COLLOQUIUM MATHEMATICUM

VOL. 124 2011 NO. 2

ON THE STABLE EQUIVALENCE PROBLEM FOR $k[x,y]$

BY

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Abstract. L. Makar-Limanov, P. van Rossum, V. Shpilrain and J.-T. Yu solved the stable equivalence problem for the polynomial ring $k[x,y]$ when k is a field of characteristic 0. In this note we give an affirmative solution for an arbitrary field k.

1. Introduction. Let k be an arbitrary commutative field and $k^{[n]}$ be the polynomial ring in n variables over k. We say that two polynomials $f, g \in k^{[n]}$ are *equivalent* (resp. *stably equivalent*) if there exists a k-automorphism φ of $k^{[n]}$ such that $\varphi(f) = g$ (resp. f, g are equivalent in $k^{[n+m]}$ for some $m > 0$). The following problem was stated by Shpilrain and Yu [11]:

Stable equivalence problem. Is it true that two stably equivalent polynomials in $k^{[n]}$ are equivalent?

An affirmative answer is known for a generic polynomial in $k^{[n]}$ of degree $> n$ when k is algebraically closed of characteristic 0 (see [11, 5]). In general, the problem remains open for $n \geq 3$. An affirmative solution in characteristic zero for $k^{[2]}$ was given by Makar-Limanov, van Rossum, Shpilrain and Yu [9]. The aim of this note is to give a solution for $k^{[2]}$ and an arbitrary field k. We prove the following:

Theorem 1. *Let $f_1, f_2 \in k^{[2]} \setminus k$ be two stably equivalent polynomials, and φ be a k-automorphism of $k^{[2+n]}$ such that $\varphi(f_1) = f_2$. If there exist coordinates t_1, t_2 of $k^{[2]}$ such that $f_i \in k[t_i]$, $i = 1, 2$, then $\varphi(k[t_1]) = k[t_2]$; otherwise $\varphi(k^{[2]}) = k^{[2]}$. In particular, f_1 and f_2 are equivalent.*

2. Proof. We start by summarizing some properties of exponential maps (see [3] for more details) and next prove some analogue of Rentschler's theorem. Then for the proof of Theorem 1 we introduce an analogue of the Makar-Limanov invariant, which will be an invariant of equivalent polynomials.

Let A be an integral finitely generated k-algebra and $\varphi : A \to A^{[1]}$ be a k-algebra homomorphism. We write $\varphi = \varphi_t : A \to A[t]$ to emphasize

2010 *Mathematics Subject Classification*: 13A50, 14R10, 14R20.
Key words and phrases: polynomial automorphisms, stable equivalence, locally nilpotent derivation, Makar-Limanov invariant.

DOI: 10.4064/cm124-2-9

a variable t. We say that φ is an *exponential map on* A if it satisfies the following two conditions:

(i) $\varphi_0(a) = a$ for all $a \in A$, where φ_0 is evaluation at $t = 0$,
(ii) $\varphi_{s+t} = \varphi_s\varphi_t$, where φ_s is extended by $\varphi_s(t) = t$ to a homomorphism $\varphi : A[t] \to A[s, t]$.

If k is algebraically closed, then there exists a one-to-one correspondence between exponential maps on A and algebraic k_+-actions on an affine variety with the coordinate ring A. Furthermore, in characteristic zero exponential maps on A correspond to locally nilpotent derivations D on A, as follows:

$$\varphi(a) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n(a)t^n.$$

The ring of φ-invariants is defined to be

$$A^{\varphi} = \{a \in A \mid \varphi(a) = a\},$$

and the *Makar-Limanov invariant of* A is

$$\mathrm{ML}(A) = \bigcap_{\varphi \in \mathrm{Exp}(A)} A^{\varphi}, \tag{2.1}$$

where $\mathrm{Exp}(A)$ is the set of all exponential maps on A.

We will need the following elementary properties of A^{φ}.

Lemma 2 ([3, Lemmas 2.1 and 2.2]).

(i) *A^{φ} is factorially closed in A (i.e., if $ab \in A^{\varphi}$ for $a, b \in A \setminus \{0\}$, then $a, b \in A^{\varphi}$). In particular, if A is a UFD, then so is A^{φ}.*
(ii) *A^{φ} is algebraically closed in A.*
(iii) *If φ is nontrivial, then there exist $c \in A^{\varphi} \setminus \{0\}$ and $x \in A$ transcendental over A^{φ} such that $A \subset A^{\varphi}[c^{-1}][x]$.*

Now we prove the following analogue of Rentschler's theorem:

Theorem 3. *Let $A = k^{[2]}$, φ be an exponential map on $A^{[n]}$, and $A^{\varphi} = \{a \in A \mid \varphi(a) = a\}$. If $A^{\varphi} \neq A$, then either $A^{\varphi} = k$ or $A^{\varphi} = k[t]$, where t is a coordinate in A.*

To prove this fact we follow the idea of the proof of Rentschler's theorem given in [4, Th. 1.2]. We will need the following lemmas.

Lemma 4 ([10, Th. 2.4.2]). *Let A be a finitely generated integral k-algebra. Suppose that $t \in A$ satisfies the following conditions:*

(i) *$A_S = k(t)^{[1]}$, where A_S is the localization of A at $S = k[t] \setminus \{0\}$;*
(ii) *$k(t) \cap A = k[t]$;*
(iii) *A is geometrically factorial over k (i.e., $A \otimes_k K$ is a UFD for any algebraic field extension K/k).*

Then $A = k[t]^{[1]}$.

LEMMA 5. *Let $L \subset K$ be a finitely generated separable field extension with* $\operatorname{trdeg}_L K = 1$. *Then there exist infinitely many discrete valuation rings (R, M) of K/L such that the residue field R/M is finite separable over L.*

Proof. It is well-known that for any DVR (R, M) of K/L the residue field R/M is finite over L (i.e., if $x \in R$ is transcendental over L and A is the integral closure of $L[x]$ in K, then $A \subset R$ is a finitely generated L-algebra of dimension 1 and R is equal to the localization of A at the maximal ideal $M \cap A$, which implies that $R/M = A/(M \cap A)$ is finite over L). If L is finite, hence perfect, then R/M is always separable over L. Suppose that L is infinite. Let $x \in K$ be a separable transcendental element over L, and $u \in K$ be a primitive element of K over $L(x)$ with the minimal polynomial f over $L(x)$. Since f has no multiple roots, $gf + hf' = 1$ for some $g, h \in L(x)^{[1]}$. Let $A = L[x]$ and B be the integral closure of A in K. Since B is a finitely generated L-algebra, there exists $v \in A$ such that $A_v[u] = B_v$ for the localizations at v, and all coefficients of f, g, h are in A_v. Since L is infinite, there exist infinitely many maximal ideals M in A such that $A/M = L$ and $v \notin M$. For each such M, let M' be a maximal ideal in B_v lying over M_v, which exists since the extension $A_v \subset B_v$ is integral. Then the field extension $L = A_v/M_v \subset B_v/M' = L[\bar{u}]$ is separable, since $\bar{g}\bar{f} + \bar{h}\bar{f}' = 1$, where the bar denotes reduction modulo M_v. Clearly, the localization of B_v at M' is a DVR of K/L with the residue field $L[\bar{u}]$. This completes the proof. ∎

The following lemma generalizes [1, (2.9)].

LEMMA 6. *Let $k \subset K$ be a separable field extension. If $A \subset K^{[1]}$ is a finitely generated normal k-algebra of dimension* 1 *such that $A \not\subset K$, then $A = k'^{[1]}$, where k' is the algebraic closure of k in A.*

Proof. We will reduce the above fact to the case when K/k is finite and separable, which was proved in [1]. Since A is finitely generated over k, we may assume the same about K. Let $t_1, \ldots, t_n$ be a separable transcendence basis of K/k and $L = k(t_1, \ldots, t_{n-1})$. Let $K[x] = K^{[1]}$ and $A = k[b_1(x), \ldots, b_s(x)]$, where $b_i \in K[x]$. If $u \in K \setminus 0$, then $u \in R \setminus M$ for all but a finite number of DVRs (R, M) of K/L (see [7, II, Lem. 6.1]). Hence by Lemma 5 there exists a DVR (R, M) of K/L such that all nonzero coefficients of $b_1, \ldots, b_s$ are in $R \setminus M$ and the residue field R/M is finite separable over L. Then $A \subset R[x]$ and the canonical homomorphism $R[x] \to (R/M)[x]$ restricted to A yields an embedding $A \to (R/M)[x]$, because $\dim A = 1$ and the image of A contains an element which depends on x. Since the extension $k \subset R/M$ is separable of transcendence degree $n - 1$, the lemma follows by induction. ∎

Now we are in a position to prove Theorem 3. Let $B = (A^{[n]})^{\varphi}$. Since $A^{\varphi} = B \cap A$, A^{φ} is factorially and algebraically closed in A by Lemma 2. It follows that A^{φ} is a UFD and $\operatorname{trdeg}_k A^{\varphi} \leq 1$. This implies that either

$A^\varphi = k$ or $A = k[t]$ (see [1, Th. 4.1] for the latter case). To show that t is a coordinate in A we apply Lemma 4. Obviously assumptions (ii) and (iii) of that lemma are satisfied. For (i) we will show that the extension $k(t) \subset \mathrm{Frac}(B)$ is separable. Assuming this fact we can complete the proof as follows. By Lemma 2(iii) there exists $x \in A^{[n]}$ transcendental over $\mathrm{Frac}(B)$ such that $A_S \subset \mathrm{Frac}(B)[x]$, where $S = k[t] \setminus 0$. Obviously, $A_S \not\subset \mathrm{Frac}(B)$, $\mathrm{trdeg}_{k(t)} A_S = 1$, and $k(t)$ is algebraically closed in A_S. This implies by Lemma 6 that $A_S = k(t)^{[1]}$. Therefore t is a coordinate in A.

It remains to show that the extension $k(t) \subset \mathrm{Frac}(B)$ is separable. Let $p = \mathrm{char}\, k > 0$ and $k[x_1, x_2] = A$. We will show that the partial derivations $\partial/\partial x_1, \partial/\partial x_2$ do not vanish simultaneously at t, which implies by [8, VIII, Cor. 5.6] that $k(t) \subset \mathrm{Frac}(A)$ is separable. Then $k(t) \subset \mathrm{Frac}(A^{[n]})$ is separable, too, and hence so is $k(t) \subset \mathrm{Frac}(B)$.

Suppose that $\partial t/\partial x_1 = \partial t/\partial x_2 = 0$. One can extend φ to an exponential $\overline{\varphi}$ on $\overline{k}[x_1, x_2]^{[n]}$, where $\overline{k}$ is the algebraic closure of k. Then the ring of $\overline{\varphi}$-invariants in $\overline{k}[x_1, x_2]$ is equal to $\overline{k}[t]$. Since $\partial t/\partial x_1 = \partial t/\partial x_2 = 0$, we have $t = s^p$ for some $s \in \overline{k}[x_1, x_2] \setminus \overline{k}[t]$, which contradicts the fact that $\overline{k}[t]$ is algebraically closed in $\overline{k}[x_1, x_2]$. This completes the proof of Theorem 3.

For the proof of Theorem 1 we introduce an analogue of the Makar-Limanov invariant, which is an invariant of equivalent polynomials. Given a k-algebra A and $a \in A$, let

$$\mathrm{ML}(A, a) = \bigcap_{\substack{\varphi \in \mathrm{Exp}(A) \\ \varphi(a) = a}} A^\varphi. \tag{2.2}$$

One easily checks that if $\psi : A \to B$ is a k-algebra isomorphism, then $\psi(\mathrm{ML}(A, a)) = \mathrm{ML}(B, \psi(a))$. Note that always

$$\mathrm{ML}(A^{[n]}, a) \subset \mathrm{ML}(A, a), \tag{2.3}$$

which is a consequence of the following two facts:

(i) there exist exponential maps φ_i on $A^{[n]} = A[t_1, \ldots, t_n]$ such that $\varphi_i(t_i) = t_i + t$ and $\varphi_i(f) = f$ for all $f \in A[t_1, \ldots, \widehat{t_i}, \ldots, t_n]$,

(ii) every exponential map on A can be extended on $A^{[n]}$ to be constant on variables.

We apply the above invariant in the proof of Theorem 1 in an analogous way as the Makar-Limanov invariant was used in [3] to prove the cancellation theorem for curves of Abhyankar, Eakin and Heinzer [1]. First we show the following:

Lemma 7. *If $f \in k^{[2]} \setminus k$, then either $\mathrm{ML}(k^{[2+n]}, f) = k^{[2]}$, or $\mathrm{ML}(k^{[2+n]}, f) = k[t]$ whenever there exists a coordinate t of $k^{[2]}$ such that $k[f] \subset k[t]$.*

Proof. By (2.3) we have $\mathrm{ML}(k^{[2+n]}, f) \subset k^{[2]}$. Suppose that this inclusion is proper. Then by Theorem 3 there exists a coordinate t of $k^{[2]}$ such that $\mathrm{ML}(k^{[2+n]}, f) \subset k[t]$. Since the extension $k[f] \subset k[t]$ is finite and $\mathrm{ML}(k^{[2+n]}, f)$ is algebraically closed, we have $\mathrm{ML}(k^{[2+n]}, f) = k[t]$. ∎

Now we prove Theorem 1 as follows. Let φ be an automorphism of $k^{[2+n]}$ such that $\varphi(f_1) = f_2$, where $f_1, f_2 \in k^{[2]} \setminus k$. Since $\varphi(\mathrm{ML}(k^{[2+n]}, f_1)) = \mathrm{ML}(k^{[2+n]}, f_2)$, it follows from Lemma 7 that either $\mathrm{ML}(k^{[2+n]}, f_1) = \mathrm{ML}(k^{[2+n]}, f_2) = k^{[2]}$ or $\mathrm{ML}(k^{[2+n]}, f_1) = k[t_1]$ and $\mathrm{ML}(k^{[2+n]}, f_2) = k[t_2]$, where t_1, t_2 are coordinates in $k^{[2]}$. In the first case φ induces the desired automorphism of $k^{[2]}$. In the second case φ induces an isomorphism $\operatorname{res}\varphi : k[t_1] \to k[t_2]$, which takes f_1 to f_2 and can be extended to an automorphism of $k^{[2]}$. This completes the proof.

REMARK 8. Note that if k is algebraically closed of characteristic 0, then one can give an alternative simple geometric proof of Theorem 1 (see also [9]). First we give some useful facts. For an isomorphism $f : X \times k^n \to Y \times k^n$, where X, Y are algebraic sets, let

$$Z_f = \{x \in X \mid f(x \times k^n) = y \times k^n \text{ for some } y \in Y\}.$$

Note that Z_f is closed, since if $Y \subset k^m$ and $f = (f_1, \dots, f_{m+n})$, then

$$Z_f = \bigcap_{i=1}^{m} \bigcap_{y,z \in k^n} \{x \in X \mid f_i(x, y) = f_i(x, z)\}.$$

Clearly, if $Z_f = X$, then f induces an isomorphism $\tilde{f} : X \to Y$ such that $\pi_Y \circ f = \tilde{f} \circ \pi_X$, where π_X, π_Y are the projections. This is always the case in the following situation:

(2.4) Let X, Y be connected affine curves one of which is not isomorphic to k. If $f : X \times k^n \to Y \times k^n$ is an isomorphism, then $Z_f = X$.

Proof. Let us recall two well-known facts. If C is an irreducible affine curve such that there exists a dominant polynomial map $k \to C$, then every such map is finite, hence surjective. Furthermore, if C is additionally smooth, then it easily follows from Lüroth's theorem that $C \cong k$. Therefore to prove (2.4) it suffices to consider two cases: Y is not dominated by k and irreducible, or Y has singularities.

In the former case, $Z_f = X$, since otherwise we would have a dominant map $\pi_Y \circ f : x \times k^n \to Y$ for some $x \in X$.

If $\mathrm{Sing}(Y) \neq \emptyset$, then $f(\mathrm{Sing}(X) \times k^n) = \mathrm{Sing}(Y) \times k^n$ and for each irreducible component X_1 of X there exists an irreducible component Y_1 of Y such that $f(X_1 \times k^n) = Y_1 \times k^n$. Obviously $\mathrm{Sing}(X) \cap X_1 \neq \emptyset$, so as above we obtain $Z_f = X$, since otherwise the map $\pi_{Y_1} \circ f : x \times k^n \to Y_1 \setminus \mathrm{Sing}(Y)$ would be dominant for some $x \in X_1 \setminus \mathrm{Sing}(X)$.

Remark. Note that (2.4) implies that affine curves have the cancellation property (see [1] for a more general result).

We will also need the following:

(2.5) Let $g \in k^{[2]} \setminus k$ and $g^{-1}(0) = \bigcup_{i=1}^{s} l_i$, where l_i are connected components of $g^{-1}(0)$. Then $l_i \cong k$ for $i = 1, \ldots, s$ if and only if there exists a coordinate t of $k^{[2]}$ such that $g \in k[t]$.

Proof. If $l_i \cong k$, $i = 1, \ldots, s$, then by the Abhyankar–Moh–Suzuki theorem [2, 12] there exists a polynomial automorphism f of k^2 such that $f(l_1) = x_1 \times k$, $x_1 \in k$. Then $f(l_i) = x_i \times k$ for some $x_i \in k$, $i = 1, \ldots, s$, since otherwise we would have a dominant map $\pi_k \circ f : l_i \to k \setminus \{x_1\}$ for some $i \geq 2$. The second implication is obvious.

Now we prove Theorem 1 as follows. Let $f_1, f_2 \in k^{[2]} \setminus k$ be stably equivalent polynomials and φ be an automorphism of $k^{[2+n]}$ such that $\varphi(f_1) = f_2$. Let f be the automorphism of k^{2+n} induced by φ. Then $f(f_2^{-1}(\lambda) \times k^n) = f_1^{-1}(\lambda) \times k^n$ for all $\lambda \in k$.

If $f_2 \notin k[t]$ for every coordinate t of $k^{[2]}$, then by (2.5) some connected component of $f_2^{-1}(\lambda)$ is not isomorphic to k for all $\lambda \in k$. By (2.4) such components are contained in Z_f, which implies that $Z_f = k^2$, since Z_f is closed. Hence there exists an induced automorphism $\tilde{f}$ of k^2 such that $\tilde{f} \circ \pi_{k^2} = \pi_{k^2} \circ f$, which means that $\varphi(k^{[2]}) = k^{[2]}$.

It remains to show that if there exist coordinates t_1, t_2 of $k^{[2]}$ such that $f_i \in k[t_i]$, $i = 1, 2$, then $\varphi(k[t_1]) = k[t_2]$. This follows from the fact that elements of $\varphi(k[t_1])$ are integral over $k[f_2]$, hence over $k[t_2]$, and similarly elements of $k[t_2]$ are integral over $\varphi(k[t_1])$. Since $k[t_i]$ is algebraically closed in $k^{[2+n]}$, we have $\varphi(k[t_1]) = k[t_2]$.

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Received 5 May 2011;
revised 1 July 2011 (5505)