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SEMI-PARALLEL CR SUBMANIFOLDS IN A COMPLEX SPACE FORM

BY

MAYUKO KON (Nagano City)

Abstract. We show that there is no proper CR submanifold with semi-flat normal connection and semi-parallel second fundamental form in a complex space form with nonzero constant holomorphic sectional curvature such that the dimension of the holomorphic tangent space is greater than 2.

1. Introduction. There are many results about real hypersurfaces immersed in a complex space form with additional conditions on the second fundamental form A. It is well known that there are no real hypersurfaces in a complex space form $M^n(c)$, $c \neq 0$, of constant holomorphic sectional curvature $4c$ with parallel second fundamental form. Maeda [M] proved that there exist no real hypersurfaces of a complex projective space $\mathbb{C}P^m$, $m \geq 3$, with semi-parallel second fundamental form.

On the other hand, Hamada [H] showed that there are no real hypersurfaces with recurrent second fundamental form in $\mathbb{C}P^m$, where recurrency of A means that $(\nabla_X A)Y = \alpha(X)AY$, α being a 1-form.

In this paper we consider the conditions of being semi-parallel or recurrent for the second fundamental form of CR submanifolds in a complex space form. The definition of a CR submanifold is given in Section 2.

By the equation of Codazzi, we easily see that there are no proper CR submanifolds in a complex space form $M^n(c)$, $c \neq 0$, with parallel second fundamental form.

In [K], we studied CR submanifolds of complex space forms with semi-parallel Ricci tensor. Moreover, we showed that there is no CR submanifold with semi-flat normal connection and with recurrent Ricci tensor in a complex space form of nonzero constant holomorphic sectional curvature, if the dimension of its holomorphic distribution is greater than 2.

The purpose of the present paper is to prove the following theorem.

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Theorem 1.1. *Let M be an n-dimensional proper CR submanifold of a complex space form $M^m(c)$, $c \neq 0$, with semi-flat normal connection. If the dimension of the holomorphic tangent space is greater than 2, then the second fundamental form A is not semi-parallel.*

2. Preliminaries. Let $M^m(c)$ denote the complex space form of complex dimension m (real dimension $2m$) with constant holomorphic sectional curvature $4c$. We denote by J the almost complex structure of $M^m(c)$. The Hermitian metric of $M^m(c)$ is denoted by G.

Let M be a real n-dimensional Riemannian manifold isometrically immersed in $M^m(c)$. We denote by g the Riemannian metric induced on M from G, and by p the codimension of M, that is, $p = 2m - n$.

We denote by $T_x(M)$ and $T_x(M)^\perp$ the tangent space and the normal space of M, respectively.

Definition. A submanifold M of a Kählerian manifold $\tilde{M}$ is called a *CR submanifold* if there exists a differentiable distribution $H : x \to H_x \subset T_x(M)$ on M satisfying the following conditions:

(i) H is holomorphic, i.e., $JH_x = H_x$ for each $x \in M$, and
(ii) the complementary orthogonal distribution $H^\perp : x \to H_x^\perp \subset T_x(M)$ is *anti-invariant*, i.e. $JH_x^\perp \subset T_x(M)^\perp$ for each $x \in M$.

We call H_x the *holomorphic tangent space* of M.

In the following, we put $\dim H_x = h$, $\dim H_x^\perp = q$ and $\operatorname{codim} M = p$. If $q = 0$ (resp. $h = 0$) for any $x \in M$, then the CR submanifold M is a holomorphic submanifold (resp. anti-invariant submanifold or totally real submanifold) of $\tilde{M}$. If a CR submanifold satisfies $p > 0$ and $q > 0$, then it is said to be *proper*.

We denote by $\tilde{\nabla}$ the covariant differentiation in $M^m(c)$, and by ∇ the one in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are respectively

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \qquad \tilde{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^\perp$ of M. We call both A and B the *second fundamental forms* of M. They are related by $G(B(X, Y), V) = g(A_V X, Y)$. The second fundamental forms A and B are symmetric.

The covariant derivative $(\nabla_X A)_V Y$ of A is defined to be

$$(\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

If $(\nabla_X A)_V Y = 0$ for any vector fields X and Y tangent to M, then the second fundamental form of M is said to be *parallel in the direction of the normal vector* V. If the second fundamental form is parallel in every direction, it is said to be *parallel*.

A nonzero tensor field K of type (r, s) on M is said to be *recurrent* if there exists a 1-form α such that $\nabla K = K \otimes \alpha$. The second fundamental form A is recurrent if A is nonzero and $(\nabla_X A)_V Y = \alpha(X) A_V Y$ for any vector fields X, Y tangent to M and any vector field V normal to M.

In what follows, we assume that M is a CR submanifold of $M^m(c)$. The tangent space $T_x(M)$ of M is decomposed as $T_x(M) = H_x + H_x^{\perp}$ at each point x of M, where $H_x^{\perp}$ denotes the orthogonal complement of H_x in $T_x(M)$. Similarly, we see that $T_x(M)^{\perp} = JH_x^{\perp} + N_x$, where N_x is the orthogonal complement of $JH_x^{\perp}$ in $T_x(M)^{\perp}$.

For any vector field X tangent to M, we put

$$JX = PX + FX,$$

where PX is the tangential part of JX and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on the tangent bundle $T(M)$. We notice that $P^3 + P = 0$.

For any vector field V normal to M, we put

$$JV = tV + fV,$$

where tV is the tangential part of JV and fV the normal part of JV. Then we see that $FP = 0$, $fF = 0$, $tf = 0$ and $Pt = 0$.

We define the covariant derivatives of P, F, t and f by $(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y$, $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$, $(\nabla_X t)V = \nabla_X(tV) - tD_X V$ and $(\nabla_X f)V = D_X(fV) - fD_X V$ respectively. We then have

$$(\nabla_X P)Y = A_{FY}X + tB(X,Y), \quad (\nabla_X F)Y = -B(X, PY) + fB(X,Y),$$
$$(\nabla_X t)V = -PA_V X + A_{fV}X, \quad (\nabla_X f)V = -FA_V X - B(X, tV).$$

For any vector fields X and Y in $H_x^{\perp} = tT(M)^{\perp}$ we obtain

$$A_{FX}Y = A_{FY}X.$$

We denote by R the Riemannian curvature tensor field of M . Then the *equation of Gauss* is

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX - g(PX,Z)PY - 2g(PX,Y)PZ\} + A_{B(Y,Z)}X - A_{B(X,Z)}Y,$$

for any X, Y and Z tangent to M.

The *equation of Codazzi* of M is

$$g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z)$$
$$= c\{g(Y, PZ)g(X, tV) - g(X, PZ)g(Y, tV) - 2g(X, PY)g(Z, tV)\}.$$

We define the curvature tensor $R^\perp$ of the normal bundle of M by

$$R^\perp(X, Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V.$$

Then we have the *equation of Ricci*

$$G(R^\perp(X, Y)V, U) + g([A_U, A_V]X, Y)$$
$$= c\{g(Y, tV)g(X, tU) - g(X, tV)g(Y, tU) - 2g(X, PY)g(V, fU)\}.$$

If $R^\perp$ vanishes identically, the normal connection of M is said to be *flat*. If $R^\perp(X, Y)V = 2cg(X, PY)fV$, then the normal connection of M is said to be *semi-flat* (see [YK2]).

We put

$$(R(X, Y)A)_V Z = R(X, Y)A_V Z - A_{R^\perp(X,Y)V}Z - A_V R(X, Y)Z.$$

If $(R(X, Y)A)_V = 0$ for any X, Y and Z tangent to M and any V normal to M, then the second fundamental form A is said to be *semi-parallel*. This condition is weaker than $\nabla A = 0$. We call M a *semi-parallel CR submanifold* if its second fundamental form A is semi-parallel.

Remark 2.1. Let S^{m+1} be a $(2m+1)$-dimensional unit sphere. For any point $z \in S^{2m+1}$ we put $\xi = JZ$, where J denotes the almost complex structure of $\mathbb{C}^{m+1}$. We consider the orthogonal projection $\pi' : T_z(\mathbb{C}^{m+1}) \to T_z(S^{2m+1})$. Putting $\phi = \pi' \cdot J$, we have a contact metric structure (ϕ, ξ, η, G) on S^{2m+1}, where η is a 1-form dual to ξ, and G the standard metric tensor field on S^{2m+1} which satisfies $G(\phi X, \phi Y) = G(X, Y) - \eta(X)\eta(Y)$.

Let M be an n-dimensional submanifold in $\mathbb{C}P^m$. Let N be an $(n+1)$-dimensional submanifold immersed in a $(2m+1)$-dimensional unit sphere S^{2m+1} such that the following diagram is commutative:

$$\begin{array}{ccc} N & \xrightarrow{i'} & S^{2m+1} \\ \downarrow & & \downarrow \pi \\ M & \xrightarrow{i} & \mathbb{C}P^m \end{array}$$

where the immersion i' is a diffeomorphism on the fibres and π is the standard fibration.

We denote the horizontal lift with respect to the connection η by *. Then the curvature tensor $K^\perp$ of the normal bundle of N satisfies

$$G(K^\perp(X^*, Y^*)V^*, U^*) = [g(R^\perp(X, Y)V, U) - 2g(X, PY)g(fV, U)]^*,$$
$$G(K^\perp(X^*, \xi)V^*, U^*) = g((\nabla_X f)V, U)^*,$$

for any vectors X and Y tangent to M and any vectors V and U normal to M. Therefore, the normal connection of N in S^{2m+1} is flat if and only if the normal connection of M in $\mathbb{C}P^m$ is semi-flat and $\nabla f = 0$ (see [YK2, pp. 223–224].

REMARK 2.2. Let M be a complex n-dimensional ($n \geq 2$) holomorphic submanifold of a complex space form $M^m(c)$. If the normal connection of M is semi-flat, then M is either totally geodesic or an Einstein Kählerian hypersurface of $M^m(c)$ with scalar curvature n^2c. The latter case occurs only when $c > 0$ (see Ishihara [I]). Then the second fundamental form of M is parallel.

Let M be an n-dimensional anti-invariant submanifold of a complex space form $M^m(c)$. If the normal connection of M is semi-flat, then the normal connection of M is flat since $P = 0$. There exists an anti-invariant submanifold with flat normal connection and parallel second fundamental form. For example, $\pi(S^1(r_1) \times \cdots \times S^1(r_{n+1}))$, $\sum r_i = 1$, is an anti-invariant submanifold of $\mathbb{C}P^m$ with flat normal connection and parallel second fundamental form (cf. Yano–Kon [YK2, p. 237, Theorem 3.17].

3. Semi-parallel second fundamental form. In this section, we prove our main theorem. First we establish some lemmas.

LEMMA 3.1. *Let M be an n-dimensional proper CR submanifold of a complex space form $M^m(c)$, $c \neq 0$, with semi-flat normal connection. If the second fundamental form A is semi-parallel, then*

$$A_{fV}X = 0 \qquad \text{for } X \in T_x(M),$$
$$g(A_V X, Y) = 0 \quad \text{for } X \in H_x,\ Y \in H_x^{\perp}.$$

Moreover, if the dimension of the holomorphic tangent space is $h > 2$, then

$$PA_V = A_V P,$$
$$g(A_V X, Y) = -\frac{1}{h}\operatorname{tr}(A_V P^2) g(X, Y) \qquad \text{for } X, Y \in H_x,$$

where tr *denotes the trace of an operator.*

Proof. Since $g((R(X,Y)A)_V Z, W) = 0$ for any vectors $X, Y, Z, W \in T_x(M)$, we have

$$R(X,Y)A_V Z = A_{R^{\perp}(X,Y)V} Z + A_V R(X,Y)Z \tag{3.1}$$
$$= 2cg(X, PY)A_{fV}Z + A_V R(X,Y)Z.$$

Thus

$$\operatorname{tr} R(X,Y)A_V A_{fV} = 2cg(X, PY)\operatorname{tr} A_{fV}^2 + \operatorname{tr} R(X,Y)A_{fV}A_V.$$

By the equation of Ricci, we have $A_{fV}A_V = A_V A_{fV}$. Thus we obtain $\operatorname{tr} A_{fV}^2 = 0$, which proves $A_{fV}X = 0$ for $X \in T_x(M)$.

The equation of Gauss and (3.1) yield

$$\begin{aligned} &c\big(g(Y,A_VZ)X - g(X,A_VZ)Y + g(PY,A_VZ)PX - g(PX,A_VZ)PY \\ &\qquad - 2g(PX,Y)PA_VZ\big) + A_{B(Y,A_VZ)}X - A_{B(X,A_VZ)}Y \\ &\quad = c\big(g(Y,Z)A_VX - g(X,Z)A_VY + g(PY,Z)A_VPX - g(PX,Z)A_VPY \\ &\qquad - 2g(PX,Y)A_VPZ\big) + A_VA_{B(Y,Z)}X - A_VA_{B(X,Z)}Y. \end{aligned}$$

We take an orthonormal basis $\{e_1,\dots,e_h, tv_1 := e_{h+1},\dots,tv_q := e_n\}$ of $T_x(M)$, where $\{e_1,\dots,e_h\}$ is an orthonormal basis of H_x and $\{v_1,\dots,v_q\}$ is an orthonormal basis of $JH_x^\perp$. Then we have

$$\begin{aligned} &c\sum_i\big(g(Pe_i,A_VX)g(e_i,Y) - g(e_i,A_VX)g(Pe_i,Y) + g(P^2e_i,A_VX)g(Pe_i,Y) \\ &\qquad - g(Pe_i,A_VX)(P^2e_i,Y) - 2g(Pe_i,Pe_i)g(PA_VX,Y)\big) \\ &\qquad + \sum_i g(A_{B(Pe_i,A_VX)}e_i,Y) - \sum_i g(A_{B(e_i,A_VX)}Pe_i,Y) \\ &\quad = c\sum_i\big(g(Pe_i,X)g(A_Ve_i,Y) - g(e_i,X)g(A_VPe_i,Y) \\ &\qquad + g(P^2e_i,X)g(A_VPe_i,Y) \\ &\qquad - g(Pe_i,X)g(A_VP^2e_i,Y) - 2g(Pe_i,Pe_i)g(A_VPX,Y)\big) \\ &\qquad + \sum_i g(A_VA_{B(Pe_i,X)}e_i,Y) - \sum_i g(A_VA_{B(e_i,X)}Pe_i,Y). \end{aligned}$$

By a straightforward computation,

$$\begin{aligned} (3.2)\qquad &(h+2)cg(A_VX,PY) + (h+2)cg(A_VPX,Y) \\ &\quad - \sum_a g(A_aPA_aA_VX,Y) + \sum_a g(A_VA_aPA_aX,Y) = 0, \end{aligned}$$

where A_a is the second fundamental form in the direction of v_a. Similarly, putting $Y = e_i$, $Z = Pe_i$ and taking the inner product with Y and summing, we obtain

$$\begin{aligned} (3.3)\qquad &c\big(g(PA_VX,Y) - \operatorname{tr}(P^2A_V)g(PX,Y) + g(P^2A_VPX,Y) \\ &\qquad - 2g(PA_VP^2X,Y) - (h+2)g(A_VPX,Y)\big) \\ &\qquad + g(A_aPA_VA_aX,Y) - g(A_VA_aPA_aX,Y) = 0. \end{aligned}$$

Since the normal connection of M is semi-flat, the equation of Ricci gives

$$A_aA_bX = A_bA_aX$$

for any $X \in H_x$. Hence, by (3.2) and (3.3),

$$\begin{aligned} -(h+1)cg(PA_VX,Y) - c\,\mathrm{tr}(P^2A_V)g(PX,Y) \\ + cg(P^2A_VPX,Y) - 2cg(PA_VP^2X,Y) = 0. \end{aligned} \tag{3.4}$$

When $X \in H_x^\perp$ and $Y \in H_x$, from (3.4),

$$g(PA_VX,Y) = -g(A_VX,PY) = 0. \tag{3.5}$$

So we have the second equation.

Next we consider the case that $X, Y \in H_x$. Since $PX, PY \in H_x$, using (3.2), we obtain

$$-(h-1)cg(PA_VX,Y) - cg(A_VPX,Y) - c\,\mathrm{tr}(P^2A_V)g(PX,Y) = 0,$$
$$-(h-1)cg(A_VPX,Y) + cg(A_VX,PY) - c\,\mathrm{tr}(P^2A_V)g(PX,Y) = 0.$$

From these equations and the assumption that $h > 2$, we get

$$g(PA_VX,Y) - g(A_VY,PX) = 0.$$

From this and (3.5), we have the third equation.

Using this, we finally obtain

$$g(A_VX,Y) = -\frac{1}{h}\,\mathrm{tr}(A_VP^2)g(X,Y) \quad \text{for } X,Y \in H_x. \ ■$$

By a method similar to that of Lemma 2.2 of [YK1], we obtain

LEMMA 3.2. *Let M be a CR submanifold of $M^m(c)$ with semi-flat normal connection. If $A_{fV} = 0$ and $PA_V = A_VP$ for any vector field V normal to M, then*

$$\begin{aligned} g(A_UX,A_VY) = cg(X,Y)g(tU,tV) - cg(FX,U)g(FY,V) \\ - \sum_i g(A_UtV,e_i)g(A_{Fe_i}X,Y). \end{aligned}$$

Using these lemmas, we prove our main theorem.

Proof of Theorem 1.1. Let M satisfy $(R(X,Y)A)_VZ = 0$. By (3.1),

$$0 = \sum_{i,j} g(R(e_i,e_j)A_ae_i,A_ae_j) - \sum_{i,j} g(A_aR(e_i,e_j)e_i,A_ae_j).$$

By a straightforward computation using the equation of Gauss,

$$\begin{aligned} 0 &= \sum_{a,i,j} g(R(e_i,e_j)A_ae_i,A_ae_j) - \sum_{a,i,j} g(A_aR(e_i,e_j)e_i,A_ae_j) \\ &= nc\sum_a \mathrm{tr}\,A_a^2 - c\sum_a(\mathrm{tr}\,A_a)^2 + \sum_{a,b}\mathrm{tr}(A_aA_b)^2 - \sum_{a,b}\mathrm{tr}\,A_a^2A_b^2 \\ &\quad - \sum_{a,b}(\mathrm{tr}\,A_aA_b)^2 + \sum_{a,b}(\mathrm{tr}\,A_a^2A_b)(\mathrm{tr}\,A_b). \end{aligned} \tag{3.6}$$

Since the normal connection is semi-flat, the Ricci equation implies

$$\sum_{a,b} \operatorname{tr} A_a^2 A_b^2 - \sum_{a,b} \operatorname{tr}(A_a A_b)^2 = \frac{1}{2}\sum_{a,b} \|[A_a, A_b]\|^2 = c^2 q(q-1).$$

By Lemma 3.2, we have

$$\begin{aligned}\sum_{a,b}(\operatorname{tr} A_a A_b)^2 &= \sum_{a,b,i}(\operatorname{tr} A_a A_b) cg(e_i, e_i) g(tv_a, tv_b) - \sum_{a,b,i} cg(Fe_i, v_a)(Fe_i, v_b) \\ &\quad - \sum_{a,b,i,j} g(A_a tv_b, e_j)(A_{Fe_j} e_i, e_i) \\ &= (n-1)c \sum_a \operatorname{tr} A_a^2 + \sum_{a,b,c}(\operatorname{tr} A_a A_b)(\operatorname{tr} A_c) g(A_a tv_b, tv_c).\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum_{a,b}(\operatorname{tr} A_a)(\operatorname{tr} A_b^2 A_a) \\ &\quad = \sum_{a,b,i}(\operatorname{tr} A_a) cg(e_i, A_b e_i) g(tv_a, tv_b) - \sum_{a,b,i} cg(Fe_i, v_a) g(FA_b e_i, v_b) \\ &\qquad - \sum_{a,b,i,j} g(A_a tv_b, e_j) g(A_{Fe_j} e_i, A_b e_i) \\ &\quad = c \sum_a (\operatorname{tr} A_a)^2 - c \sum_{a,b}(\operatorname{tr} A_a) g(A_b tv_a, tv_b) \\ &\qquad + \sum_{a,b,c}(\operatorname{tr} A_a)(\operatorname{tr} A_b A_c) g(A_a tv_b, tv_c).\end{aligned}$$

From Lemma 3.2, we have

$$-\sum_{a,b}(\operatorname{tr} A_a) g(A_b tv_a, tv_b) = c(n-1)q - \sum_a \operatorname{tr} A_a^2.$$

Putting these equations into (3.6), we obtain

$$q(n-q) = 0.$$

This is a contradiction. ∎

4. Recurrent second fundamental form. In this section, we study the case where the second fundamental form is recurrent.

LEMMA 4.1. *Let M be an n-dimensional proper CR submanifold of a complex space form $M^m(c)$, $c \neq 0$ with semi-flat normal connection. If the second fundamental form A is recurrent, then A is semi-parallel.*

Proof. By the definition of "recurrent", the second fundamental form A of M satisfies $(\nabla_X A)_V Y = \alpha(X) A_V Y$ for any $X \in T_x M$ and $V \in T_x M^\perp$.

Then we have

$$\begin{aligned}(\nabla_X\nabla_Y A)_V Z &= (\nabla_X\alpha)(Y)A_V Z + \alpha(Y)(\nabla_X A)_V Z + \alpha(\nabla_X Y)A_V Z\\ &= (\nabla_X\alpha)(Y)A_V Z + \alpha(Y)\alpha(X)A_V Z + \alpha(\nabla_X Y)A_V Z,\\ (\nabla_Y\nabla_X A)_V Z &= (\nabla_Y\alpha)(X)A_V Z + \alpha(X)\alpha(Y)A_V Z + \alpha(\nabla_Y X)A_V Z,\\ (\nabla_{[X,Y]}A)_V Z &= \alpha([X,Y])A_V X.\end{aligned}$$

So we obtain

$$(R(X,Y)A)_V Z = (\nabla_X\alpha)(Y)A_V Z - (\nabla_Y\alpha)(X)A_V Z.$$

On the other hand, taking an eigenvector Z of A_V, that is, $A_V Z = \beta Z$ for some β, we have

$$\begin{aligned}&g((R(X,Y)A)_V Z, Z)\\ &\qquad= g(R(X,Y)A_V Z, Z) - g(A_{R^\perp(X,Y)V}Z, Z) - g(A_V R(X,Y)Z, Z)\\ &\qquad= -g(A_{R^\perp(X,Y)V}Z, Z) = -2cg(A_{fV}Z, Z)g(X, PY).\end{aligned}$$

From these equations, we obtain

$$((\nabla_X\alpha)(Y) - (\nabla_Y\alpha)(X))g(A_V Z, Z) = -2cg(X, PY)g(A_{fV}Z, Z).$$

Hence

$$((\nabla_X\alpha)(Y) - (\nabla_Y\alpha)(X))g(A_{FW}Z, Z) = 0.$$

So we see that either $(\nabla_X\alpha)(Y) - (\nabla_Y\alpha)(X) = 0$ for any X and Y tangent to M or $g(A_{FW}Z, Z) = 0$ for any eigenvector Z of A_{FW}. We remark that the latter condition implies that $A_{FW} = 0$.

By the equation of Codazzi,

$$\begin{aligned}&\alpha(X)g(A_V Y, Z) - \alpha(Y)g(A_V X, Z)\\ &\qquad= cg(Y, PZ)g(X, tV) - cg(X, PZ)g(Y, tV) - 2cg(X, PY)g(Z, tV)\end{aligned}$$

for any X, Y, Z tangent to M and V normal to M. If $A_{FW} = 0$, putting $X = PY$, $Z = W$ and $V = FW$, we have

$$g(PY, PY)g(W, W) = 0$$

for any $Y, W \in T_xM$. This is a contradiction. So $(\nabla_X\alpha)(Y) - (\nabla_Y\alpha)(X) = 0$. Hence $(R(X,Y)A)_V = 0$. ∎

Lemma 4.1 gives the relation between being recurrent and being semi-parallel for the second fundamental form of proper CR submanifolds in $M^m(c)$. Thus, using Theorem 3.3, we have

Theorem 4.2. *Let M be an n-dimensional proper CR submanifold of a complex space form $M^m(c)$, $c \neq 0$, $h > 2$, with semi-flat normal connection. Then the second fundamental form A is not recurrent.*

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Mayuko Kon
Faculty of Education
Shinshu University
6-Ro, Nishinagano, Nagano City 380-8544, Japan
E-mail: mayuko_k@shinshu-u.ac.jp

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