COLLOQUIUM MATHEMATICUM

VOL. 124 2011 NO. 2

ON TERMS OF LINEAR RECURRENCE SEQUENCES WITH ONLY ONE DISTINCT BLOCK OF DIGITS

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Abstract. In 2000, Florian Luca proved that $F_{10} = 55$ and $L_5 = 11$ are the largest numbers with only one distinct digit in the Fibonacci and Lucas sequences, respectively. In this paper, we find terms of a linear recurrence sequence with only one block of digits in its expansion in base $g \geq 2$. As an application, we generalize Luca's result by finding the Fibonacci and Lucas numbers with only one distinct block of digits of length up to 10 in its decimal expansion.

1. Introduction. A sequence $(G_n)_{n\geq 1}$ is a *linear recurrence sequence* with coefficients $c_0, c_1, \ldots, c_{k-1}$, with $c_0 \neq 0$, if

$$G_{n+k} = c_{k-1}G_{n+k-1} + \cdots + c_1 G_{n+1} + c_0 G_n \tag{1.1}$$

for all positive integers n. A recurrence sequence is therefore completely determined by the *initial values* $G_0, \ldots, G_{k-1}$, and by the coefficients $c_0, c_1, \ldots, c_{k-1}$. The integer k is called the *order* of the linear recurrence. The *characteristic polynomial* of the sequence $(G_n)_{n\geq 0}$ is given by

$$G(x) = x^k - c_{k-1}x^{k-1} - \cdots - c_1 x - c_0.$$

It is well-known that for all n,

$$G_n = g_1(n)r_1^n + \cdots + g_l(n)r_l^n,$$

where r_j is a root of $G(x)$ and $g_j(x)$ is a polynomial over a certain number field, for $j = 1, \ldots, l$. A root r_j of the recurrence is called a *dominant root* if $|r_j| > |r_i|$ for all $j \neq i \in \{1, \ldots, l\}$. The corresponding polynomial $g_j(n)$ is named the *dominant polynomial* of the recurrence. In this paper, we consider only integer recurrence sequences, i.e. recurrence sequences whose coefficients and initial values are integers. Hence, $g_j(n)$ is an algebraic number for all $j = 1, \ldots, l$ and $n \in \mathbb{Z}$.

A general Lucas sequence $(C_n)_{n\geq 1}$ given by $C_{n+2} = C_{n+1} + C_n$ for $n \geq 1$, where the values C_0 and C_1 are previously fixed, is an example of a linear recurrence of order 2 (also called *binary*). For instance, if $C_0 = 0$ and $C_1 = 1$,

2010 *Mathematics Subject Classification*: 11A63, 11B37, 11B39, 11J86.
Key words and phrases: Fibonacci, Lucas, digits, linear forms in logarithms, reduction method, linear recurrence sequence.

DOI: 10.4064/cm124-2-1

then $(C_n)_{n\geq 1} = (F_n)_{n\geq 1}$ is the well-known *Fibonacci sequence*:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots.$$

Also, if $C_0 = 2$ and $C_1 = 1$, the sequence $C_n = L_n$ gives the *Lucas numbers*:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \dots.$$

In 2000, F. Luca [2] proved that $F_{10} = 55$ and $L_5 = 11$ are the largest numbers with only one distinct digit in the Fibonacci and Lucas sequences, respectively. A question arises: is there any Fibonacci or Lucas number of the form $1212\dots 12$? And of the form $175175\dots 175$? And so on? More generally, let B be a natural number with l digits. One can think of a string of B's, that is,

$$B \cdot \frac{10^{lm} - 1}{10^l - 1} = B \cdots B \qquad (m \text{ times}).$$

In particular, Luca's result concerns the case $l = 1$. Moreover, it seems to be harder to answer the previous questions when we replace Fibonacci and Lucas numbers by a term of a general linear recurrence sequence.

The aim of this paper is to determine terms of an integer linear recurrence sequence with only B in its expansion in a base $g \geq 2$. More precisely, our main result is the following.

Theorem 1. *Let $(G_n)_{n\geq 1}$ be an integer linear recurrence sequence whose characteristic polynomial has a positive dominant root. Let $g \geq 2$ and $l \geq 1$ be integers. Then there exists an effectively computable constant C such that if n, m, B are solutions of the Diophantine equation*

$$G_n = B \cdot \frac{g^{lm} - 1}{g^l - 1} \tag{1.2}$$

such that $0 < B < g^l$, then $n, m \leq C$. The constant C depends only on g, l and the parameters of G_n.

As an application, we use our method to find Fibonacci and Lucas numbers with only B in their decimal expansion, where the number B has at most 10 digits.

Theorem 2. *Let B be a natural number with l digits. The only solutions of the Diophantine equations*

$$F_n = B \cdot \frac{10^{lm} - 1}{10^l - 1} \quad \text{and} \quad L_n = B \cdot \frac{10^{lm} - 1}{10^l - 1}, \tag{1.3}$$

in positive integer numbers m, n and l, with $m > 1$ and $1 \leq l \leq 10$, are $(m, n, l) = (2, 10, 1)$ and $(m, n, l) = (2, 5, 1)$ in the Fibonacci and Lucas cases, respectively.

We organize this paper as follows. In Section 2, we recall some useful properties such as a result of Matveev on linear forms in three logarithms and the reduction method of Baker–Davenport that we use to prove Theorems 1 and 2. The third section is devoted to the proof of Theorem 1. In the last section, for each particular case (Fibonacci and Lucas), we first use Baker's method to obtain a bound for n, then we completely solve the problem by means of the Baker–Davenport reduction method. Thus, we prove Theorem 2.

2. Auxiliary results. In this section, we recall some results that will be useful for the proof of the above theorems. Let $G(x)$ be the characteristic polynomial of a linear recurrence G_n. One can factor $G(x)$ over the set of complex numbers as

$$G(x) = (x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_l)^{m_l},$$

where $r_1, \ldots, r_l$ are distinct non-zero complex numbers (called the *roots* of the recurrence) and $m_1, \ldots, m_l$ are positive integers. A fundamental result in the theory of recurrence sequences asserts that there exist uniquely determined polynomials $g_1, \ldots, g_l \in \mathbb{Q}(\{r_j\}_{j=0}^{l})[x]$, with $\deg g_j \leq m_j - 1$ for $j = 1, \ldots, l$, such that

$$G_n = g_1(n)r_1^n + \cdots + g_l(n)r_l^n \quad \text{for all } n. \tag{2.1}$$

For more details, one can refer to [4, Theorem C.1].

In the case of Fibonacci and Lucas sequences, the above formula is known as *Binet's formulas*:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where $\alpha = (1 + \sqrt{5})/2$ (the golden number) and $\beta = (1 - \sqrt{5})/2 = -1/\alpha$. Moreover, one can easily prove by induction that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1}, \quad \alpha^{n-1} \leq L_n \leq 2\alpha^n$$

for all $n \geq 1$.

The first lemma will be useful in the proof of Theorem 1.

Lemma 1. *Let $(G_n)_{n\geq 1}$ be a linear recurrence having a dominant root r_1 and an infinite subsequence of positive terms. Denote by $g_1(n)$ the dominant polynomial of $(G_n)_{n\geq 1}$. Then $g_1(n)$ is a non-zero constant. Moreover, if $t \in \{0, 1\}$ and $G_{2n+t} > 0$ for infinitely many integers n, then $g_1(n)r_1^t > 0$.*

Proof. We know that

$$G_n = g_1(n)r_1^n + \cdots + g_l(n)r_l^n,$$

where each r_j is a root of the characteristic polynomial of G_n, with multiplicity m_j, and each $g_j(n)$ is a non-zero polynomial of degree $\leq m_j - 1$.

Suppose that r_1 is the dominant root; then we immediately see that $r_1 \neq r_j$ for all $j \neq i$. Thus $m_1 = 1$ and then the degree of the dominant polynomial is at most $m_1 - 1 = 0$, so it is a constant, say g_1. Now, dividing G_n by r_1^n, we get

$$\frac{G_n}{r_1^n} = g_1 + \sum_{j=2}^{l} \frac{g_j(n)}{\kappa_j^n},$$

where $\kappa_j = r_1/r_j$. Since $|\kappa_j| > 1$, we have

$$\lim_{n\to\infty} \frac{g_j(n)}{\kappa_j^n} = 0 \quad \text{for all } 2 \leq j \leq l,$$

and so

$$\lim_{n\to\infty} \frac{G_n}{r_1^n} = g_1 \neq 0.$$

Now, if $t \in \{0, 1\}$ and $G_{2n+t} > 0$ for infinitely many integers n, then

$$0 \leq \lim_{n\to\infty} \sup \frac{G_{2n+t}}{r_1^{2n}} = g_1 r_1^t.$$

Therefore, $g_1 r_1^t > 0$ as $g_1 r_1^t \neq 0$ and the result follows by distinguishing the cases $t = 0$ and $t = 1$. ∎

In order to prove Theorems 1 and 2, we will need to use a lower bound for a linear form in three logarithms *à la Baker*, and such a bound was given by the following result of Matveev [3].

LEMMA 2. *Let $\alpha_1, \alpha_2, \alpha_3$ be non-zero algebraic numbers and let b_1, b_2, b_3 be non-zero integer rational numbers. Define*

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3.$$

Let D be the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$. Put

$$\chi = [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}].$$

Let A_1, A_2, A_3 be real numbers which satisfy

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\} \quad \textit{for } j = 1, 2, 3.$$

Assume that

$$B' \geq \max\{1, \max\{|b_j| A_j / A_1 : 1 \leq j \leq 3\}\}.$$

Define also

$$C_1 = \frac{5 \cdot 16^5}{6\chi} \cdot e^3 (7 + 2\chi)(20.2 + \log(3^{5.5} D^2 \log(eD))).$$

If $\Lambda \neq 0$, then

$$\log |\Lambda| \geq -C_1 D^2 A_1 A_2 A_3 \log(1.5eDB' \log(eD)).$$

As usual, in the above statement, the *logarithmic height* of an s-degree algebraic number α is defined as

$$h(\alpha) = \frac{1}{s}\Big(\log|a| + \sum_{j=1}^{s}\log\max\{1, |\alpha^{(j)}|\}\Big),$$

where a is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$), $(\alpha^{(j)})_{1\le j\le s}$ are the conjugates of α and, as usual, the absolute value of the complex number $z = a + bi$ is $|z| = \sqrt{a^2+b^2}$.

After finding an upper bound on n which is in general too large, the next step is to reduce it. For that, we need a variant of the famous Baker–Davenport lemma, which is due to Dujella and Pethő [1, Lemma 5, a)]. For a real number x, we use $\|x\| = \min\{|x-n| : n \in \mathbb{N}\}$ for the distance from x to the nearest integer.

Lemma 3. *Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of γ such that $q > 6M$ and let $\epsilon = \|\mu q\| - M\|\gamma q\|$, where μ is a real number. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-m}$$

in positive integers m, n with

$$\frac{\log(Aq/\epsilon)}{\log B} \le m < M.$$

Now, we are ready to prove our results.

3. The proof of Theorem 1. Equations (1.2) and (2.1) give

$$G_n = g_1(n)r_1^n + \cdots + g_s(n)r_s^n = B\cdot\frac{g^{lm}-1}{g^l-1}, \tag{3.1}$$

where $g_1(n),\dots,g_s(n)$ are polynomials of degree at most $k-1$. Without loss of generality, we may suppose that $|r_1| > |r_t| := \max_{2\le j\le s}|r_j|$. So r_1 is the dominant root.

Suppose that G_n has only finitely many positive numbers. So there exists a positive integer n_0 such that $G_n \le 0$ for all $n \ge n_0$. Applying the absolute value, the triangular inequality and the fact that $B \ge 1$, from equation (3.1) we deduce

$$(g^{lm}-1)/(g^l-1) \le |g_1(n)|\,|r_1|^n + \cdots + |g_s(n)|\,|r_s|^n.$$

Since r_1 is the dominant root, we have

$$(g^{lm}-1)/(g^l-1) \le (|g_1(n)| + \cdots + |g_s(n)|)|r_1|^n.$$

Now, $|r_1| > 1$ and $n \le n_0$, hence $|r_1|^n \le |r_1|^{n_0}$, yielding

$$(g^{lm}-1)/(g^l-1) \le K|r_1|^{n_0},$$

where $K = \max_{1\le n\le n_0} \sum_{j=1}^{s} |g_j(n)|$. Thus,

$$g^{lm} \le K|r_1|^{n_0}(g^l - 1) + 1,$$

and by applying the log function, we finally conclude that

$$m \le \log(K|r_1|^{n_0}(g^l - 1) + 1)/l \log g =: M.$$

Thus, $n \le n_0$ and $m \le M$, hence $n, m < \max\{n_0, M\} = C$ and the theorem is proved in this case.

Now, we suppose that G_n has infinitely many positive numbers. By Lemma 1, the dominant polynomial $g_1(n)$ is a constant, say g_1. Thus, we obtain

$$\left| r_1^n - \frac{B}{g^l - 1} \cdot \frac{g^{lm}}{g_1} \right| \le |r_t|^n \cdot \sum_{j=2}^{s} \left| \frac{g_j(n)}{g_1} \right| + 1.$$

For all sufficiently large n, say $n \ge n_1$, we have

$$\left| r_1^n - \frac{B}{g^l - 1} \cdot \frac{g^{lm}}{g_1} \right| < |r_t|^n (s-1) n^k$$

and so

$$\left| 1 - \frac{B}{g^l - 1} \cdot \frac{g^{lm} r_1^{-n}}{g_1} \right| < \kappa^{-n}(s-1)n^k,$$

where $\kappa = |r_1/r_t| > 1$. Observe that

$$\kappa^{-n}(s-1)n^k = \frac{1}{\kappa^{n/2}} \cdot \frac{(s-1)n^k}{\kappa^{n/2}} < \kappa^{-n/2}$$

for all sufficiently large n, say $n > n_2 \ge n_1$. Therefore,

$$|1 - e^{\Lambda}| < \kappa^{-n/2}, \tag{3.2}$$

where

$$\Lambda = \log\left(\frac{B}{(g^l - 1)g_1} \right) + lm \log g - n \log r_1. \tag{3.3}$$

Now, we claim that $\Lambda \ne 0$. Suppose $\Lambda = 0$. Thus, $Bg^{lm}/(g^l - 1) = g_1 r_1^n$ and then equation (3.1) leads to an absurdity as $\sum_{j=2}^{s} g_j(n) r_j^n = -B/(g^l - 1)$ for all $n \in \mathbb{N}$. Hence $\Lambda \ne 0$ as desired.

Note that according to the signs of g_1 and r_1, the number Λ can be real or complex. In order to deal with this problem, we will split our proof according to the positivity of G_n.

CASE 1. If $G_{2n} \le 0$ for all sufficiently large n, then we repeat the above argument in order to find a constant C such that if

$$G_{2n} = B \cdot \frac{g^{lm} - 1}{g^l - 1}$$

then $m, 2n < C$.

Thus, we only need to consider the odd indices (as the even ones are already bounded). Therefore, we replace n by $2n+1$ in equation (3.1) to get

$$G_{2n+1} = B \cdot \frac{g^{lm} - 1}{g^l - 1}.$$

Note that $G_{2n+1} > 0$ for infinitely many n (since $G_n > 0$ for infinitely many n). Therefore, by Lemma 1, we have $r_1 g_1 > 0$. Thus, equation (3.3) becomes

$$\Lambda_0 = \log\left(\frac{B}{(g^l - 1)g_1 r_1}\right) + lm \log g - n \log r_1^2,$$

which is a real number as $r_1 g_1$ and r_1^2 are positive numbers.

Case 2. If $G_{2n} > 0$ for infinitely many n (note that $g_1 > 0$ by Lemma 1), then we consider two subcases:

Case 2.1. If $G_{2n+1} > 0$ for infinitely many n, then again Lemma 1 yields $r_1 g_1 > 0$. So $r_1 > 0$ as $g_1 > 0$. Therefore, the linear form Λ given by equation (3.3) is a real number.

Case 2.2. If $G_{2n+1} \le 0$, for all sufficiently large n, then we proceed as before to get a bound C for the $G_{2n+1} > 0$, such that if

$$G_{2n+1} = B \cdot \frac{g^{lm} - 1}{g^l - 1},$$

then $m, 2n+1 < C$. So, we must bound only the even indices, by considering the equation

$$G_{2n} = B \cdot \frac{g^{lm} - 1}{g^l - 1}.$$

Thus, (3.3) becomes

$$\Lambda_1 = \log\left(\frac{B}{(g^l - 1)g_1}\right) + lm \log g - 2n \log r_1,$$

which is also a real number.

Summarizing all the above cases, we take the real linear form

$$\Lambda_t = \log\left(\frac{B}{(g^l - 1)g_1 r_1^t}\right) + lm \log g - n \log r_1^2, \qquad t \in \{0, 1\}.$$

If $\Lambda_t > 0$, then $\Lambda_t < e^{\Lambda_t} - 1 < \kappa^{-n/2}$. In the case of $\Lambda_t < 0$, we get

$$1 - e^{-|\Lambda_t|} = |e^{\Lambda_t} - 1| < \kappa^{-n/2}.$$

Therefore,

$$|\Lambda_t| < e^{|\Lambda_t|} - 1 < \frac{\kappa^{-n/2}}{1 - \kappa^{-n/2}} < \kappa^{-n/2+1}$$

for all sufficiently large $n > n_3 \geq n_2$. Hence, $|\Lambda_t| < \kappa^{-n/2+1}$, i.e.

$$-\log|\Lambda_t| > \left(\frac{n}{2} - 1\right)\log\kappa. \tag{3.4}$$

To apply Lemma 2, we take

$$\alpha_1 = \frac{B}{(g^l - 1)g_1 r_1^t}, \quad \alpha_2 = g, \quad \alpha_3 = r_1^2, \quad b_1 = 1, \quad b_2 = lm, \quad b_3 = -n.$$

Then, we can choose

$$D = k, \quad A_1 = kh_1 + 0.16, \quad A_2 = k\log g, \quad A_3 = kh_3 + 0.16,$$

where k is the degree of the number field $\mathbb{Q}(g_1, r_1^2)$ over $\mathbb{Q}$, and h_1 and h_3 are the logarithmic height of α_1 and α_3, respectively. Moreover, we have

$$B' = \max\left\{1, \frac{lmk\log g}{kh_1 + 0.16}, \frac{n(kh_3 + 0.16)}{kh_1 + 0.16}\right\}.$$

Since $\chi = 1$, we obtain

$$\begin{aligned} -\log|\Lambda| < 1.6 \cdot 10^8 (20.2 + \log(3^{5.5}k^2\log(ek))) \\ \cdot k^5(\max\{h_1, \log g, h_3\})^3 \log(1.5ekB'\log(ek)). \end{aligned} \tag{3.5}$$

Combining estimates (3.4) and (3.5), we get a constant $C > 0$, which depends only on g, l, and the parameters of G_n, such that $m, n < C$. ∎

4. The proof of Theorem 2. The aim of this section is to prove Theorem 2, which is an application of Theorem 1 to the Fibonacci and Lucas sequences. Note that the dominant root for Fibonacci and Lucas sequences is $\alpha = (1+\sqrt{5})/2$. The dominant polynomials are respectively $g(n) = 1/\sqrt{5}$ and $g(n) = 1$. Since α^n is irrational for all non-zero integer number n, we can apply Theorem 1 to conclude that the Diophantine equations (1.3) have only finitely many solutions. So our goal in this section is to improve our estimates in Section 3 and therefore to completely solve these equations. First, we prove Theorem 2 in the Fibonacci case. The proof of the Lucas case will be handled in a similar way.

4.1. The Fibonacci case

4.1.1. *Finding a bound on n.* We assume that $n > 47$. By Binet's formula and equation (1.3), we have

$$\alpha^n - \beta^n = \frac{\sqrt{5}B}{10^l - 1}(10^{ml} - 1);$$

that is,

$$\alpha^n - \frac{\sqrt{5}B}{10^l - 1} \cdot 10^{ml} = \beta^n - \frac{B\sqrt{5}}{10^l - 1}, \tag{4.1}$$

and hence

$$\left|\alpha^n - \frac{\sqrt{5}B}{10^l - 1} \cdot 10^{ml}\right| \leq \alpha^{-47} + \sqrt{5} < 2.4. \tag{4.2}$$

Define $\Lambda_F = \log(\sqrt{5}B/(10^l - 1)) - n\log\alpha + ml\log 10$. Then (4.2) becomes

$$|e^{\Lambda_F} - 1| < \frac{2.4}{\alpha^n} < \alpha^{-n+2}. \tag{4.3}$$

We also claim that $\Lambda_F > 0$. In fact, from equation (4.1), we deduce that

$$1 - e^{\Lambda_F} = \frac{1}{\alpha^n}\left(\beta^n - \frac{B\sqrt{5}}{10^l - 1}\right) \leq \frac{1}{\alpha^n}\left(\alpha^{-47} - \frac{\sqrt{5}}{10^{10} - 1}\right) < 0;$$

so $\Lambda_F > 0$. Thus $\Lambda_F < e^{\Lambda_F} - 1 < \alpha^{-n+2}$ (see (4.3)). Therefore,

$$\log|\Lambda_F| < -(n-2)\log\alpha. \tag{4.4}$$

Now, we will apply Lemma 2, but first we must be sure that $\Lambda_F \neq 0$. Indeed, if $(\sqrt{5}B/(10^l - 1))10^{ml}\alpha^{-n} = 1$ then $\alpha^{2n} \in \mathbb{Q}$, which is absurd. So $\Lambda_F \neq 0$. To apply Lemma 2, we take

$$\alpha_1 = \sqrt{5}B/(10^l - 1), \quad \alpha_2 = \alpha, \quad \alpha_3 = 10, \quad b_1 = 1, \quad b_2 = -n, \quad b_3 = ml.$$

Observe that $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(\sqrt{5})$ and then $D = 2$. The conjugates of $\alpha_1, \alpha_2, \alpha_3$ are $\alpha_1' = -\alpha_1$, $\alpha_2' = \beta$, $\alpha_3' = \alpha_3$, respectively. Surely, α_2 and α_3 are algebraic integers, while the minimal polynomial of α_1 is

$$(X - \alpha_1)(X - \alpha_1') = X^2 - \frac{5B^2}{(10^l - 1)^2}.$$

Thus, the minimal polynomial of α_1 is a divisor of $(10^l - 1)^2X^2 - 5B^2$. Therefore,

$$h(\alpha_1) < \frac{1}{2}(2\log(10^l - 1) + 2\log\sqrt{5}) < 23.84.$$

Also, $h(\alpha_2) = \log\alpha/2 < 0.25$ and $h(\alpha_3) = \log 10 < 2.31$. We take $A_1 = 47.68$, $A_2 = 0.5$ and $A_3 = 4.62$. Since $n > 47$, we have

$$\max\{1, \max\{|b_j|A_j/A_1 : 1 \leq j \leq 3\}\} = \max\{n/12, 5ml/48\},$$

and so it suffices to choose $B' = 5n/48$ as $n > ml$. Since $C_1 < 4.45 \cdot 10^9$, Lemma 2 yields

$$\log|\Lambda_F| > -1.97 \cdot 10^{12}\log(1.439n). \tag{4.5}$$

Combining the estimates (4.4) and (4.5), we get

$$1.97 \cdot 10^{12}\log(1.439n) > (n-2)\log\alpha,$$

and this inequality implies $n < 1.4 \cdot 10^{14}$.

Now, let us determine some estimates for m in terms of n that will be useful later. Equation (1.3) yields

$$ml = \left\lfloor \frac{\log F_n}{\log 10} \right\rfloor + 1.$$

Hence

$$(n-2)\frac{\log\alpha}{\log 10} < ml \le (n-1)\frac{\log\alpha}{\log 10} + 1. \tag{4.6}$$

Thus, we deduce from the estimate on n that $m < 3 \cdot 10^{13}$.

4.1.2. *Reducing the bound.* We know that $0 < \Lambda_F < \alpha^{-n+2}$. Since $m \ge 2$ and $\alpha^c = 10$, where $c = \log 10/\log\alpha$, we have

$$\alpha^{n-2} \ge \alpha^{cml-6} > (\alpha^c)^m 10^{-6} = 10^{m-6}.$$

Therefore,

$$0 < ml\log\alpha_3 - n\log\alpha_2 + \log\alpha_1 < 10^{-m+6}.$$

Dividing by $\log\alpha_2$, we get

$$0 < ml\gamma - n + \mu < 3 \cdot 10^6 \cdot 10^{-m}, \tag{4.7}$$

with $\gamma = \log\alpha_3/\log\alpha_2$ and $\mu = \log\alpha_1/\log\alpha_2$.

Surely, γ is an irrational number ([1]) (because α and 10 are multiplicatively independent). So, let p_n/q_n denote the nth convergent of its continued fraction. In order to reduce our bound on m (which is too large!), we will use Lemma 3. For that, taking $M = 3 \cdot 10^{13}$, we have

$$\frac{p_{34}}{q_{34}} = \frac{9146274886090674}{1911458405521733};$$

then $q_{34} \ge 1911458405521733 > 1.9 \cdot 10^{15} > 6M$. Moreover, we get

$$M\|q_{34}\gamma\| = 0.00736166\ldots < 0.0075,$$

and the minimal value of $\|q_{34}\mu\|$ is at least 0.008. Hence

$$\epsilon = \|\mu q\| - M\|\gamma q\| > 0.008 - 0.0075 = 0.0005.$$

Thus all the hypotheses of Lemma 3 are satisfied and we take $A = 3 \cdot 10^6$ and $B = 10$. It follows from Lemma 3 that there is no solution of inequality (4.7) (and hence of the Diophantine equation (1.3)) in the range

$$\left[\left\lfloor \frac{\log(Aq_{34}/\epsilon)}{\log B} \right\rfloor + 1, M\right] = [26, 3 \cdot 10^{13}].$$

Therefore $m \le 26$ and then inequality (4.6) tells us that $n < 1246$. To finish, we use Mathematica to print the values of all Fibonacci numbers in

([1]) Actually, this number is transcendental by the Gelfond–Schneider theorem: if α and β are algebraic numbers, with $\alpha \ne 0$ or 1, and β irrational, then α^β is transcendental.

the range $47 < n < 1246$ and find that there are no Fibonacci numbers as desired in the theorem. This completes the proof.

4.2. The Lucas case. From Binet's formula $L_n = \alpha^n + \beta^n$, we take

$$\Lambda_L = \log(B/(10^l - 1)) - n \log\alpha + ml \log 10.$$

Since $B/(10^l - 1) < \sqrt{5}B/(10^l - 1)$, we get the same estimates as in (4.3) and (4.4), so the possible solutions appear when $n < 1.4 \cdot 10^{14}$. Therefore, $m < 3 \cdot 10^{13}$. Then the Baker–Davenport reduction method can be applied to prove that actually $n < 1245$. Finally, we again use Mathematica to complete the proof of Theorem 2.

Acknowledgements. The authors would like to express their gratitude to the anonymous referee for carefully examining this paper and providing a number of important comments and suggestions. One of his/her suggestions led us to Theorem 1. The first author is also grateful to CNPq and FEMAT for financial support. The work on this paper was completed during a very enjoyable visit of the second author at *l'Institut de Mathématiques de Bordeaux*. He thanks the people of this institution for their hospitality. He was also supported in part by Purdue University North Central.

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Received 5 September 2010;
revised 23 May 2011 (5424)