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THE GEOMETRIC REDUCTIVITY OF THE QUANTUM GROUP $SL_q(2)$

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Abstract. We introduce the concept of geometrically reductive quantum group which is a generalization of the Mumford definition of geometrically reductive algebraic group. We prove that if G is a geometrically reductive quantum group and acts rationally on a commutative and finitely generated algebra A, then the algebra of invariants A^G is finitely generated. We also prove that in characteristic 0 a quantum group G is geometrically reductive if and only if every rational G-module is semisimple, and that in positive characteristic every finite-dimensional quantum group is geometrically reductive. Both the concept of geometrically reductive quantum group and the above mentioned theorems are formulated in the language of Hopf algebras and generalize the results of Borsai and Ferrer Santos. The main theorem of the paper says that in positive characteristic the quantum group $SL_q(2)$ is geometrically reductive for any parameter q.

1. Introduction and a generalization of the Borsari–Ferrer Santos results. Throughout the paper K denotes a fixed field which will serve as the ground field for all vector spaces, algebras, bialgebras, Hopf algebras and algebraic groups under consideration. All tensor products are supposed to be defined over K. Given vector spaces V and W, $\mathrm{Hom}(V,W)$ stands for the vector space of all linear maps $V \to W$. As usual, V^* and $\mathrm{End}(V)$ denote the space dual to V and the space $\mathrm{Hom}(V,V)$, respectively.

Let H be a fixed Hopf algebra with comultiplication $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to K$, and antipode $S : H \to H$; for basic facts concerning Hopf algebras and their (co)actions, see [7]. We use the following notation: $\sum h_1 \otimes h_2 = \Delta(h)$, and inductively, $\sum h_1 \otimes \cdots \otimes h_{n+1} = \sum h_1 \otimes \cdots \otimes h_{n-1} \otimes \Delta(h_n)$. By an H-comodule we mean a right H-comodule. The field K will be viewed as an H-comodule, via $\rho(\alpha) = \alpha \otimes 1$ for $\alpha \in K$. For any H-comodules V, W the vector space $V \otimes W$ will be viewed as an H-comodule, via $\rho : V \otimes W \to (V \otimes W) \otimes H$ with $\rho(v \otimes w) = \sum v_i \otimes w_j \otimes h_i h'_j$ for $v \in V$, $w \in W$, where $\sum v_i \otimes h_i = \rho(v)$ and $\sum_j w_j \otimes h'_j = \rho(w)$.

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Given an H-comodule (V,ρ), we denote by $V^{\operatorname{co}H}$ the *space of coinvariants*, that is, $V^{\operatorname{co}H} = \{v \in V \mid \rho(v) = v \otimes 1\}$. If A is an H-comodule algebra, then A^H is a subalgebra of A called the *algebra of coinvariants*. A *graded H-comodule algebra* is meant to be an H-comodule algebra A together with an algebra grading $A = \bigoplus_{i\geq 0} A_i$ such that all A_i's are subcomodules of A. Recall that a graded algebra $A = \bigoplus_i A_i$ is called *connected* if $A_0 = K$. If (A,ρ) is an H-comodule algebra, then an ideal I in A is called a *comodule ideal* if $\rho(I) \subset I \otimes H$. Observe that if H is commutative (as an algebra) and (V,ρ) is an H-comodule, then the map $\rho : V \to V \otimes H$ induces a morphism of algebras $\rho : S(V) \to S(V) \otimes H$ which makes the symmetric algebra $S(V)$ a graded H-comodule algebra. If G is an affine algebraic group and $K[G]$ is the Hopf algebra of all regular functions $G \to K$, then it is well known that a $K[G]$-comodule is nothing other than a rational (left) G-module, and a $K[G]$-comodule algebra is an algebra A endowed with a rational action of the group G on A. Moreover, $A^{\operatorname{co}K[G]} = A^G$. So, of interest is the following generalization of the fundamental problem of classical invariant theory.

Problem. Assume that A is a commutative and finitely generated H-comodule algebra. When is the algebra of coinvariants $A^{\operatorname{co}H}$ also finitely generated?

As is known, in some cases the algebra of coinvariants is not finitely generated. For instance, if $\operatorname{char}(K) = 0$, $H = K[T]$ with T primitive, and $A = K[X_1,\dots,X_5]$, then the (locally nilpotent) derivation $d = X_1^2\frac{\partial}{\partial X_3} + (X_1X_3 + X_2)\frac{\partial}{\partial X_4} + X_4\frac{\partial}{\partial X_5}$ makes A an H-comodule algebra (via $\rho(a) = \sum_{i\geq 0} d^i(a) \otimes T^i/i!$) such that the algebra of coinvariants $A^{\operatorname{co}H}$ $(= \ker d)$ is not finitely generated [2].

In order to formulate our positive results recall (see [4, V-8], [1]) that an algebraic group is *geometrically reductive* (in the sense of Mumford) if for each epimorphism of rational G-modules $\lambda : V \to K$ there are $r > 0$ and $f \in S^r(V)^G$ with $\tilde{\lambda}(f) \neq 0$, where $\tilde{\lambda} : S(V) \to K$ is the algebra morphism induced by λ. If $\lambda(v) \neq 0$ for some $v \in V^G$, then G is *linearly reductive*. It is known that G is linearly reductive if and only if every rational G-module is semisimple, and that in characteristic 0 each geometrically reductive algebraic group is linearly reductive. Examples of geometrically reductive algebraic groups are finite groups and the classical groups $GL(n,K)$, $SL(n,K)$, $O(n,K)$ and $Sp(2n,K)$. All the algebraic tori T^n are linearly reductive.

H. Borsari and W. Ferrer Santos [1] carried over the above definition of geometric reductivity to all commutative Hopf algebras.

Definition ([1, Def. 1.1]). A commutative Hopf algebra H is said to be *geometrically reductive* (for coactions) if for every epimorphism of H-

comodules $\lambda : V \to K$ there are $r > 0$ and $f \in S^r(V)^{\operatorname{co} H}$ with $\tilde{\lambda}(f) \neq 0$. If $\lambda(v) \neq 0$ for some $v \in V^{\operatorname{co} H}$, then H is called *linearly reductive.*

REMARK. Throughout the paper we write "geometrically coreductive" (resp., "linearly coreductive") instead of "geometrically reductive for coactions" (resp., "linearly reductive for coactions").

Notice that if G is an algebraic group, then the Hopf algebra $K[G]$ is geometrically coreductive (resp., linearly coreductive) if and only if G is geometrically reductive (resp., linearly reductive). The following results are proved in [1] (for commutative H).

THEOREM 1.1. *If the Hopf algebra H is geometrically coreductive and A is a commutative H-comodule algebra, then the algebra of coinvariants $A^{\operatorname{co} H}$ is finitely generated provided so is A.*

Theorem 1.1 is a generalization of the Nagata theorem [8] for geometrically reductive algebraic groups.

THEOREM 1.2. *The Hopf algebra H is linearly coreductive if and only if H is cosemisimple, i.e., if every H-comodule is semisimple.*

THEOREM 1.3. *If* $\operatorname{char}(K) = 0$, *then H is geometrically coreductive if and only if H is linearly coreductive.*

Furthermore, Ferrer Santos proved in [3] the following result.

THEOREM 1.4. *If* $\operatorname{char}(K) > 0$, *then every finite-dimensional* (*commutative*) *Hopf algebra is geometrically coreductive.*

The first objective of this paper is to extend the concept of geometrically coreductive Hopf algebra to *all Hopf algebras* (i.e., not necessarily commutative), and then to generalize Theorems 1.1–1.4 to this case.

Notice that the Borsari–Ferrer Santos definition of geometrically coreductive Hopf algebra cannot be repeated for an arbitrary Hopf algebra, because, if V is an H-comodule, then the symmetric algebra $S(V)$ does not, in general, admit any natural H-comodule algebra structure ("natural" means here that $V = S^1(V)$ is a subcomodule of $S(V)$).

EXAMPLE 1.5. Let H be the group algebra kG, where G is an (abstract) group with $g_1g_2 \neq g_2g_1$ for some $g_1, g_2 \in G$, and let V be a vector space with a basis v_1, v_2. Then $\rho : V \to V \otimes H$, $\rho(v_i) = v_i \otimes g_i$, $i = 1, 2$, makes V an H-comodule. Suppose that $S(V)$ admits a natural H-comodule algebra structure $\rho : S(V) \to S(V) \otimes H$. Then $0 = v_1v_2 - v_2v_1 \in S(V)$, whence $0 = \rho(v_1v_2 - v_2v_1) = v_1v_2 \otimes (g_1g_2 - g_2g_1) \neq 0$, which is impossible.

In order to overcome this difficulty, we proceed as follows. Given an H-comodule W and an element $w \in W$, we denote by $H(w)$ the smallest subcomodule of W containing w. Now let (V, ρ) be an H-comodule. Recall that the tensor algebra $T(V) = \bigoplus_{i\geq 0} T^i(V)$ is a (connected) graded H-comodule algebra, via

$$\rho(v_1 \otimes \cdots \otimes v_n) = \rho(v_1)\cdots\rho(v_n) \in T(V)\otimes H, \qquad v_i \in V.$$

It is easy to see that the ideal $I_H(V)$ in $T(V)$ generated by the set $\bigcup_{v,v'\in V} H(v\otimes v' - v'\otimes v)$ is a homogeneous comodule ideal, so that we have the quotient connected graded H-comodule algebra

$$S_H(V) = T(V)/I_H(V) = \bigoplus_{i\geq 0} S_H^i(V).$$

Also it is easy to verify that $S_H(V)$ has the following properties.

LEMMA 1.6.

(1) *$S_H(V)$ is a commutative algebra with $S_H^1(V) = V$. Furthermore, if H is commutative, then $S_H(V)$ is the ordinary symmetric algebra $S(V)$.*
(2) *If V is finite-dimensional, then the algebra $S_H(V)$ is finitely generated.*
(3) *If A is a commutative H-comodule algebra and $\lambda : V \to A$ is a morphism of H-comodules, then there exists a unique morphism of H-comodule algebras $\tilde{\lambda} : S_H(V) \to A$ (called the induced morphism) such that $\tilde{\lambda}|V = \lambda$. In particular, for any morphism of H-comodules $f : V \to W$ we have the induced morphism of graded H-comodule algebras $S_H(f) : S_H(V) \to S_H(W)$ such that $S_H(f)(v) = f(v)$ for $v \in V$.*

In view of the above properties, $S_H(V)$ can be called the *symmetric H-comodule algebra of the comodule V*.

Now we introduce the main concept of the paper.

DEFINITION. The Hopf algebra H is called *geometrically coreductive* if for any epimorphism of H-comodules $j : V \to K$ there exist $r > 0$ and $f \in S_H^r(V)^{\mathrm{co}\,H}$ such that $\tilde{j}(f) \neq 0$, where $\tilde{j} : S_H(V) \to K$ is the morphism of comodule algebras induced by j. If $\lambda(v) \neq 0$ for some $v \in V^{\mathrm{co}\,H}$ $(= S_H^1(V)^{\mathrm{co}\,H})$, then H is called *linearly coreductive.*

Notice that for commutative Hopf algebras this definition and that of Borsari–Ferrer Santos coincide. It is clear that each linearly coreductive Hopf algebra is geometrically coreductive. For later use, observe also that the Hopf algebra H is geometrically coreductive if for any epimorphism of finite-dimensional H-comodules $\lambda : V \to K$ there exist $r > 0$ and $f \in S_H^r(V)^{\mathrm{co}\,H}$ such that $\tilde{\lambda}(f) \neq 0$.

Now we show that Theorems 1.1–1.4 hold for all Hopf algebras. Theorem 1.1 can be proved similarly to [1, Theorem 4.3], replacing $S(V)$ by $S_H(V)$ (also it can be deduced by duality from the proof of Theorem 3.1 in [5]).

Theorem 1.2 can be proved in the same way as [1, Observation 2.1], using the following well known facts.

LEMMA 1.7. *Let V be a finite-dimensional H-comodule and let $v_1, \ldots, v_n$ be a basis of V with $\rho(v_i) = \sum_j v_j \otimes h_{ji}$. Moreover, let $v_1^*, \ldots, v_n^*$ be the dual basis of the dual vector space V^*.*

(i) *The map $\rho : V^* \to V^* \otimes H$, $\rho(v_i^*) = \sum_t v_t^* \otimes S(h_{it})$, $i = 1, \ldots, n$, makes V^* an H-comodule (and does not depend on the choice of a basis). Moreover, if $S^2 = \mathrm{Id}$, then the evaluation map $e : V \to V^{**}$ is an isomorphism of H-comodules.*

(ii) *For any H-comodule W the vector space $\mathrm{Hom}(V, W)$ admits a unique H-comodule structure such that the natural map $\Phi : W \otimes V^* \to \mathrm{Hom}(V, W)$, $\Phi(w \otimes v^*)(v) = v^*(v)w$, is an isomorphism of H-comodules. If $\{w_j \mid j \in J\}$ is a basis of W with $\rho(w_j) = \sum_{s \in J} w_s \otimes h'_{sj}$ and $\{x_{ij} \mid i = 1, \ldots, n,\ j \in J\}$ is the basis of $\mathrm{Hom}(V, W)$ defined by $x_{ij}(v_r) = \delta_{ir} w_j$, then the structure map $\rho : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W) \otimes H$ is given by*

$$\rho(x_{ij}) = \sum_{t,s} x_{ts} \otimes h'_{sj} S(h_{it}).$$

Furthermore, $\mathrm{Hom}(V, W)^{\mathrm{co}\, H} = \mathrm{Hom}_H(V, W)$, where $\mathrm{Hom}_H(V, W)$ is the vector space of all morphisms of H-comodules $V \to W$.

Theorem 1.3 is proved below. As for Theorem 1.4, its proof is a simple modification of the proof of [5, Theorem 5.12], applying the main results of [9]. Observe that Theorem 1.4 is not true if $\mathrm{char}(K) = 0$, because, in characteristic 0, not all finite-dimensional Hopf algebras are cosemisimple.

Proof of Theorem 1.3. Suppose that $\mathrm{char}(K) = 0$ and let $\lambda : V \to K$ be an epimorphism of H-comodules. By assumption, there are $r > 0$ and $f \in S_H^r(V)^{\mathrm{co}\, H}$ such that $\tilde{\lambda}(f) \neq 0$. It suffices to construct a morphism of H-comodules $\gamma' : S_H^+(V) = \bigoplus_{i \geq 1} S_H^i(V) \to V$ such that $\lambda\gamma'(y) = n\tilde{\lambda}(y)$ for $y \in S_H^n(V)$, $n \geq 1$. To this end, we define as in [1, Observation 1.3] a linear map $\gamma : T^+(V) = \bigoplus_{i \geq 1} T^i(V) \to V$ by $\gamma(v) = v$ and

$$\gamma(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^n \Big[\prod_{j \neq i} \lambda(v_j)\Big] v_i, \qquad n \geq 2,$$

where $v, v_1, \ldots, v_n \in V$. Notice that for each $n \geq 1$ and $1 \leq i \leq n$ the map $\gamma_i^{(n)} : T^n(V) \to V$ given by $\gamma_i^{(n)}(v_1 \otimes \cdots \otimes v_n) = [\prod_{j \neq i} \lambda(v_j)] v_i$ is a morphism of H-comodules, because it can be identified with the map $\lambda \otimes \cdots \otimes \lambda \otimes \mathrm{Id}_V \otimes \lambda \otimes \cdots \otimes \lambda$, where Id_V is at the ith position. As $\gamma | T^n(V) = \sum_{i=1}^n \gamma_i^{(n)}$, this implies that $\gamma : T^+(V) \to V$ is a morphism of H-comodules.

Now we show that $I_H(V) \subset \ker \gamma$. Recall that the ideal $I_H(V)$ is generated by the set $X = \bigcup_{v, v' \in V} H(v \otimes v' - v' \otimes v)$ and observe that $\gamma(X) = 0$, because $\gamma(v\otimes v'-v'\otimes v)=0$ for all $v,v'\in V$ and γ is a morphism of H-comodules. Hence $I_H(M)\subset\ker\gamma$ provided $\gamma(u\otimes x)=0=\gamma(x\otimes u)$ for all $x\in\ker\gamma\cap V^{\otimes n}$, $n\ge 2$, and $u\in V$. Let $x=\sum v_{i_1}\otimes\cdots\otimes v_{i_n}\in\ker\gamma$, $v_{i_j}\in V$. Then

$$\begin{aligned}\gamma(x\otimes u)&=\sum\gamma(v_{i_1}\otimes\cdots\otimes v_{i_n}\otimes u)\\&=\sum\Big(\lambda(u)\sum_{k=1}^{n}\Big[\prod_{j\neq k}\lambda(v_{i_j})\Big]v_{i_k}+\lambda(v_{i_1})\ldots\lambda(v_{i_n})u\Big)\\&=\lambda(u)\gamma(x)+\frac{1}{n}\lambda(\gamma(x))u=0.\end{aligned}$$

Similarly, $\gamma(u\otimes x)=0$. Thus we have shown that there exists a morphism of H-comodules $\gamma':S_H^+(V)\to V$ such that $\gamma'(v_1\otimes\cdots\otimes v_n+I_H(V))=\gamma(v_1\otimes\cdots\otimes v_n)$ for $v_i\in V$ and $n\ge 1$. Certainly $\lambda\gamma'(y)=n\tilde{\lambda}(y)$ for $y\in S_H^n(V)$. The theorem follows.

Remark 1.8. In [5] the authors introduced the notion of a geometrically reductive Hopf algebra L (geometrically reductive for actions) in which the category of L-modules is used. It is not difficult to see that a Hopf algebra L is geometrically reductive if and only if its finite dual L^0 is geometrically coreductive and that Theorems 2–4 in [5, Introduction] are consequences of the above mentioned Theorems 1.1, 1.3, 1.4.

In the following section, we study the natural question when the Hopf algebra $K[SL_q(2)]$ of the quantum group $SL_q(2)=SL_q(2,K)$ is geometrically coreductive. Since the algebraic group $SL(2)=SL_1(2)$ is geometrically reductive, we know that the Hopf algebra $K[SL_1(2)]$ is geometrically coreductive. Further, it follows from [11] that the Hopf algebra $K[SL_q(2)]$ is cosemisimple (= linearly coreductive) whenever the parameter $q\in K$ is not a root of unity or if $q=1$ and $\mathrm{char}(K)=0$. Our main result is the following theorem.

Theorem 1.9. *If* $\mathrm{char}(K)>0$, *then the Hopf algebra* $K[SL_q(2)]$ *is geometrically coreductive for each parameter* q $(\neq 0)$. *Moreover, the Hopf algebra* $K[SL_{-1}(2)]$ *is geometrically coreductive in any characteristic.*

2. On the geometric coreductivity of $K[SL_q(2)]$. We begin with some auxiliary results.

Let $f:H\to D$ be a morphism of Hopf algebras. It is clear that every H-comodule (V,ρ) can be considered as a D-comodule, via $(1\otimes f)\rho:V\to V\otimes D$ (and then every morphism of H-comodules becomes a morphism of D-comodules). In particular, we have the space of coinvariants $V^{\mathrm{co}\,D}$ which will be denoted by V^f. If (A,ρ) is an H-comodule algebra, then $(1\otimes f)\rho$ makes A a D-comodule algebra and we have the algebra of coinvariants A^f.

In particular, we have the algebra $H^f = \{h \in H \mid (1 \otimes f)\Delta(h) = h \otimes 1\}$. Recall that for a given H-comodule U and $u \in U$ we denote by $H(u)$ the smallest subcomodule of U containing u.

LEMMA 2.1. *With the above notation, if* (U, ρ) *is an* H*-comodule, then* $\rho(U^f) \subset U \otimes H^f$. *Moreover, if* H^f *is a Hopf subalgebra of* H, *then* $\rho(H(u)) \subset H(u) \otimes H^f$ *for each* $u \in U^f$.

The proof of the lemma is an easy calculation and we omit it.

The following theorem (and its proof) is similar to [1, Theorem 2.3].

THEOREM 2.2. *Assume that* $f : H \to D$ *is a morphism of Hopf algebras such that* H^f *is a Hopf subalgebra of* H. *Furthermore, assume that* D *and* H^f *are geometrically coreductive* (*resp., linearly coreductive*). *Then the Hopf algebra* H *is geometrically coreductive* (*resp., linearly coreductive*).

Proof. Let $\lambda : V \to K$ be an epimorphism of H-comodules, and let $\lambda_H : S_H(V) \to K$ and $\lambda_D : S_D(V) \to K$ denote the induced morphisms of H-comodule and D-comodule algebras, respectively. Obviously the natural inclusion $V \subset S_H(V)$ viewed as a morphism of D-comodules induces a morphism of graded D-comodule algebras $\pi : S_D(V) \to S_H(V)$. Since $\lambda_H\pi(v) = \lambda(v) = \lambda_D(v)$ for $v \in V$, we have $\lambda_H\pi = \lambda_D$. By the geometric coreductivity of D, there are $r > 0$ and $\zeta \in S_D^r(V)^{\operatorname{co} D}$ such that $\lambda_D(\zeta) = 1$. Let $x = \pi(\zeta)$. Then $x \in S_H^r(V)^f$ and $\lambda_H(x) = \lambda_H(\pi(\zeta)) = \lambda_D(\zeta) = 1$. Now set $U = H(x) \subset S_H^r(V)$ and $L = H^f$. In view of Lemma 2.1, if $\rho : S_H^r(V) \to S_H^r(V) \otimes H$ is the structure map, then $\rho(U) \subset U \otimes L$, so that (U, ρ') with $\rho'(u) = \rho(u)$ is an L-comodule. Let $\omega = \lambda_H|U : U \to K$. Then $\omega(x) = 1$ and by the geometric coreductivity of L, there are $l > 0$ and $y \in S_L^l(U)^{\operatorname{co} L}$ with $\tilde{\omega}(y) = 1$, where $\tilde{\omega} : S_L(U) \to K$ is the induced morphism of L-comodule algebras. Furthermore, the inclusion $U \subset S_H(V)$ induces a morphism of H-comodule algebras

$$g : S_L(U) \to S_H(V)$$

such that $g(S_L^t(U)) \subset S_H^{rt}(V)$ for all $t \geq 0$. Hence $g(y) \in S_H^{rl}(V)^{\operatorname{co} H}$ and $\lambda_H(g(y)) = \omega(y) = 1$. Consequently, the Hopf algebra H is geometrically coreductive. Moreover, if H^f and D are linearly coreductive, then so is H. ∎

Now observe that if the antipode $S : H \to H$ is involutive, i.e., $S^2 = \mathrm{Id}$, and if V is a finite-dimensional H-comodule, then by Lemma 1.7, the evaluation map $e : V \to V^{**}$ is a morphism of H-comodules and for any $v \in V^{\operatorname{co} H}$ the map $e(v) : V^* \to K$ is a morphism of H-comodules. In particular, we can take the induced morphism of H-comodule algebras $\widetilde{e(v)} : S_H(V^*) \to K$. Hence one easily obtains the following theorem.

THEOREM 2.3. *If* $S^2 = \mathrm{Id}$, *then the Hopf algebra* H *is geometrically coreductive if and only if for any finite-dimensional* H*-comodule* V *and* *any nonzero* $v_0 \in V^{\operatorname{co} H}$ *there exist* $r > 0$ *and* $T \in S^r_H(V^*)^{\operatorname{co} H}$ *such that* $\widetilde{e(v_0)}(T) \neq 0$.

REMARK 2.4. If G is an affine algebraic group, then the above theorem applies to the (commutative) Hopf algebra $K[G]$ and amounts to the well known fact that G is geometrically reductive if and only if for any finite-dimensional, rational G-module V and any nonzero $v_0 \in V^G$ there is a nonconstant and G-invariant regular function $T : V \to K$ such that $T(v_0) \neq 0$ (see, e.g., [10]).

For later use we also need the following theorem.

THEOREM 2.5. *Let* V *be an* H*-comodule of dimension* $n > 0$ *and let* $v_1, \ldots, v_n$ *be a basis of* V *with* $\rho(v_i) = \sum_j v_j \otimes h_{ji}$, $i = 1, \ldots, n$. *Moreover, let* E *denote the* H*-comodule* $\operatorname{End}(V)$ (*see Lemma* 1.7) *and let* $\{x_{ij} \mid i, j = 1, \ldots, n\}$ *be the basis of* E *given by* $x_{ij}(v_r) = \delta_{ir} v_j$.

(i) $\Delta(h_{ij}) = \sum_s h_{is} \otimes h_{sj}$ *and* $\varepsilon(h_{ij}) = \delta_{ij}$ *for all* i, j.

(ii) *Suppose that the set* $\{h_{ij}, S(h_{ij}) \mid i, j = 1, \ldots, n\}$ *is contained in a commutative subalgebra* B *of* H. *Then we have:*

  (a) *The element* $F = \det(x_{ij}) \in S_H(E)$ *is a homogeneous coinvariant* (*of degree* n) *such that* $\tilde{j}(F) = \det(j(x_{ij}))$ *for any morphism of* H*-comodules* $j : E \to K$.

  (b) *If* $S^2 = \mathrm{Id}$, *then the map* $\Psi : E \to E^*$ *defined by* $\Psi(x_{ij}) = x^*_{ji}$ *is an isomorphism of* H*-comodules. Furthermore,* $F^* = \det(x^*_{ij}) \in S_H(E^*)$ *is a homogeneous coinvariant* (*of degree* n^2) *such that for any* $f \in E^{\operatorname{co} H}$ *the map* $e(f) : E^* \to K$ *is a morphism of* H*-comodules and* $\widetilde{e(f)}(F^*) = \det(f)$.

Proof. Part (i) is a (well known) simple exercise. For (ii), if A denotes the matrix $(h_{ij}) \in M_n(H)$ and $S(A) = (S(h_{ij}))$, then $AS(A) = I$, by (i) ($I = (\delta_{ij})$). Furthermore, from Lemma 1.7 we deduce that

$$\rho(x_{ij}) = \sum_{t,s} x_{ts} \otimes h_{sj} S(h_{it}) = \sum_{s,t} x_{ts} \otimes S(h_{it}) h_{sj}.$$

In particular, all $\rho(x_{ij})$'s belong to the commutative subalgebra $S_H(E) \otimes B$ of $S_H(E) \otimes H$. Further, the matrix $(\rho(x_{ij}))$ equals $(1 \otimes S(A))(X \otimes 1)(1 \otimes A)$, where $X = (x_{ij})$ and $1 \otimes C = (1 \otimes c_{rs})$, $C \otimes 1 = (c_{rs} \otimes 1)$ for any matrix $C = (c_{rs})$. Hence

$$\begin{aligned} \rho(F) = \det(\rho(x_{ij})) &= \det(1 \otimes S(A)) \det(X \otimes 1) \det(1 \otimes A) \\ &= \det(X \otimes 1) \det(1 \otimes S(A)A) = \det(X \otimes 1) \det(1 \otimes I) \\ &= \det(X) \otimes 1 = F \otimes 1, \end{aligned}$$

which means that $F \in S_H(E)^{\operatorname{co} H}$. The second statement of (ii)(a) is obvious.

It remains to prove (ii)(b). Suppose that $S^2 = \mathrm{Id}$. By Lemma 1.7,

$$\rho(x^*_{ij}) = \sum_{t,s} x^*_{ts} \otimes S(h_{js}S(h_{ti})) = \sum_{t,s} x^*_{ts} \otimes h_{ti}S(h_{js}) = \sum_{s,t} x^*_{st} \otimes h_{si}S(h_{jt}).$$

It follows that the map $\Psi : E \to E^*$, $\Psi(x_{ij}) = x^*_{ji}$, is an isomorphism of H-comodules, which in turn implies that $F^* = S_H(\Psi)(F)$ is a coinvariant. Now let $f \in E^{\mathrm{co}\,H}$. Then $e(f) : E^* \to K$ is a morphism of H-comodules, again by Lemma 1.7, so that we have the induced morphism of H-comodule algebras $\widetilde{e(f)} : S_H(E^*) \to K$. Furthermore, $f = \sum_{i,j} \alpha_{ij}x_{ij}$ for some $\alpha_{ij} \in K$, which means that $f(v_t) = \sum_j \alpha_{tj}v_j$ for $t = 1, \ldots, n$. Hence $\widetilde{e(f)}(F^*) = \det(e(f)(x^*_{ij})) = \det(x^*_{ij}(f)) = \det(\alpha_{ij}) = \det(f)$. ■

Remark 2.6. Part (ii)(a) of the above theorem and its proof are a simple generalization of [3, Lemma 2.1].

Now let $0 \neq q \in K$ be a fixed parameter. Following [6, Section IV], we denote by $K[M_q(2)]$ the algebra generated by the symbols a, b, c, d subject to the relations

$$ba = qab, \quad db = qbd, \quad ca = qac, \quad dc = qcd,$$
$$bc = cb, \quad ad - da = (q^{-1} - q)bc.$$

Observe that the algebra $K[M_q(2)]$ has a natural grading such that the degree of the generators a, b, c, d is equal to 1. Furthermore, the following lemma holds.

Lemma 2.7 ([6, Theorem IV.4.1]). *The set* $\{a^{n_0}b^{n_1}c^{n_2}d^{n_3} \mid n_i \geq 0\}$ *is a* (*linear*) *basis of the algebra* $K[M_q(2)]$.

As is known, the algebra $K[M_q(2)]$ is a bialgebra with comultiplication and counit defined by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d$$
$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d,$$
$$\varepsilon(a) = 1 = \varepsilon(d), \quad \varepsilon(b) = 0 = \varepsilon(c).$$

It is easy to verify that the element $\det_q = ad - q^{-1}bc \in K[M_q(2)]$ (called the *quantum determinant*) is in the center of $K[M_q(2)]$ and that $\Delta(\det_q) = \det_q \otimes \det_q$, $\varepsilon(\det_q) = 1$. Hence we obtain the quotient bialgebra $K[SL_q(2)] = K[M_q(2)]/(\det_q - 1)$, which turns out to be a Hopf algebra with antipode S defined by

$$S(a) = d, \quad S(b) = -qb, \quad S(c) = -q^{-1}c, \quad S(d) = a.$$

$K[SL_q(2)]$ is called the *Hopf algebra of the quantum group* $SL_q(2)$.

An easy consequence of Lemma 2.7 is the following.

LEMMA 2.8. *The set* $\{a^n b^i c^j,\, b^i c^j d^m \mid i, j, n \geq 0,\, m \geq 1\}$ *is a basis of* $K[SL_q(2)]$.

The above basis will be called the *standard basis* of $K[SL_q(2)]$.

Obviously the Hopf algebra $K[SL_q(2)]$ is geometrically coreductive for $q = 1$, because the algebraic group $SL(2) = SL_1(2)$ is geometrically reductive. Moreover, the following result is due to M. Takeuchi.

THEOREM 2.9 ([11]). *The Hopf algebra* $K[SL_q(2)]$ *is linearly coreductive* (= *cosemisimple*) *whenever* q *is not a root of unity.*

Below we are going to prove that in positive characteristic the Hopf algebra $K[SL_q(2)]$ is geometrically coreductive for any q. The idea of the proof is as follows. Making use of Theorem 2.2, we will construct a morphism of Hopf algebras $f : K[SL_q(2)] \to D$ such that D is a finite-dimensional Hopf algebra and $K[SL_q(2)]^f$ is a Hopf subalgebra of $K[SL_q(2)]$ isomorphic to $K[SL_\varepsilon(2)]$ for some $\varepsilon \in \{1, -1\}$. Then we prove that for both $\varepsilon \in \{1, -1\}$ the Hopf algebra $K[SL_\varepsilon(2)]$ is geometrically coreductive (in any characteristic). The conclusion will follow by Theorem 1.4.

Let us start by recalling the definition and properties of the Gauss polynomials $\binom{n}{r}_t$. Denote by $\mathbb{Q}(t)$ the field of fractions of the polynomial ring $\mathbb{Z}[t]$ and set

$$(n)_t = \frac{t^n - 1}{t - 1}, \qquad n \geq 1, \qquad (0)_t = 1,$$

$$(n)!_t = (1)_t (2)_t \ldots (n)_t = \frac{(t-1)(t^2-1)\ldots(t^n-1)}{(t-1)^n}, \qquad n \geq 0,$$

$$\binom{n}{r}_t = \frac{(n)!_t}{(r)!_t (n-r)!_t} \in \mathbb{Q}(t), \qquad 0 \leq r \leq n.$$

It is clear that $\binom{n}{n}_t = \binom{n}{0}_t = 1$ and $\binom{n}{r}_1 = \binom{n}{r} = \frac{n!}{(n-r)!r!}$. The following lemma lists the basic properties of $\binom{n}{r}_t$.

LEMMA 2.10.

(i) $\displaystyle\binom{n}{r}_t \in \mathbb{Z}[t]$.

(ii) $\displaystyle\binom{n}{r}_t = \binom{n}{n-r}_t$.

(iii) $\displaystyle\binom{n}{r}_t = \binom{n-1}{r-1}_t + t^r \binom{n-1}{r}_t = \binom{n-1}{r}_t + t^{n-r} \binom{n-1}{r-1}_t$ *for* $0 < r \leq n$.

(iv) *Assume that* $\lambda \in K$ *is a primitive* m*th root of unity. Then* $\binom{m}{r}_\lambda = 0$ *for* $0 < r < m$. *Moreover,* $(n)_\lambda = 0$ *if and only* $m \mid n$ *for* $n \geq 1$.

Parts (i)–(iii) of the lemma are contained in [6, Proposition IV.2.1]. Part (iv) is obvious.

Below we shall also need the following lemmas.

LEMMA 2.11 ([6, Proposition IV.2.2]). *Let A be an algebra and let $x, y \in A$ be such that $yx = \lambda xy$ for some nonzero $\lambda \in K$. Then*

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r}_{\lambda} x^r y^{n-r}, \qquad n \geq 0.$$

LEMMA 2.12. *The following equalities hold in $K[SL_q(2)]$:*

(i) *We have*

$$(1) \qquad a^s d^s = \sum_{k=0}^{s} \binom{s}{k}_{q^2} q^{k^2-2ks}(bc)^k, \quad d^s a^s = \sum_{k=0}^{s} \binom{s}{k}_{q^2} q^{k^2}(bc)^k, \quad s \geq 0.$$

(ii) $a^m d^m = d^m a^m$ *and* $a^m d^m - q^{-m^2} b^m c^m = 1$ *whenever q^2 is a primitive mth root of unity.*

Part (i) of the lemma easily follows by induction on s and by Lemma 2.10(iii). Part (ii) is a consequence of (i) and Lemma 2.10(iv), because $q^{m^2} = q^{-m^2}$.

Now assume that the parameter q is a root of unity and let

$$m = \min\{k \geq 1 \mid (q^2)^k = 1\}.$$

Moreover, let

$$\varepsilon = q^{m^2} = q^{-m^2}.$$

Note that $\varepsilon \in \{1, -1\}$.

LEMMA 2.13. *There exists a unique morphism of Hopf algebras*

$$\phi : K[SL_\varepsilon(2)] \to K[SL_q(2)]$$

such that $\phi(u) = u^m$ for $u \in \{a, b, c, d\}$. Moreover, ϕ is injective.

Proof. By Lemma 2.12(ii), there exists a unique morphism of algebras

$$\phi : K[SL_\varepsilon(2)] \to K[SL_q(2)]$$

satisfying the above conditions. By Lemmas 2.10 and 2.11, ϕ is a morphism of Hopf algebras. In view of Lemma 2.8, ϕ is injective. ∎

REMARK 2.14. The above lemma can be deduced from [12, Section 5].

Now let $L = \operatorname{im}\phi$ and let $H = K[SL_q(2)]$ for simplicity. In view of the above lemma, L is a Hopf subalgebra of H isomorphic to $K[SL_\varepsilon(2)]$ with $\varepsilon \in \{1, -1\}$. Furthermore, the Hopf ideal $L^+ = \ker(\varepsilon : L \to K)$ in L is generated by the set $\{a^m - 1, b^m, c^m, d^m - 1\}$. This implies that $J =$

$H(a^m-1, b^m, c^m, d^m-1)H$ is a Hopf ideal in H, so that we have the quotient Hopf algebra $D = H/J$ and the natural projection

$$f : H \to D.$$

This gives the algebra of D-coinvariants $H^f = \{h \in H \mid (1\otimes f)\Delta(h) = h\otimes 1\}$. Of importance is the following theorem.

THEOREM 2.15. *In any characteristic we have $H^f = L$. In particular, H^f is a Hopf subalgebra of H isomorphic to $K[SL_\varepsilon(2)]$ for some $\varepsilon \in \{1, -1\}$.*

Proof. If $m = 1$, that is, $q^2 = 1$, then $D = K$, $f = \varepsilon$ and $H^f = H = L$. So we can assume that $m \geq 2$. Observe that $L \subset H^f$, because

$$(\mathrm{Id} \otimes f)\Delta(a^m) = a^m \otimes f(a^m) + b^m \otimes f(c^m) = a^m \otimes 1$$

(see Lemmas 2.10 and 2.11), and similarly $b^m, c^m, d^m \in H^f$.

For $k = (k_1, k_2, k_3)$, $n = (n_1, n_2, n_3) \in \mathbb{N}^3$ we write $k \leq n$ when $k_i \leq n_i$ for $i = 1, 2, 3$. Moreover, for $\lambda \in K$, by $\binom{n}{k}_\lambda$ we mean $\binom{n_1}{k_1}_\lambda \binom{n_2}{k_2}_\lambda \binom{n_3}{k_3}_\lambda$.

Now let $x = \sum_{s\in\mathbb{N}^4} \lambda_s a^{s_1} b^{s_2} c^{s_3} d^{s_4} \in H^f$ ($\lambda_s \in K$). We are going to show that $x \in L$. Set $t = \max\{s_1 \mid \lambda_s \neq 0\}$ and choose an integer $k \geq 0$ such that $mk \geq t$. Then clearly $y = d^{mk}x = \sum_{n\in\mathbb{N}^3} \lambda_n b^{n_1} c^{n_2} d^{n_3} \in H^f$ for some $\lambda_n \in K$, by (1). From Lemma 2.11 and (1) we infer that

$$\begin{aligned}
(2)\qquad \Delta(y) &= \sum_n \lambda_n \sum_{k\leq n} \binom{n}{k}_{q^2} c^{k_2} d^{n_2-k_2} a^{k_1} b^{n_1-k_1} c^{k_3} d^{n_3-k_3} \\
&\qquad\qquad \otimes a^{k_2} c^{n_2-k_2} b^{k_1} d^{n_1-k_1} b^{k_3} d^{n_3-k_3} \\
&= \sum_n \lambda_n \sum_{k\leq n} \tau_1 \binom{n}{k}_{q^2} a^{k_1} b^{n_1-k_1} c^{k_2+k_3} d^{n_2+n_3-(k_2+k_3)} \\
&\qquad\qquad \otimes a^{k_2} b^{k_1+k_3} c^{n_2-k_2} d^{n_1+n_3-(k_1+k_3)} \\
&= \sum_n \lambda_n \sum_{\substack{k\leq n\\ k_1+k_2+k_3\leq n_1+n_3}} \tau_2 \binom{n}{k}_{q^2} \sum_{u=0}^{k_2} \alpha_u a^{k_1} b^{n_1-k_1} c^{k_2+k_3} d^{n_2+n_3-(k_2+k_3)} \\
&\qquad\qquad \otimes b^{k_1+k_3} c^{n_2-k_2} (bc)^u d^{n_1+n_3-(k_1+k_2+k_3)} \\
&+ \sum_n \lambda_n \sum_{\substack{k\leq n\\ k_1+k_2+k_3> n_1+n_3}} \tau_2 \binom{n}{k}_{q^2} \sum_{u=0}^{n_1+n_3-(k_1+k_3)} \beta_u a^{k_1} b^{n_1-k_1} c^{k_2+k_3} d^{n_2+n_3-(k_2+k_3)} \\
&\qquad\qquad \otimes a^{k_1+k_2+k_3-(n_1+n_3)} b^{k_1+k_3+u} c^{n_2+u-k_2}
\end{aligned}$$

for some $\alpha_u, \beta_u \in K$ (depending on n, k) such that $\alpha_0 = \beta_0 = 1$. Moreover, $\tau_1, \tau_2 \in K$ are some integral powers of the parameter q (also depending on n, k).

Now observe that all elements of H appearing in the above formula on the right hand side of the tensor products belong to the standard basis

$$B_1 = \{a^s b^i c^j,\, b^i c^j d^t \mid 0 \le i, j, s,\, 1 \le t\}$$

of the Hopf algebra H. It is obvious that

$$B_2 = \{a^s b^i c^j, b^i c^j d^t \mid 0 \le i, j, s \le m-1,\, 1 \le t \le m-1\}$$

is a basis of D. Furthermore, if $w \in B_1$ and $f(w) \neq 0$, then $f(w) \in B_2$. Consider the set

$$T = \{a^s c, cd^t \mid 0 \le s,\, 1 \le t \le m-1\} \subset B_2$$

and notice that $1 \notin T$ and $f^{-1}(T) \cap B_1 = S$, where

$$S = \{a^s c, cd^t \mid 0 \le s, t\} \subset B_1.$$

Further, by (2), $\Delta(y) = \sum_{\gamma \in B_1} y_\gamma \otimes \gamma$ for some $y_\gamma \in H$ and

$$[(\mathrm{Id} \otimes f)\Delta]\,(y) = \sum_{\gamma \in S} y_\gamma \otimes f(\gamma) + \sum_{\gamma \notin S} y_\gamma \otimes f(\gamma) = y \otimes 1,$$

because $y \in H^f$. But $f(\gamma) \in T$ if and only if $\gamma \in S$ for $\gamma \in B_1$, whence $\sum_{\gamma \in S} y_\gamma = 0$. Again by (2), this implies that

$$0 = \sum_{\gamma \in S} y_\gamma = \sum_{n,\, n_2 \neq 0} \lambda_n \tau_2 \binom{n_1}{0}_{q^2} \binom{n_2}{n_2 - 1}_{q^2} \binom{n_3}{0}_{q^2} b^{n_1} c^{n_2 - 1} d^{n_3 + 1}.$$

Hence $\lambda_n \binom{n_2}{n_2-1}_{q^2} = \lambda_n \binom{n_2}{1}_{q^2} = \lambda_n (n_2)_{q^2} = 0$ for $n \ge 0$ with $n_2 \neq 0$. Consequently, if $\lambda_n \neq 0$, then $q^{2n_2} = 1$, which means that $m \mid n_2$.

Now consider the sets

$$T = \{d^i,\, a^i \mid 1 \le i \le m-1\} \subset B_2,$$
$$S = f^{-1}(T) \cap B_1 = \{d^i,\, a^i \mid m \nmid i\} \subset B_1.$$

Similarly to the above, $f(\gamma) \in T$ if and only if $\gamma \in S$ for $\gamma \in B_1$. It follows that $\sum_{\gamma \in S} y_\gamma = 0$, whence

$$\sum_{n,\, m \nmid (n_1 - n_2 + n_3)} \lambda_n \tau_2 \binom{n_1}{0}_{q^2} \binom{n_2}{n_2}_{q^2} \binom{n_3}{0}_{q^2} b^{n_1} c^{n_2} d^{n_3} = 0.$$

Since we know that $m \mid n_2$ for $\lambda_n \neq 0$, the above equality reduces to

$$\sum_{n,\, m \nmid (n_1 + n_3)} \lambda_n \tau_2 b^{n_1} c^{n_2} d^{n_3} = 0.$$

Therefore, if $\lambda_n \neq 0$, then $n_1 + n_3$ is divisible by m. Further, by considering the sets

$$T = \{a^i b,\, bd^i \mid 0 \le i \le m-1\} \subset B_2,$$
$$S = \pi^{-1}(T) \cap B_1 = \{a^i b,\, bd^i \mid i \ge 0\} \subset B_1,$$

one can verify that

$$
\begin{aligned}
0 = & \sum_{n,\, n_3\neq 0} \lambda_n \tau_2 \binom{n_1}{0}_{q^2} \binom{n_2}{n_2}_{q^2} \binom{n_3}{1}_{q^2} b^{n_1} c^{n_2+1} d^{n_3-1} \\
& + \sum_{n,\, n_1\neq 0} \lambda_n \tau_2 \binom{n_1}{1}_{q^2} \binom{n_2}{n_2}_{q^2} \binom{n_3}{0}_{q^2} a b^{n_1-1} c^{n_2} d^{n_3}.
\end{aligned}
$$

In view of the relation $ad = 1 + q^{-1}bc$ it follows that

$$
\begin{aligned}
0 = & \sum_{n_1,n_2\geq 0,\, n_3\geq 1} \lambda_n \tau_2 (n_3)_{q^2} b^{n_1} c^{n_2+1} d^{n_3-1} \\
& + \sum_{n_2\geq 0,\, n_1\geq 1} \lambda_{(n_1,n_2,0)} \tau_2 (n_1)_{q^2} a b^{n_1-1} c^{n_2} \\
& + \sum_{n_2\geq 0,\, n_1,n_3\geq 1} \lambda_n \tau_3 (n_1)_{q^2} b^{n_1-1} c^{n_2} d^{n_3-1} \\
& + \sum_{n_2\geq 0,\ n_1,n_3\geq 1} \lambda_n \tau_4 (n_1)_{q^2} b^{n_1} c^{n_2+1} d^{n_3-1} \\
= & \sum_{n_1\geq 0,\, n_2,n_3\geq 1} \lambda_{(n_1,n_2-1,n_3)} \tau_2 (n_3)_{q^2} b^{n_1} c^{n_2} d^{n_3-1} \\
& + \sum_{n_2\geq 0,\, n_1\geq 1} \lambda_{(n_1,n_2,0)} \tau_2 (n_1)_{q^2} a b^{n_1-1} c^{n_2} \\
& + \sum_{n_1,n_2\geq 0,\, n_3\geq 1} \lambda_{(n_1+1,n_2,n_3)} \tau_3 (n_1+1)_{q^2} b^{n_1} c^{n_2} d^{n_3-1} \\
& + \sum_{n_1,n_2,n_3\geq 1} \lambda_{(n_1,n_2-1,n_3)} \tau_4 (n_1)_{q^2} b^{n_1} c^{n_2} d^{n_3-1}.
\end{aligned}
$$

where $\tau_2, \tau_3, \tau_4 \in K$ are some integral powers of q. Suppose that $\lambda_n \neq 0$. We already know that $m \mid n_2$ and $m \mid (n_1 + n_3)$. For the proof that $y \in L$ we have to show that $m \mid n_1$ and $m \mid n_3$. To this end, it clearly suffices to check that $m \mid n_1$. One can assume that $n_1, n_3 \geq 1$. In the above sum the coefficient of $b^{n_1-1}d^{n_3-1}$ equals $\tau_3\lambda_{(n_1,0,n_3)}(n_1)_{q^2}$, whence $m \mid n_1$ whenever $n_2 = 0$. So let $n_2 \geq 1$. But the coefficient of the monomial $b^{n_1-1}c^{n_2}d^{n_3-1}$ (again in the above sum) is equal to

$$\lambda_{(n_1-1,n_2-1,n_3)}\tau_2(n_3)_{q^2} + \lambda_{(n_1,n_2,n_3)}\tau_3(n_1)_{q^2} + \lambda_{(n_1-1,n_2-1,n_3)}\tau_4(n_1-1)_{q^2} = 0.$$

Hence $\lambda_n\tau_3(n_1)_{q^2} = 0$, because $\lambda_{(n_1-1,n_2-1,n_3)} = 0$ $(m \nmid (n_1 - 1 + n_3))$. Consequently, $(n_1)_{q^2} = 0$ and $m \,|\, n_1$.

Thus we have proved that $y = d^{mk}x \in L$ for some $k \geq 0$. In view of Lemma 2.12(ii) it follows that

$$a^{mk}y = a^{mk}d^{mk}x = (a^m d^m)^k x = (1 \pm (bc)^m)^k x = \sum_{i=0}^{k} \alpha_i (bc)^{im} x \in L$$

for some $\alpha_i \in K$ with $\alpha_0 = 1$. Using the standard basis of H, we can write

$$x = \sum_{j \in \mathbb{N}^4,\ j_1 j_4 = 0} \beta_j a^{j_1} b^{j_2} c^{j_3} d^{j_4}$$

for some $\beta_j \in K$. Let $T = \{j \in \mathbb{N}^4 \mid \beta_j \neq 0 \wedge \exists_{r=1,\dots,4}\, m \nmid j_r\}$. If the set T is not empty, choose a $t \in T$ with $t_2 = \min\{j_2 \mid j \in T\}$. Then the element $\sum_{i=0}^{k} \alpha_i (bc)^{im} x$ does not belong to L, because its presentation in the standard basis contains the summand $\alpha_0 \beta_t a^{t_1} b^{t_2} c^{t_3} d^{t_4}$. This contradiction makes it clear that the set T is empty, and therefore $x \in L$. The theorem follows. ∎

THEOREM 2.16. *Suppose that* $\operatorname{char}(K) > 0$. *Then the Hopf algebra* $K[SL_q(2)]$ *is geometrically coreductive for each* q, *provided the Hopf algebra* $K[SL_\varepsilon(2)]$ *is geometrically coreductive for both* $\varepsilon \in \{1, -1\}$.

Proof. In view of Theorem 2.9, we can assume that q is a root of unity. In that case we have the natural morphism of Hopf algebras $f : H \to D$, where $H = K[SL_q(2)]$ and $D = H/(a^m - 1, b^m, c^m, d^m - 1)$ for some $m \geq 1$. By Theorem 2.15, H^f is a Hopf subalgebra of H isomorphic to $K[SL_\varepsilon(2)]$ for some $\varepsilon \in \{1, -1\}$. Furthermore, it is easy to see that the Hopf algebra D is finite-dimensional. The conclusion now follows, using Theorems 1.4 and 2.2. ∎

We are now going to prove that if $\varepsilon^2 = 1$, then in any characteristic the Hopf algebra $K[SL_\varepsilon(2)]$ is geometrically coreductive (obviously only the case $\varepsilon = -1$ requires proof). The proof given below is patterned on Springer's proof of the geometric reductivity of the algebraic group $SL(2)$ presented in [10]. Again some preparations are needed.

Let $K_q[x, y]$ be the algebra generated by the symbols x, y subject to the relation $yx = qxy$ (the algebra $K_q[x, y]$ is called the *quantum plane*). It is easy to see that the algebra $K_q[x, y]$ is a $K[SL_q(2)]$-comodule algebra, via

$$\rho(x) = x \otimes a + y \otimes c, \qquad \rho(y) = x \otimes b + y \otimes d.$$

By Lemma 2.11 it follows that

$$\rho(x^s y^t) = \sum_{i=0}^{s} \sum_{j=0}^{t} q^{j(s-i)} \binom{s}{i}_{q^2} \binom{t}{j}_{q^2} x^{i+j} y^{s+t-(i+j)} \otimes a^i b^j c^{s-i} d^{t-j} \tag{3}$$

for any $s, t \geq 0$. Given an $n \geq 0$, we denote by $K_q[x, y]_n$ the subspace of $K_q[x, y]$ spanned by the set $\{x^i y^{n-i} \mid i = 0, 1, \dots, n\}$. By the above formula, $K_q[x, y]_n$ is a subcomodule of $K_q[x, y]$.

REMARK 2.17. If $q = 1$, then $K_q[x, y]$ is nothing other than the symmetric algebra $S(K^2)$ with the $SL(2)$ action induced by the standard action of the group $SL(2)$ on K^2 given by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right](x, y) = (ax + by, cx + dy)$.

Let, as above, $H = k[SL_q(2)]$ and fix an $n \geq 0$. Below $e_k = x^k y^{n-k} \in K_q[x, y]_n$ for $k = 0, \ldots, n$, and the elements $\{h_{sk} \mid s, k = 0, \ldots, n\} \subset H$ are defined by

$$\rho(e_k) = \sum_{s=0}^{n} e_s \otimes h_{sk}, \qquad k = 0, \ldots, n.$$

Furthermore, we set $\binom{n}{i}_\lambda = 0$ for $\lambda \in K$ whenever $i > n$ or $i < 0$. From (3) we obtain

$$h_{sk} = \sum_{i=0}^{n} q^{(s-i)(k-i)} \binom{k}{i}_{q^2} \binom{n-k}{s-i}_{q^2} a^i b^{s-i} c^{k-i} d^{n-k-s+i} \tag{4}$$

for $s, k = 0, \ldots, n$. As $ad - q^{-1}bc = 1$, it follows that

$$h_{sk} = \sum_{i=0}^{n} \sum_{r=0}^{i} \tau_1 \binom{i}{r}_{q^2} \binom{k}{i}_{q^2} \binom{n-k}{s-i}_{q^2} b^{s-i+r} c^{k-i+r} d^{n-k-s} \tag{5}$$

whenever $n - k - s \geq 0$, and similarly

$$h_{sk} = \sum_{i=0}^{n} \sum_{r=0}^{n-k-s+i} \tau_2 \binom{n-k-s+i}{r}_{q^2} \binom{k}{i}_{q^2} \binom{n-k}{s-i}_{q^2} \cdot a^{-(n-k-s)} b^{s-i+r} c^{k-i+r} \tag{6}$$

whenever $n - k - s < 0$ (again $\tau_1, \tau_2 \in K$ are some integral powers of q).

LEMMA 2.18. *Suppose that M is a nonzero subcomodule of the H-comodule $K_q[x, y]_n$. Then $x^n, y^n \in M$.*

Proof. Let $0 \neq m = \sum_{k=0}^{n} \alpha_k e_k \in M$. By (5) and (6), we know that

$$\begin{aligned}
\rho(m) &= \sum_{k=0}^{n} \alpha_k \sum_{s=0}^{n} e_s \otimes h_{sk} \\
&= \sum_{k+s \leq n} \alpha_k e_s \otimes \sum_{i=0}^{n} \sum_{r=0}^{i} \tau_1 \binom{i}{r}_{q^2} \binom{k}{i}_{q^2} \binom{n-k}{s-i}_{q^2} b^{s-(i-r)} c^{k-(i-r)} d^{n-k-s} \\
&\quad + \sum_{k+s > n} \alpha_k e_s \otimes \sum_{k-n+s \leq i} \sum_{r=0}^{n-k-s+i} \tau_2 \lambda(k, s, i, r) a^{-(n-k-s)} b^{s-(i-r)} c^{k-(i-r)},
\end{aligned}$$

where

$$\lambda(k, s, i, r) = \binom{n-k-s+i}{r}_{q^2} \binom{k}{i}_{q^2} \binom{n-k}{s-i}_{q^2}.$$

Now set $k_0 = \max\{k \mid \alpha_k \neq 0$ and write $\rho(m)$ as the sum $\sum_{\gamma\in\Gamma} x_\gamma \otimes y_\gamma$, where Γ is the standard basis of H and $x_\gamma \in M$. If $k_0 = 0$, i.e., $m = \alpha_0 e_0 = \alpha_0 y^n$, then it is easily seen that $x_\gamma = \alpha_0 \tau_1 x^n$ for $\gamma = b^n$. Similarly, if $k_0 > 0$, then $x_\gamma = \alpha_{k_0} \tau_1 y^n$ for $\gamma = c^{k_0} d^{n-k_0}$ and $x_\gamma = \alpha_{k_0} \tau_2 x^n$ for $\gamma = a^{k_0} b^{n-k_0}$. Hence $x^n, y^n \in M$. ■

Corollary 2.19.

(i) *Suppose that q^2 is a primitive mth root of unity. Then the Hopf algebra $H = K[SL_q(2)]$ is not cosemisimple in the following cases:* (a) $m \geq 2$, (b) $m = 1$ *and* $\mathrm{char}(K) > 0$.

(ii) *If $q^2 = 1$, then the H-comodule $K_q[x, y]_n$ is simple whenever* $\mathrm{char}(K) = 0$ *or if* $\mathrm{char}(K) > 0$ *and $n = p^r - 1$ for some $r \geq 0$.*

(iii) *If q is not a root of unity, then the H-comodule $K_q[x, y]_n$ is simple for each $n \geq 0$.*

Proof. (i) Let $T = Kx^m + Ky^m$. Then T is a subcomodule of $K_q[x, y]_m$, by Lemmas 2.10 and 2.11. If $m \geq 2$, then clearly $T \neq K_q[x, y]_m$. Suppose that $K_q[x, y]_m = T \oplus M$ for some subcomodule $M \subset K_q[x, y]_m$. Then $x^m, y^m \in M$, by the above lemma, which is impossible. If $p = \mathrm{char}(K) > 0$ and $m = 1$ (i.e., $q^2 = 1$), then $\rho(x^p) = (x \otimes a + y \otimes c)^p = x^p \otimes a^p + y^p \otimes c^p$ and $\rho(y^p) = (x \otimes b + y \otimes d)^p = x^p \otimes b^p + y^p \otimes d^p$, because $y \otimes c$ commutes with $x \otimes a$ and $y \otimes d$ commutes with $x \otimes b$. This means that $T = Kx^p + Ky^p$ is a proper subcomodule of $K_q[x, y]_p$, and as above, we show that T is not a direct summand of $K_q[x, y]_p$. Therefore, in either case H is not cosemisimple.

(ii) Suppose that $q^2 = 1$ and let M be a nonzero subcomodule of $K_q[x, y]_n$. By the above lemma, $x^n \in M$. Since

$$\rho(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \otimes a^i c^{n-i}$$

and, under our assumption, $\binom{n}{i} 1_K \neq 0$ for $i = 0, \ldots, n$, it follows that $x^i y^{n-i} \in M$ for $i = 0, \ldots, n$. Consequently, $M = K_q[x, y]_n$. This means that $K_q[x, y]_n$ is a simple H-comodule.

(iii) can be proved in the same way as (ii). ■

Lemma 2.20.

(i) *We have*

$$\binom{n}{k}_{q^2} S(h_{sk}) = (-q)^{s-k} \binom{n}{s}_{q^2} h_{n-k,n-s}, \qquad 0 \leq k, s \leq n.$$

(ii) *If $\binom{n}{k}_{q^2} \neq 0$ for $k = 0, \ldots, n$, then the map $f : K_q[x, y]_n \to (K_q[x, y]_n)^*$*

given by

$$f(e_k) = (-q)^k \binom{n}{k}_{q^2}^{-1} e^*_{n-k}, \qquad k = 0, \dots, n,$$

is an isomorphism of H-comodules.

Proof. By (3),

$$\begin{aligned} S(h_{sk}) &= \sum_{i=0}^{n} q^{(s-i)(k-i)} \binom{k}{i}_{q^2} \binom{n-k}{s-i}_{q^2} S(a^i b^{s-i} c^{k-i} d^{n-k-s+i}) \\ &= (-q)^{s-k} \sum_{i} q^{(s-i)(k-i)} \binom{k}{i}_{q^2} \binom{n-k}{s-i}_{q^2} a^{n-k-s+i} b^{s-i} c^{k-i} d^i. \end{aligned}$$

Set $i' = n-k-s+i$, $s' = n-k$, $k' = n-s$. Then

$$\begin{aligned} s-i &= n-k' - (i'+n-k'-s') = s'-i', \\ k-i &= n-s' - (i'+n-k'-s') = k'-i' \end{aligned}$$

and

$$\begin{aligned} \binom{n-k}{s-i}_{q^2} &= \binom{s'}{s'-i'}_{q^2} = \binom{s'}{i'}_{q^2}, \\ \binom{k}{i}_{q^2} &= \binom{n-s'}{n'-s'-(k'-i')}_{q^2} = \binom{n-s'}{k'-i'}_{q^2}, \end{aligned}$$

which implies that

$$\begin{aligned} S(h_{sk}) = (-q)^{(s-k)} \sum_{i'=0}^{n} q^{(s'-i')(k'-i')} &\binom{n-s'}{k'-i'}_{q^2} \binom{s'}{i'}_{q^2} \\ &\cdot a^{i'} b^{s'-i'} c^{k'-i'} d^{i'+n-k'-s'}. \end{aligned}$$

Further, one easily checks that

$$\binom{n}{s}_{\lambda} \binom{n-s}{k-i}_{\lambda} \binom{s}{i}_{\lambda} = \binom{n}{k}_{\lambda} \binom{n-k}{s-i}_{\lambda} \binom{k}{i}_{\lambda}$$

for all $\lambda \in K$ and $s, k, i \geq 0$. Hence

$$\begin{aligned} \binom{n}{k}_{q^2} S(h_{sk}) &= \binom{n}{s'}_{q^2} S(h_{sk}) \\ &= (-q)^{s'-k'} \binom{n}{k'}_{q^2} \sum_{i'} q^{(s'-i')(k'-i')} \binom{k'}{i'}_{q^2} \binom{n-k'}{s'-i'}_{q^2} \\ &\qquad \cdot a^{i'} b^{s'-i'} c^{k'-i'} d^{n-k'-s'+i'} \\ &= (-q)^{s'-k'} \binom{n}{k'}_{q^2} h_{s'k'} = (-q)^{s-k} \binom{n}{s}_{q^2} h_{n-k,n-s}. \end{aligned}$$

Thus (i) is proved. Since

$$\begin{aligned}
(\rho f)(e_k) &= (-q)^k \binom{n}{k}_{q^2}^{-1} \rho(e^*_{n-k}) = (-q)^k \binom{n}{k}_{q^2}^{-1} \sum_{s=0}^{n} e^*_s \otimes S(h_{n-k,s}) \\
&= (-q)^k \binom{n}{k}_{q^2}^{-1} \sum_{s=0}^{n} e^*_s \otimes (-q)^{n-s-k} \binom{n}{n-k}_{q^2} \binom{n}{s}_{q^2}^{-1} h_{n-s,k} \\
&= \sum_{s=0}^{n} (-q)^{n-s} e^*_s \otimes \binom{n}{s}_{q^2}^{-1} h_{n-s,k} = \sum_{s=0}^{n} (-q)^{s} \binom{n}{s}_{q^2}^{-1} e^*_{n-s} \otimes h_{sk} \\
&= \sum_{s} f(e_s) \otimes h_{sk} = [(f \otimes \mathrm{Id})\rho](e_k)
\end{aligned}$$

for $k = 0, \ldots, n$, also (ii) is true. ∎

COROLLARY 2.21. *Assume that $q^2 = 1$. Moreover, assume that either* $\mathrm{char}(K) = 0$ *and $n \geq 0$ is arbitrary, or* $\mathrm{char}(K) = p > 0$ *and $n = p^r - 1$ for some $r \geq 0$. Then $\binom{n}{k}_{q^2} = \binom{n}{k} 1_K \neq 0$ for $k = 0, \ldots, n$ and the linear map*

$$f : K_q[x, y]_n \to (K_q[x, y]_n)^*,\ f(e_k) = (-q)^k \binom{n}{k}^{-1} e^*_{n-k},\ k = 0, \ldots, n,$$

is an isomorphism of H-comodules.

Proof. If $\mathrm{char}(K) = p > 0$, then $\binom{n}{k} 1_K \neq 0$ since $\binom{p^r-1}{k} = (-1)^k \bmod p$. Therefore, the corollary is a consequence of the above lemma. ∎

Now fix an $\varepsilon \in \{1, -1\}$ and set $H = K[SL_\varepsilon(2)]$. Note that the antipode S of H has order 2, that is, $S^2 = \mathrm{Id}$. This will allow us to apply Theorem 2.2 to H. Also notice that the algebra H admits a $\mathbb{Z}$-grading determined by $\deg(a) = \deg(b) = 1$, $\deg(c) = \deg(d) = -1$. In particular,

$$H = \bigoplus_{n \in \mathbb{Z}} H_n,$$

where H_n is the vector subspace of H spanned by the set

$$\{a^{n_1} b^{n_2} c^{n_3} d^{n_4} \mid n_1 + n_2 - (n_3 + n_4) = n\}.$$

It is easy to see that each H_n is a subcomodule of the H-comodule (H, Δ).

LEMMA 2.22. *The subalgebra $B = \bigoplus_{n \in \mathbb{Z}} H_{2n}$ of H is commutative.*

Proof. Let $X = a^{s_1} b^{s_2} c^{s_3} d^{s_4} \in H_{2s}$ and $Y = a^{t_1} b^{t_2} c^{t_3} d^{t_4} \in H_{2t}$. Then clearly

$$\begin{aligned}
XY &= \varepsilon^{t_1(s_3+s_2)+t_2(s_4+s_1)+t_3(s_4+s_1)+t_4(s_3+s_2)} YX \\
&= \varepsilon^{(t_1+t_4)(s_3+s_2)+(t_2+t_3)(s_4+s_1)} YX.
\end{aligned}$$

Since the numbers $s_1 + s_4 + s_2 + s_3$ and $t_1 + t_4 + t_2 + t_3$ are even, $s_1 + s_4$ is even if and only if $s_2 + s_3$ is even, and $t_1 + t_4$ is even if and only if $t_2 + t_3$

is even. Consequently, the number $(t_1+t_4)(s_3+s_2)+(t_2+t_3)(s_4+s_1)$ is even, whence $XY=YX$. ■

For $u\ge n$ we define $H_{n,u}$ to be the subspace of H_n spanned by the set

$$\{a^{n_1}b^{n_2}c^{n_3}d^{n_4}\mid n_1+n_2-(n_3+n_4)=n,\ 0\le n_1+n_2\le u\}.$$

Observe that $H_{n,u}\subset H_{n,u+1}$ and $H_n=\bigcup_{u\ge n}H_{n,u}$. Moreover, $H_{n,u}$ is a subcomodule of H_n.

Lemma 2.23. *The set $B_u=\{a^ib^{u-i}c^jd^{u-n-j}\mid 0\le i\le u,\ 0\le j\le u-n\}$ is a basis of $H_{n,u}$.*

Proof. Let us assume that $\sum_{i,j}\alpha_{ij}a^ib^{u-i}c^jd^{u-n-j}=0$ for some $\alpha_{ij}\in K$. As $H=K[M_\varepsilon(2)]/(ad-\varepsilon bc-1)$, it follows that in the algebra $K[M_\varepsilon(2)]$ we have the equality

$$\sum_{0\le i\le u,\,0\le j\le u-n}\alpha_{ij}a^ib^{u-i}c^jd^{u-n-j}=h(ad+bc-1)$$

(for some $h\in K[M_\varepsilon(2)]$). This implies that $h=0$, using the natural grading in $K[M_\varepsilon(2)]$ given by $\deg(\delta)=1$ for $\delta\in\{a,b,c,d\}\subset K[M_\varepsilon(2)]$. Therefore,

$$\sum_{0\le i\le u,\,0\le j\le u-n}\alpha_{ij}a^ib^{u-i}c^jd^{u-n-j}=0$$

in $K[M_\varepsilon(2)]$, so that $\alpha_{ij}=0$ for all i,j, by Lemma 2.7. It remains to prove that the set B_u spans the subspace $H_{n,u}$. Notice that given $n_1,n_2,n_3,n_4\ge 0$,

$$\begin{aligned}a^{n_1}b^{n_2}c^{n_3}d^{n_4}&=a^{n_1}b^{n_2}(ad-\varepsilon bc)c^{n_3}d^{n_4}\\&=\alpha a^{n_1+1}b^{n_2}c^{n_3}d^{n_4+1}+\beta a^{n_1}b^{n_2+1}c^{n_3+1}d^{n_4}\end{aligned}$$

for some $\alpha,\beta\in K$. By induction on $u-(n_1+n_2)$, it follows that $a^{n_1}b^{n_2}c^{n_3}d^{n_4}\in H_{n,u}$ is a linear combination of elements from B_u. ■

Lemma 2.24. *For each $n\ge 0$ the linear map $g: H_{0,n}\to K_\varepsilon[x,y]_n\otimes K_\varepsilon[x,y]_n$ given by*

$$g(a^kb^{n-k}c^sd^{n-s})=e_k\otimes e_s,\qquad k=0,\dots,n,$$

is an isomorphism of H-comodules.

The proof is straightforward computation, using (3) and Lemma 2.23.

Now we are ready to prove the announced theorem.

Theorem 2.25. *The Hopf algebra $H=K[SL_\varepsilon(2)]$ is geometrically coreductive for any field K.*

Proof. Let (V,ρ) be a finite-dimensional H-comodule and let $0\ne v_0\in V^{\mathrm{co}\,H}$. As $S^2=\mathrm{Id}$, by Theorem 2.3, it suffices to find $r>0$ and $T\in S_H^r(V^*)^{\mathrm{co}\,H}$ such that $\widetilde{e(v_0)}(T)\ne 0$. Choose a linear map $l: V\to K$ with

$l(v_0) = 1$. Then we have the morphism of H-comodules

$$\psi : V \to H \xrightarrow{\pi} H_0, \qquad \psi(v) = \pi\Big(\sum l(v_i)h_i\Big),$$

where $\pi : H \to H_0$ is the projection on the 0-component of the grading $H = \bigoplus_{n\in\mathbb{Z}} H_n$ and $\sum v_i \otimes h_i = \rho(v)$. Certainly $\psi(v_0) = 1$. Since $H_0 = \bigcup_n H_{0,n}$, $\operatorname{im}\psi \subset H_{0,n}$ for some $n \geq 0$, and we can assume that $n = p^m - 1$ for some $m \geq 0$, provided $\operatorname{char}(K) = p > 0$. By Lemmas 2.24, 1.7 and Corollary 2.21, it follows that the H-comodules $H_{0,n}$ and $E = \operatorname{End}(K_\varepsilon[x,y]_n)$ are isomorphic. Therefore, there exists a morphism of H-comodules

$$\varphi : V \to E$$

such that $u = \varphi(v_0) \neq 0$. It is clear that $u \in E^{\operatorname{co}H} = \operatorname{End}_H(K_\varepsilon[x,y]_n)$, because $v_0 \in V^{\operatorname{co}H}$. Furthermore, in view of Corollary 2.19, the H-comodule $K_\varepsilon[x,y]_n$ is simple. Hence, u is an isomorphism of H-comodules. The morphism φ induces a morphism of graded H-comodule algebras

$$S_H(\varphi^*) : S_H(E^*) \to S_H(V^*)$$

(determined by $S_H(\varphi^*)(g) = g\varphi$ for $g \in E^*$). By Theorem 2.5(ii)(b), we know that there exists a coinvariant $F^* \in S_H(E^*)$ of degree $r = \dim V > 0$ such that for any $f \in E^{\operatorname{co}H}$ the map $e(f) : E^* \to K$, $e(f)(e^*) = e^*(f)$, is a morphism of H-comodules and $\widetilde{e(f)}(F^*) = \det(f)$. Set $T = S_H(\varphi^*)(F^*)$. Then $T \in S^r_H(V^*)^{\operatorname{co}H}$ and it is easily seen that $\widetilde{e(v_0)}(T) = \det(u) \neq 0$, because $S_H(\varphi^*)\widetilde{e(v_0)} = \widetilde{e(u)}$. ■

The main result of the paper is the following theorem.

THEOREM 2.26.

(i) *If* $\operatorname{char}(K) > 0$, *then the Hopf algebra* $K[SL_q(2)]$ *is geometrically coreductive for any parameter* q. *Moreover, the Hopf algebra* $K[SL_{-1}(2)]$ *is geometrically coreductive for any field* K.

(ii) *Assume that* q *is a primitive* m*th root of unity. Then* $K[SL_q(2)]$ *is not linearly coreductive if* $m \geq 2$ *or if* $m = 1$ *and* $\operatorname{char}(K) > 0$.

Part (i) is a consequence of Theorems 2.16 and 2.25. As for (ii), it follows from Corollary 2.19.

REMARK 2.27. From [11, Theorem 5.8] it follows that if q is not a root of unity, then for any $n \geq 2$ the Hopf algebra $K[SL_q(n)]$ of the quantum group $SL_q(n,K)$ is cosemisimple (= linearly coreductive). Furthermore, if $\operatorname{char}(K) > 0$, then we know that the Hopf algebra $K[SL_1(n)]$ is geometrically coreductive, because the algebraic group $SL_1(n) = SL(n)$ is geometrically reductive. So it is natural to conjecture that if $\operatorname{char}(K) > 0$ and q is a root of unity, then the Hopf algebra $K[SL_q(n)]$ is geometrically coreductive for each q and $n \geq 2$.

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