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# *AN UNCOUNTABLE PARTITION CONTAINED IN THE ATOMLESS σ-FIELD*

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**Abstract.** This short note considers the question of whether every atomless $\sigma$-field contains an uncountable partition. The paper comments the situation for a couple of known $\sigma$-fields. A negative answer to the question is the main result.

**Definitions and basic facts.** Throughout the note the very basic set-theoretical notation is used. A natural number $i$ is understood to be equal to $\{0, 1, \ldots, i-1\}$. The set of finite sequences of natural numbers is denoted by $\mathbb{N}^{<\aleph_0}$. If $s, k \in \mathbb{N}^{<\aleph_0}$ then $s \prec k$ means that there exists $i \in \mathbb{N}$ such that $k{\restriction}_i = s$ (simply, $k$ is extension of $s$). The predecessor of $s \in \mathbb{N}^{<\aleph_0}$ is denoted by $\hat{s}$.

If $\mathcal{G} \subseteq \mathcal{P}(X)$ then $\sigma(\mathcal{G})$ stands for the smallest $\sigma$-field on $X$ containing $\mathcal{G}$ (called the $\sigma$-field *generated* by $\mathcal{G}$). If $\mathcal{G}$ is countable then $\sigma(\mathcal{G})$ is said to be *countably generated.*

Let $\mathcal{A}$ be a $\sigma$-field on $X$. A nonempty set $A \in \mathcal{A}$ is an *atom* of $\mathcal{A}$ if $A \subseteq B$ or $A \cap B = \emptyset$ for any $B \in \mathcal{A}$. If no element of $\mathcal{A}$ is an atom of $\mathcal{A}$ then $\mathcal{A}$ is said to be *atomless.* If all the atoms of $\mathcal{A}$ form a partition of $X$, then $\mathcal{A}$ is *atomic.* For a reference on $\sigma$-fields see [1] and [2].

**1. The problem.** It is well known that every atomless $\sigma$-field contains an uncountable subfamily of nonempty and pairwise disjoint sets. A simple construction of such a subfamily can be found in [1]. The family constructed there does not, however, cover the whole space. The following question appeared during the Set Theory seminar at IM UG and was open for some time: *Does every atomless $\sigma$-field contain an uncountable subfamily which is a partition of the space*? The answer turns out to be negative (see Section 3). The next section exhibits complicated atomless $\sigma$-fields that do contain an uncountable partition, which contrasts with the final result.

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## 2. Examples of $\sigma$-fields

EXAMPLE 1. $\mathcal{F}\subset\mathcal{P}(X)$ is a *$\sigma$-independent family* if for distinct $F_0, F_1, \ldots \in \mathcal{F}$ and disjoint $I, J \subset \mathbb{N}$ (not both empty) the intersection $(\bigcap_{i\in I} F_i) \cap (\bigcap_{j\in J} X \setminus F_j)$ is not empty. The $\sigma$-field generated by an uncountable $\sigma$-independent family is the simplest and most common example of an atomless $\sigma$-field ([1]).

LEMMA 2.1. *If $\mathcal{A} \subseteq \mathcal{P}(X)$ is a $\sigma$-field that contains an infinite $\sigma$-independent family $\mathcal{F}$ then $\mathcal{A}$ also contains an uncountable partition of $X$.*

*Proof.* Let $F_0, F_1, \ldots$ be distinct members of $\mathcal{F}$. For $I \subset \mathbb{N}$, define $A_I$ as $(\bigcap_{i\in I} F_i) \cap (\bigcap_{j\in\mathbb{N}\setminus I} X \setminus F_j)$. Of course, every $A_I$ is in $\mathcal{A}$. Note that $A_I$ and $A_J$ are disjoint if $I, J \subseteq \mathbb{N}$ are different. Indeed, if $i \in I$ and $i \notin J$ then $A_I \subset F_i$ and $A_J \subset X \setminus F_i$. One can also see that for every $x \in X$, there exists $I$ (defined as $\{i \in \mathbb{N} : x \in F_i\}$) such that $x \in A_I$. These two facts mean that $\{A_I : I \subset \mathbb{N}\}$ is an uncountable partition of $X$. ∎

EXAMPLE 2 (CH). Let

$$X = \{a \in [0,1]^\omega : |a[\omega]| < \aleph_0\}.$$

Define $A_t = \{a \in X : t \in a[\omega]\}$ for $t \in [0,1]$. It is shown in [1] that $\mathcal{A} = \sigma(\{A_t\}_{t\in[0,1]})$ is an atomless $\sigma$-field that does not contain an infinite $\sigma$-independent family. If the Continuum Hypothesis is assumed, $[0,1]$ may be represented as $\{t_\alpha : \alpha \in \omega_1\}$. Set $B_\alpha = A_{t_\alpha} \setminus \bigcup_{\beta<\alpha} A_{t_\beta}$ for $\alpha \in \omega_1$. It is evident that $\{B_{t_\alpha} : \alpha \in \omega_1\}$ is a partition of $X$ and is contained in $\mathcal{A}$.

EXAMPLE 3. The previous example can be generalized as follows. Let $\kappa$ be any cardinal. Define

$$Z = \{z \in \mathcal{P}(\kappa) : 0 < |z| < \aleph_0\}.$$

Define $G_\alpha = \{z \in Z : \alpha \in z\}$ for $\alpha \in \kappa$. Let $\mathcal{C} = \sigma(\{G_\alpha : \alpha \in \kappa\})$. We now recall some known general properties of $\sigma$-fields.

LEMMA 2.2. *Let $\mathcal{G} \subset \mathcal{P}(X)$ and $\mathcal{A} = \sigma(\mathcal{G})$. For any $A \in \mathcal{A}$:*

(i) *There exists a countable $\mathcal{G}_0 \subset \mathcal{G}$ such that $A \in \sigma(\mathcal{G}_0)$.*

(ii) *For $\mathcal{G}_0$ as above, $A$ is a union of sets of the form*

$$\bigcap_{i\in I} G_i \cap \bigcap_{j\in\mathbb{N}\setminus I} G_j^c$$

*for $G_0, G_1, \ldots$ being all elements of $\mathcal{G}_0$ and $I \subset \mathbb{N}$.*

*Proof.* Let $\mathcal{Z}$ be the subfamily of $\mathcal{A}$ consisting of all elements satisfying (i). Note that $\mathcal{G} \subset \mathcal{Z}$, since $G \in \sigma(\{G\})$ for every $G \in \mathcal{G}$. It is easy to check that $\mathcal{Z}$ is closed under complements and countable unions. Hence $\mathcal{Z}$ is a $\sigma$-field. Since $\mathcal{A}$ is the smallest $\sigma$-field that contains $\mathcal{G}$, we have $\mathcal{Z} = \mathcal{A}$, which proves (i).

Similarly, to prove (ii), define $\mathcal{W}$ as the subfamily of $\mathcal{A}$ of all elements satisfying (ii). Again, one can easily prove that $\mathcal{G} \subset \mathcal{W}$ and $\mathcal{W}$ is a $\sigma$-field, which means $\mathcal{W} = \mathcal{A}$. ■

We will now show that the above $\sigma$-field $\mathcal{C}$ has the same properties as $\mathcal{A}$ in the previous example. The proof of the fact below is similar to the proof for $\mathcal{A}$ in [1].

PROPOSITION 2.3. *$\mathcal{C}$ does not contain an infinite $\sigma$-independent family. If $\kappa$ is uncountable, then $\mathcal{C}$ is atomless.*

*Proof.* Suppose that there exists an infinite, countable $\sigma$-independent family $\mathcal{F} = \{F_1, F_2, \ldots\}$ contained in $\mathcal{C}$. By Lemma 2.2, for every $n \in \mathbb{N}$, there exists a countable $\mathcal{G}_n \subset \{G_\alpha : \alpha \in \kappa\}$ such that $F_n \in \sigma(\mathcal{G}_n)$. Clearly, $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$ is countable and $\mathcal{F} \subset \sigma(\mathcal{G})$. By Lemma 2.1, $\sigma(\mathcal{G})$ contains an uncountable family $\mathcal{R}$ with pairwise disjoint elements.

By Lemma 2.2(ii), all elements of $\mathcal{R}$ are unions of sets of the form $A_I = \bigcap_{i\in I} G_{\alpha_i} \cap \bigcap_{j\in\mathbb{N}\setminus I} G^c_{\alpha_j}$, where $G_{\alpha_0}, G_{\alpha_1}, \ldots$ are all elements of $\mathcal{G}$ and $I \subset \mathbb{N}$. Note that if $I$ is infinite then $A_I$ is empty since each of its elements has to be infinite and so cannot be in $Z$. Thus $|\{A_I : I \in \mathbb{N}\}|$ is not greater than $|\mathbb{N}^{<\aleph_0}| = \aleph_0$. It is not possible to write each element of the uncountable $\mathcal{R}$ as a union of some subfamily of $\{A_I : I \in \mathbb{N}\}$ because the elements of $\mathcal{R}$ are pairwise disjoint. This contradiction means that $\mathcal{C}$ does not contain any infinite $\sigma$-independent family.

Assume that $\kappa$ is uncountable. Suppose $A$ is an atom of $\mathcal{C}$. By Lemma 2.2, $A$ equals $\bigcap_{i\in I} G_{\alpha_i} \cap \bigcap_{j\in\mathbb{N}\setminus I} G^c_{\alpha_j}$ for some $\alpha_0, \alpha_1, \ldots \in \kappa$ and $I \subset \mathbb{N}$. There exists $\beta \in \kappa$ that is not in $\{\alpha_0, \alpha_1, \ldots\}$ because $\kappa$ is uncountable. Note that $G_\beta$ and $A$ are not disjoint since $\{\alpha_i : i \in I\} \cup \{\beta\}$ is in both of these sets. The element $\{\alpha_i : i \in I\}$ is in $A$ but not in $G_\beta$, so $G_\beta$ does not contain $A$. Hence, $A$ is not an atom of $\mathcal{C}$, and $\mathcal{C}$ is an atomless $\sigma$-field. ■

If $\kappa = \omega_1$ then, as in Example 2, the family $\{G_{t_\alpha} \setminus \bigcup_{\beta<\alpha} G_{t_\beta}\}_{\alpha\in\omega_1}$ is an uncountable partition of $Z$. Note that this example works without the Continuum Hypothesis. We do not know if such a partition exists for $\kappa > \omega_1$.

**3. The counter-example.** For an uncountable cardinal $\kappa$ let us define

$$X = \{f \in 2^\kappa : \mathrm{supp}(f) < \aleph_0\};$$

supp$(f)$ stands for $\{\alpha \in \kappa : f(\alpha) \neq 0\}$, the *support* of $f$. For $\alpha \in \kappa$ define $G_\alpha = \{f \in X : f(\alpha) = 1\}$. Let $\mathcal{G} = \{G_\alpha : \alpha \in \kappa\}$ and $\mathcal{A} = \sigma(\mathcal{G})$. It is shown in [1] that $\mathcal{A}$ is an atomless $\sigma$-field that does not contain any infinite $\sigma$-independent family.

For $E \subset F \in [\kappa]^{\leq\aleph_0}$ define

$$\mathrm{A}(E, F) = \{f \in X : f[E] = \{1\} \wedge f[F \setminus E] = \{0\}\}.$$

If $G_{\alpha_0}, G_{\alpha_1}, \ldots \in \mathcal{G}$ and $I \subset \mathbb{N}$, the set $\bigcap_{i\in I} G_{\alpha_i} \cap \bigcap_{j\in\mathbb{N}\setminus I} G^c_{\alpha_j}$ can be represented as $\mathrm{A}(\{\alpha_i\}_{i\in I}, \{\alpha_j\}_{j\in\mathbb{N}})$. Using Lemma 2.2(ii) one can conclude that every set in $\mathcal{A}$ is a union of some $\mathrm{A}(\cdot,\cdot)$ sets. Note that, by the definition of $X$, if $E$ is infinite then $\mathrm{A}(E,\cdot)$ is empty.

THEOREM 3.1. *For every family $\mathcal{F} \subset \mathcal{A}$ which covers $X$ there exists a countable $\mathcal{F}_0 \subset \mathcal{F}$ that also covers $X$.*

*Proof.* Without losing generality it can be assumed that every set in $\mathcal{F}$ is an $\mathrm{A}(\cdot,\cdot)$ set. By induction we now construct countable set $B_s, D_s \subset \omega_1$ and finite $A_s, C_{s\frown 0}, C_{s\frown 1}, \ldots \subset \omega_1$ for every $s \in \mathbb{N}^{<\aleph_0}$.

Define $C_\emptyset = \emptyset$. There exists $B_\emptyset \in [\omega_1]^{\le\aleph_0}$ such that $\mathrm{A}(\emptyset, B_\emptyset) \in \mathcal{F}$ and the constant zero function is in $\mathrm{A}(\emptyset, B_\emptyset)$. Let $C_{\langle 0\rangle}, C_{\langle 1\rangle}, \ldots$ be all the finite subsets of $D_\emptyset = B_\emptyset$. Define $A_\emptyset = \emptyset$.

Assume that the following sets are defined for given $s \in \mathbb{N}^{<\aleph_0}$: finite $C_s$ and for every $k \prec s$, countable $B_k$, $D_k$, and finite $C_k$, $A_k$.

Let $a_s \in 2^{\omega_1}$ be such that $\mathrm{supp}(a_s) = \bigcup_{k\preceq s} C_k$. The set $\mathrm{supp}(a_s)$ is finite so $a_s \in X$. Thus, there exists a countable $B_s \subset \omega_1$ and finite $A_s \subset \omega_1$ such that $a_s \in \mathrm{A}(A_s, B_s) \in \mathcal{F}$. Define $D_s = B_s \setminus \bigcup_{k\prec s} D_k$. The set $D_s$ is countable. Let $\{C_{s\frown n}\}_{n\in\mathbb{N}}$ be all the finite subsets of $D_s$. The construction is finished.

To sum up, the constructed sets have the following properties for any $s \in \mathbb{N}^{<\aleph_0}$:

(i) $A_s \subset B_s \in [\omega_1]^{\le\aleph_0}$ are such that $\mathrm{A}(A_s, B_s) \in \mathcal{F}$, the set $D_s = B_s \setminus \bigcup_{k\prec s} D_k$ is countable and $\{C_{s\frown n}\}_{n\in\mathbb{N}}$ is the set of all finite subsets of $D_s$.
(ii) $D_k \cap D_s = \emptyset$ for $k \prec s$.
(iii) $A_s = (\bigcup_{k\preceq s} C_k) \cap B_s$.

Properties (i) and (ii) are immediate consequences of the construction. Each $a_s$ is in $\mathrm{A}(A_s, B_s)$, so $A_s = \mathrm{supp}(a_s) \cap B_s = (\bigcup_{k\preceq s} C_k) \cap B_s$. Hence, (iii) is true.

Note that $\mathcal{R} = \{\mathrm{A}(A_s, B_s) : s \in \mathbb{N}^{<\aleph_0}\}$ is countable, being indexed by finite sequences of natural numbers. All of its nonempty elements are in $\mathcal{F}$. It is sufficient to show that $\mathcal{R}$ covers $X$.

Suppose that $x \in X$ is not in $\bigcup\mathcal{R}$. We will now construct by induction two sequences, $i_0, i_1, \ldots \in \mathbb{N}$ and $\alpha_0, \alpha_1, \ldots \in \kappa$.

$x \notin \bigcup\mathcal{R}$ means that $x \notin \mathrm{A}(A_\emptyset, B_\emptyset)$. There exists $\alpha_0 \in D_\emptyset = B_\emptyset$ for which $x(\alpha_0) = 1$ because $A_\emptyset = \emptyset$. Let $i_0 \in \mathbb{N}$ be such that $\mathrm{supp}(x\restriction_{D_\emptyset}) = C_{\langle i_0\rangle}$.

Assume that $s = \langle i_0, i_1, \ldots, i_{l-1}\rangle$ is such that

$$\mathrm{supp}(x) \cap \bigcup_{k\prec s} D_k = \bigcup_{k\preceq s} C_k. \tag{$*$}$$

Since $x \notin \bigcup \mathcal{R}$, we have $x \notin \mathrm{A}(A_s, B_s)$, so $\mathrm{supp}(x) \cap B_s \neq A_s$. By (iii), $A_s = \bigcup_{k \preceq s} C_k \cap B_s$ and hence $\mathrm{supp}(x) \cap B_s \neq \bigcup_{k \preceq s} C_k \cap B_s$. Together with $(*)$, this yields

$$\mathrm{supp}(x) \cap D_s \neq \bigcup_{k \preceq s} C_k \cap D_s.$$

By $(*)$ and (ii), the right hand side above is an empty set. Therefore, $\mathrm{supp}(x) \cap D_s \neq \emptyset$. There exists $\alpha_l \in D_s$ such that $x(\alpha_l) = 1$. Let $i_l \in \mathbb{N}$ be such that $\mathrm{supp}(x) \cap D_s = C_{s \frown i_l}$. This ends the construction.

$\alpha_0, \alpha_1, \ldots \in \kappa$ are distinct because they are in $D_\emptyset, D_{\langle i_0 \rangle}, D_{\langle i_0, i_1 \rangle}, \ldots$ respectively and these sets are pairwise disjoint. We have $x(\alpha_n) = 1$ for all $n \in \mathbb{N}$, hence $\mathrm{supp}(x)$ is infinite, which contradicts $x \in X$. ∎

This implies that every partition contained in $\mathcal{A}$ has to be countable, which gives a negative answer to the question considered in this note. The theorem is equivalent to the statement that $X$ with the topology generated by the $\mathrm{A}(\cdot, \cdot)$ sets is a Lindelöf space. Andrzej Nowik has recently given a topological proof of the above fact for $\kappa = \omega_1$ using Fodor's lemma.

*REFERENCES*

[1] K. P. S. Bhaskara Rao and B. V. Rao, *Borel spaces*, Dissertationes Math. 190 (1981).
[2] K. P. S. Bhaskara Rao and R. M. Shortt, *Borel spaces 2*, ibid. 372 (1998).

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