\\
&\le d_1^2 e^{\nu_1 t} + |v(0)|_{2,\Omega}^2.
\end{aligned}
\tag{3.34}
$$

Applying this inequality to estimate the second factor in the first term on the r.h.s. of (3.33) we obtain

$$
\begin{aligned}
&\left|\frac{\chi(t)}{r}\right|_{2,\Omega}^2 e^{\nu_1 t} + \nu_3\int_0^t \left\|\frac{\chi(t')}{r}\right\|_{1,\Omega}^2 e^{\nu_1 t'}\,dt'\\
&\qquad\le cd_1^2(d_1^2 e^{\nu_1 t} + |v(0)|_{2,\Omega}^2) + \frac{c}{\nu_1}a_3^2 e^{\nu_1 t} + \left|\frac{\chi(0)}{r}\right|_{2,\Omega}^2.
\end{aligned}
\tag{3.35}
$$

Omitting the second term on the l.h.s. of (3.35) we obtain for any $t > 0$ the estimate

$$
\begin{aligned}
\left|\frac{\chi(t)}{r}\right|_{2,\Omega}^2 &\le cd_1^2(d_1^2 + |v(0)|_{2,\Omega}^2 e^{-\nu_1 t}) + \frac{c}{\nu_1}\,a_3^2 + \left|\frac{\chi(0)}{r}\right|_{2,\Omega}^2 e^{-\nu_1 t}\\
&\le cd_1^2(d_1^2 + |v(0)|_{2,\Omega}^2) + \frac{c}{\nu_1}\,a_3^2 + \left|\frac{\chi(0)}{r}\right|_{2,\Omega}^2 \equiv d_3^2.
\end{aligned}
\tag{3.36}
$$

Now we consider (3.33) in $(kT,(k+1)T)$. For t in this interval, we get

$$(3.37)\quad \left|\frac{\chi(t)}{r}\right|_{2,\Omega}^2 e^{\nu_1 t}+\nu_3\int\limits_{kT}^{t}\left\|\frac{\chi(t')}{r}\right\|_{1,\Omega}^2 e^{\nu_1 t'}\,dt' \le c\sup_{kT\le t'\le(k+1)T}|w(t')|_{2,\Omega}^2\int\limits_{kT}^{t}\|w(t')\|_{1,\Omega}^2 e^{\nu_1 t'}\,dt' + ca_3^2\int\limits_{kT}^{t}e^{\nu_1 t'}\,dt'+\left|\frac{\chi(kT)}{r}\right|_{2,\Omega}^2 e^{\nu_1 kT}.$$

Considering (3.34) in $(kT,(k+1)T)$ we obtain

$$(3.38)\quad |v(t)|_{2,\Omega}^2 e^{\nu_1 t}+\nu_2\int\limits_{kT}^{t}\|v(t')\|_{1,\Omega}^2 e^{\nu_1 t'}\,dt' \le cd_1^2 e^{\nu_1 t}+|v(kT)|_{2,\Omega}^2 e^{\nu_1 kT}\le cd_1^2 e^{\nu_1 t}.$$

Applying (3.36) and (3.38) in (3.37) yields

$$(3.39)\quad \left|\frac{\chi(t)}{r}\right|_{2,\Omega}^2 e^{\nu_1 t}+\nu_3\int\limits_{kT}^{t}\left\|\frac{\chi(t')}{r}\right\|_{1,\Omega}^2 e^{\nu_1 t'}\,dt' \le c(d_1^4+a_3^2)e^{2\nu_1 t}+cd_3^2 e^{\nu_1 kT}.$$

Hence (3.39) implies

$$(3.40)\quad \int\limits_{kT}^{t}\left\|\frac{\chi(t')}{r}\right\|_{1,\Omega}^2 dt' \le c(d_1^4+a_3^2)e^{\nu_1 T}+cd_3^2\equiv d_4(T).$$

Inequalities (3.36) and (3.40) give (3.31). This ends the proof.

Expressing $(1.1)_2$ in polar coordinates

$$v_{r,r}+v_{z,z}+\frac{v_r}{r}=0$$

we obtain the following elliptic problem:

$$(3.41)\quad \begin{aligned} &v_{r,z}-v_{z,r}=\chi,\\ &v_{r,r}+v_{z,z}=-v_r/r,\\ &\overline{n}\cdot v|_S=0. \end{aligned}$$

If we introduce new quantities $u_r=rv_r$, $u_z=rv_z$, then (3.41) takes the form

$$(3.42)\quad \begin{aligned} &u_{r,r}+u_{z,z}=0,\\ &u_{r,z}-u_{z,r}=r\chi-v_z,\\ &\overline{n}\cdot u|_S=0. \end{aligned}$$

Equation $(3.42)_1$ implies the existence of a potential ψ such that $u_r = \psi_{,z}$, $u_z = -\psi_{,r}$, and $(3.42)_3$ gives

$$(n_1\psi_{,z} - n_2\psi_{,r})|_S = \overline{\tau}\cdot\nabla\psi|_S = 0, \quad \text{so} \quad \psi|_S = \text{const}\,.$$

Since ψ is determined up to an arbitrary constant, we can assume that $\psi|_S = 0$. Therefore (3.42) implies the following problem:

$$\begin{aligned} &\Delta\psi = r\chi - v_z,\\ &\psi|_S = 0, \end{aligned} \tag{3.43}$$

where $\Delta = \partial_r^2 + \partial_z^2$.

Let us consider the interval $J_k \equiv (kT,(k+1)T) = (k_0, k_0+T)$. From (3.18) and (3.31) we have the estimates

$$\begin{aligned} &\|v\|_{L_2(J_k;H^1(\Omega))} + \|v\|_{L_4(J_k;L_4(\Omega))} + \|v\|_{L_\infty(J_k;L_2(\Omega))} \le d_2(T),\\ &\|\chi\|_{L_2(J_k;H^1(\Omega))} + \|\chi\|_{L_4(J_k;L_4(\Omega))} + \|\chi\|_{L_\infty(J_k;L_2(\Omega))} \le d_4(T). \end{aligned} \tag{3.44}$$

Then we have

LEMMA 3.5. *Let the assumptions of Lemmas* 3.3 *and* 3.4 *hold. Let* $t \in J_k = (k_0, k_0+T)$, $k_0 = kT$, $k \in \mathbb{N}$. *Then* (3.43) *implies*

$$\begin{aligned} \|v'\|_{L_2(J_k;H^2(\Omega))} &+ \|v'\|_{L_4(J_k;W_4^1(\Omega))}\\ &+ \|v'\|_{L_\infty(J_k;W_2^1(\Omega))} \le c(d_2(T)+d_4(T)), \end{aligned} \tag{3.45}$$

where $v' = (v_r, v_z)$.

From (3.45) we have

$$\|v'\|_{L_\infty(J_k;L_q(\Omega))} \le c(d_2(T)+d_4(T)), \qquad q\in(1,\infty). \tag{3.46}$$

REMARK 3.6. Since we examine problem (3.43) in a rectangle in the (r,z)-plane, some compatibility conditions at its vertices must be satisfied. Since $\psi_{,r} = v_z$ and $\psi_{,z} = -v_r$, we have $\psi_{,rr} = v_{z,r} = 0$ and $\psi_{,zz} = -v_{r,z} = 0$ at vertices, so $\Delta\psi|_{\text{vertex}} = 0$. On the other hand $\chi = v_{r,z} - v_{z,r}$, so $\chi|_{\text{vertex}} = 0$ and $v_z|_{\text{vertex}} = 0$. Hence $(3.43)_1$ is satisfied in the rectangle.

To prove global existence we need additional estimates for w.

LEMMA 3.7. *Suppose the assumptions of Lemma* 3.5, $f_\varphi(0) \in L_s(\Omega)$, $(1.6)_1$, *and* $w(kT) \in W_s^{2-2/s}(\Omega)$, *hold for some* $s > 1$. *Then there exists a solution to problem* (1.5) *such that* $w \in W_s^{2,1}(\Omega\times(kT,(k+1)T))$, *and there exist positive increasing functions* $\varphi_1(\sigma)$, $c_3(\sigma)$, $\sigma\in\mathbb{R}_+$ *such that*

$$\begin{aligned} \|w\|_{2,s,\Omega\times(kT,(k+1)T)} \le c_3(T)[&\varphi_1(d_2+d_4)d_2 + |f_\varphi(0)|_{s,\Omega}\\ &+ \|w(kT)\|_{2-2/s,s,\Omega}]. \end{aligned} \tag{3.47}$$

Proof. The existence of w follows from potential theory. We only show the estimate. Set again $J_k = (kT,(k+1)T)$. For solutions of (1.5) we have

$$(3.48)\qquad \|w\|_{2,s,\Omega\times J_k} \le c(T)\Big[|v\cdot\nabla w|_{s,\Omega\times J_k} + \Big|\frac{v_r}{r}w\Big|_{s,\Omega\times J_k} + \Big|\frac{w}{r^2}\Big|_{s,\Omega\times J_k} + \|w\|_{1-1/s,s,S\times J_k} + \|w(kT)\|_{2-2/s,s,\Omega} + |f_\varphi|_{s,\Omega\times J_k}\Big].$$

In virtue of (3.46) and by the interpolation inequality (2.5), the first two terms on the r.h.s. of (3.48) are estimated by

$$\varepsilon_1\|w\|_{2,s,\Omega\times J_k} + c(1/\varepsilon_1, T, d_2+d_4)|w|_{\sigma,\Omega\times J_k} \equiv I,$$

for any $\varepsilon_1 \in (0,1)$, where c is an increasing function and $\sigma < s$. If we choose $\sigma \le 4$ and apply $(3.18)_2$, the second term is bounded by cd_2.

Similarly, by $(3.18)_2$ and (2.5), the third and the fourth terms on the r.h.s. of (3.48) are bounded by I. Hence for sufficiently small ε_1 we conclude the proof.

Finally, we examine problem (1.1).

LEMMA 3.8. *Let the assumptions of Lemma* 3.7, $v(kT) \in W_r^{2-2/r}(\Omega)$ *and* $f|_{t=0} \equiv f(0) \in L_r(\Omega)$, $(1.6)_1$ *hold. Assume that* $\sigma < r$ *and* $v \in L_\sigma(\Omega \times (kT,(k+1)T))$, $\sigma \le 4$. *Then there exist increasing positive functions* c_4, φ_2 *such that*

$$(3.49)\qquad \|v\|_{2,r,\Omega\times(kT,(k+1)T)} \le c_4(T)[\varphi_2(d_2+d_4,T)d_2 + |f(0)|_{r,\Omega} + \|v(kT)\|_{2-2/r,r,\Omega}].$$

Proof. For solutions of (1.1) we have (see Lemma 1.4)

$$(3.50)\qquad \|v\|_{2,r,\Omega\times J_k} \le c(T)[|v\cdot\nabla v|_{r,\Omega\times J_k} + |f(0)|_{r,\Omega} + \|v(kT)\|_{2-2/r,r,\Omega}].$$

We estimate the first term on the r.h.s. by $|v'\cdot\nabla v|_{r,\Omega\times J_k}$, which is bounded in the same way as the first term on the r.h.s. of (3.48). This concludes the proof.

To prove global existence we need global estimates for $\|w(kT)\|_{2-2/s,s,\Omega}$ and $\|v(kT)\|_{2-2/r,r,\Omega}$ for any $k\in\mathbb{N}$.

To show this, we introduce a smooth function $\zeta_k = \zeta_k(t)$ such that $\zeta_k(t) = 0$ for $t \le kT - T_1$, $\zeta_k(t) = 1$ for $kT - \frac{1}{2}T_1 \le t \le kT + \frac{1}{2}T_1$, and $\zeta_k(t) = 0$ for $t \ge kT + T_1$, where T_1 is small compared to T.

Let us introduce $w_k = w\zeta_k$ and $v_k = v\zeta_k$. In view of (1.1) and (1.5), they solve the problems

$$(3.51)\qquad \begin{aligned}
&v_{k,t} + v\cdot\nabla v_k - \operatorname{div}\mathbb{T}(v_k,p_k) = f_k + v\dot{\zeta}_k && \text{in } \Omega\times J_k',\\
&\operatorname{div} v_k = 0 && \text{in } \Omega\times J_k',\\
&v_k\cdot\overline{n} = 0 && \text{on } S\times J_k',\\
&\overline{n}\cdot\mathbb{T}(v_k,p_k)\cdot\overline{\tau}_\alpha = 0,\quad \alpha=1,2, && \text{on } S\times J_k',\\
&v_k|_{t=kT-T_1} = 0 && \text{in } \Omega,
\end{aligned}$$

where

$$J'_k = (kT - T_1, kT + T_1).$$

Here $p_k = p\zeta_k$, $f_k = f\zeta_k$, $\dot\zeta_k$ is the derivative of ζ_k, and

$$\begin{aligned} & w_{k,t} + v\cdot\nabla w_k + \frac{v_r}{r} w_k - \nu\Delta w_k + \nu\frac{w_k}{r^2} = f_{\varphi k} + w\dot\zeta_k && \text{in } \Omega\times J'_k,\\ & w_{k,r}|_{r=R_i} = \frac{1}{R_i} w_k|_{r=R_i},\ i=1,2, && \text{on } S_1\times J'_k,\\ & w_{k,z} = 0 && \text{on } S_2\times J'_k,\\ & w_k|_{t=kT-T_1} = 0 && \text{in } \Omega, \end{aligned} \tag{3.52}$$

with $f_{\varphi k} = f_\varphi \zeta_k$. Repeating the proofs of Lemmas 3.7 and 3.8 we obtain

$$\|w_k\|_{2,s,\Omega\times J'_k} \le c(T_1)(\varphi_1(d_2+d_4)d_2 + d_2 + |f(0)|_{s,\Omega}) \equiv I_1, \tag{3.53}$$

$$\|v_k\|_{2,r,\Omega\times J'_k} \le c(T_1)(\varphi_2(d_2+d_4)d_2 + d_2 + |f(0)|_{r,\Omega}) \equiv I_2, \tag{3.54}$$

where $s\le 4$ and $r\le 4$. Hence

$$\|w(kT)\|_{2-2/s,s,\Omega} \le I_1, \quad \|v(kT)\|_{2-2/r,r,\Omega} \le I_2, \quad k\in\mathbb{N}. \tag{3.55}$$

In view of (3.55), Lemmas 3.7 and 3.8 give global estimates for w and v.

4. Existence and uniqueness. To prove the existence of solutions to problem (1.1), we shall distinguish two approaches. First we introduce weak solutions to (1.1) and prove their existence. We follow Ladyzhenskaya [6, Ch. 6].

DEFINITION 4.1. By a *weak solution* to problem (1.1) we mean a function v satisfying the integral identity

$$\int_{\Omega^t} [-v\cdot\varphi_{,t'} - v_i v_j \varphi_{j,x_i} + \mathbb{D}(v)\cdot\mathbb{D}(\varphi)]\,dx\,dt' + \int_\Omega v\cdot\varphi|_t\,dx - \int_\Omega v(0)\varphi|_{t=0}\,dx = \int_{\Omega^t} f\cdot\varphi\,dx\,dt' \tag{4.1}$$

for any $\varphi\in W_2^{1,1}(\Omega^T)$ with

$$\begin{aligned} \varphi\cdot\overline{n}|_S &= 0,\\ \operatorname{div}\varphi &= 0. \end{aligned} \tag{4.2}$$

LEMMA 4.2. *Suppose the assumptions of Lemma* 3.3 *hold. Then there exists a weak solution to problem* (1.1) *such that conditions* (3.18) *are satisfied.*

Proof. To prove the existence we use the Galerkin method. Hence, we are looking for approximate solutions in the form

$$v^n = \sum_{i=1}^n \alpha_{ni}(t) a_i(x), \tag{4.3}$$

where the functions a_i form a fundamental system in $H^1(\Omega)$, orthonormal in $L_2(\Omega)$, and such that $\max_{x\in\Omega}|a_i(x)|<\infty$. Moreover, the functions a_i satisfy

$$\begin{aligned}&\operatorname{div} a_i = 0,\\ &a_i\cdot\overline{n}|_S = 0,\\ &\overline{n}\cdot\mathbb{D}(a_i)\cdot\overline{\tau}_\alpha|_S = 0,\qquad \alpha=1,2.\end{aligned}\tag{4.4}$$

The functions $\alpha_{ni}=\alpha_{ni}(t)$, $i=1,\dots,n$, are solutions of the following Cauchy problem:

$$\begin{aligned}&\frac{d}{dt}(v^n,a_i) = -\nu(v^n_{,x},a_{i,x}) - (v^n_k v^n_{,x_k},a_i)+(f,a_i),\\ &\alpha_{ni}(0)=(v(0),a_i),\end{aligned}\tag{4.5}$$

$i=1,\dots,n$, and $(\cdot,\cdot)$ is the scalar product in $L_2(\Omega)$.

The remainder of the proof is exactly the same as in [6, Ch. 6, proof of Theorem 20]. However, to prove the global existence we use Lemma 3.3. Hence, we have to consider system (4.5) in the interval (k_0,k_0+T), $k_0\in\mathbb{R}_+\cup\{0\}$, $k_0=kT$, $k\in\mathbb{N}$. Therefore, we need appropriate initial conditions for $t=k_0$. For this purpose, we examine the functions

$$\psi_{n,l}(t)=(v^n(\cdot,t),a_l(\cdot)).$$

For a fixed l, and $n\ge l$, they form a set of uniformly bounded and uniformly continuous functions on $[k_0,k_0+T]$. The boundedness follows from the estimate $(3.18)_2$:

$$\|v^n\|_{V_2^1(\Omega^T)}\le c.\tag{4.6}$$

To show the uniform continuity we integrate (4.5) with respect to time from t to $t+\Delta t$. Applying the Cauchy inequality we obtain

$$\begin{aligned}&|\psi_{n,l}(t+\Delta t)-\psi_{n,l}(t)|\\ &\quad\le \nu\int_t^{t+\Delta t}|v^n_{,x}|_{2,\Omega}|a_{l,x}|_{2,\Omega}\,dt' + \max_{x\in\Omega}|a_l|\int_t^{t+\Delta t}(|v^n|_{2,\Omega}|v^n_{,x}|_{2,\Omega}+|f|_{2,\Omega})\,dt'\\ &\quad\le \nu|a_{l,x}|_{2,\Omega}|v^n_{,x}|_{2,\Omega\times(k_0,k_0+T)}\sqrt{\Delta t}\\ &\qquad + \max_x|a_l|\Big(\sup_t|v^n|_{2,\Omega}|v^n_{,x}|_{2,\Omega\times(k_0,k_0+T)}\sqrt{\Delta t}+\int_t^{t+\Delta t}|f(t')|_{2,\Omega}\,dt'\Big)\\ &\quad\le c(l)\Big(\sqrt{\Delta t}+\int_t^{t+\Delta t}|f(t')|_{2,\Omega}\,dt'\Big).\end{aligned}$$

In view of the uniform continuity of the sequence $\{\psi_{n,l}\}$, we can choose a subsequence $\{\psi_{n_k,l}\}$ which converges uniformly to a continuous function

$\psi_l(t)$. We see that v is determined for all $t \in [kT, (k+1)T]$ in the form

$$v(x,t) = \sum_l \psi_l(t) a_l(x).$$

In view of the above construction, we can determine initial data for the interval $((k+1)T, (k+2)T)$ having proved the existence of weak solutions in $(kT, (k+1)T)$. This way of proving the existence of global weak solutions is motivated by the proof of Lemma 3.3, where estimate (3.18) is obtained step by step. This ends the proof.

Proof of Theorem 1. To increase regularity of the weak solutions we consider problem (1.1) in the form

$$\begin{aligned} & v_{,t} - \operatorname{div} \mathbb{T}(v,p) = -v' \cdot \nabla v + f, \\ & \operatorname{div} v = 0, \\ & v \cdot \overline{n}|_S = 0, \\ & \overline{n} \cdot \mathbb{T}(v,p) \cdot \overline{\tau}_\alpha|_S = 0, \qquad \alpha = 1,2, \\ & v|_{t=0} = v(0), \end{aligned} \tag{4.7}$$

where $v' = (v_r, v_z)$. By Lemmas 4.1 and 3.3 we have

$$|v' \cdot \nabla v|_{4/3,\Omega^T} \le |v'|_{4,\Omega^T} |\nabla v|_{2,\Omega^T} \le d_2^2.$$

Hence from (4.7) we have existence of weak solutions such that $v \in W^{2,1}_{4/3}(\Omega^T)$. However, this implies only that $v \in L_q(\Omega^T)$, $q \le 4$, so there is no increase of regularity. Since $W^{2,1}_{4/3}(\Omega^T) \subset L_4(\Omega^T)$, [11] yields uniqueness. This implies Theorem 1.

To increase regularity of the weak solutions, we are looking for solutions to problem (1.4) in the form $\chi^n = \sum_{i=1}^n \beta_{ni}(t) b_i(x)$, where β_{ni}, $i \le n$, are solutions to the Cauchy problem

$$\begin{aligned} \frac{d}{dt}(\chi^n, b_i) &+ (v^n \cdot \nabla \chi^n, b_i) + ((v^n_{r,r} + v^n_{z,z})\chi^n, b_i) \\ &= \nu\bigg(\bigg(r\bigg(\frac{\chi^n}{r}\bigg)_{,r}\bigg)_{,r} + \chi^n_{,zz} + 2\bigg(\frac{\chi^n}{r}\bigg)_{,r}, b_i\bigg) \\ &\quad + \bigg(\frac{2}{r} w^n w^n_{,z}, b_i\bigg) + (F_\varphi, b_i), \beta_{ni}(0) = (\chi(0), b_i), \qquad i \le n, \end{aligned} \tag{4.8}$$

and the functions b_i such that $b_i|_S = 0$ form a basis in $H^1(\Omega)$, and v^n, w^n are approximate solutions of v.

In view of Lemma 3.4 we have the estimate

$$\|\chi^n\|_{V_2^1(\Omega^T)} \le d_4. \tag{4.9}$$

Hence, by the usual passage to the limit we have, for a weak solution $v \in V_2^1(\Omega^T)$, also a solution of (1.4) such that $\chi \in V_2^1(\Omega^T)$.

We prove the existence of solutions to problem (1.1) by the Leray–Schauder fixed point theorem. For this purpose, we consider the problem

$$\begin{aligned} &v_{,t} - \operatorname{div} \mathbb{T}(v,p) = -\lambda\overline{v}\cdot\nabla\overline{v} + f && \text{in } \Omega\times J_0,\\ &\operatorname{div} v = 0 && \text{in } \Omega\times J_0,\\ &v\cdot\overline{n} = 0 && \text{on } S\times J_0,\\ &\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S\times J_0,\\ &v|_{t=0} = v(k_0) && \text{in } \Omega, \end{aligned} \tag{4.10}$$

where $\lambda\in[0,1]$, $k_0\in\mathbb{R}_+$ and $J_0 = (k_0, k_0+T)$.

We shall try to prove existence of a fixed point of (4.10) for $\lambda = 1$ with the least possible regularity. Let

$$v = T(\overline{v},\lambda) \tag{4.11}$$

be the transformation determined by (4.10).

Lemma 4.3. *Assume that* $\overline{v}\in L_q(J_0; W_p^1(\Omega))$ *with* $1/p - 1/4 = 1/q$, $p > 8/5$, $q > 8/3$, $f(0)\in L_r(\Omega)$, $v(k_0)\in W_r^{2-2/r}(\Omega)$, $r > 4/3$. *Then the transformation*

$$T : L_q(J_0; W_p^1(\Omega))\times[0,1]\to L_q(J_0; W_p^1(\Omega))$$

is compact for $4/r - 2/p - 2/q < 1$.

Proof. Applying [1] we have (see Lemma 1.4)

$$\|v\|_{2,r,\Omega\times J_0} \le c(|\overline{v}\cdot\nabla\overline{v}|_{r,\Omega\times J_0} + |f(0)|_{r,\Omega} + \|v(k_0)\|_{2-2/r,r,\Omega}), \tag{4.12}$$

where the first term is bounded by

$$|\overline{v}|_{\lambda_1 r,\mu_1 r,\Omega\times J_0}|\nabla\overline{v}|_{\lambda_2 r,\mu_2 r,\Omega\times J_0} \equiv I,$$

where $1/\lambda_1 + 1/\lambda_2 = 1$, $1/\mu_1 + 1/\mu_2 = 1$, and $\lambda_1 r = 2p/(2-p)$, $\lambda_2 r = p$, $\mu_1 r = q = \mu_2 r$. In the above we used the fact that $\overline{v}\in L_q(J_0; L_\sigma(\Omega))$ with $\sigma = 2p/(2-p)$ for $p < 2$ and $\sigma = \infty$ for $p > 2$. In view of the compact imbedding

$$W_r^{2,1}(\Omega\times J_0)\subset L_q(J_0; W_p^1(\Omega))$$

(which holds under the assumptions of the lemma) and

$$I \le c\|\overline{v}\|^2_{L_q(J_0; W_p^1(\Omega))},$$

the transformation (4.11) is compact. This concludes the proof.

Proof of Theorem 2. The uniform continuity of T with respect to $\overline{v}$ in $L_q(J_0; W_p^1(\Omega))$ and with respect to $\lambda\in[0,1]$ is evident. The a priori bound for a fixed point of (4.11) with $\lambda = 1$ is found in Lemma 3.8. For $\lambda = 0$ we have the unique existence (see [1]). Compactness of T follows from

Lemma 4.3. Hence, by the Leray–Schauder fixed point theorem, we have existence of solutions to problem (1.1). This ends the proof.

Proof of Theorem 3. We assume that we have two solutions v_i, p_i, $i = 1,2$, of problem (1.1). Then $V = v_1 - v_1$, $P = p_1 - p_2$ are solutions to the problem

$$
\begin{aligned}
&V_{,t} - \operatorname{div} \mathbb{T}(V,P) = -(V\cdot\nabla v_1 + v_2\cdot\nabla V) && \text{in } \Omega\times J_0,\\
&\operatorname{div} V = 0 && \text{in } \Omega\times J_0,\\
&V\cdot\overline{n} = 0 && \text{on } S\times J_0,\\
&\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha = 0,\quad \alpha=1,2, && \text{on } S\times J_0,\\
&V|_{t=k_0} = 0 && \text{in } \Omega.
\end{aligned}
\tag{4.13}
$$

Multiplying $(4.13)_1$ by V and integrating over Ω implies

$$
\frac{1}{2}\frac{d}{dt}|V|_{2,\Omega}^2 + \nu\|V\|_{1,\Omega}^2 \le |v_1|_{\infty,\Omega}^2 |V|_{2,\Omega}^2. \tag{4.14}
$$

From (4.14) we have uniqueness of solutions of problem (1.1) such that $v\in L_2(J_0;L_\infty(\Omega))$ for any $k_0\in\mathbb{R}_+$. This proves Theorem 3.

Remark 4.4. Weak solutions determined by Lemma 4.2 satisfy the Serrin condition (see [10]).

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