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AN AREA FORMULA IN METRIC SPACES

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Abstract. We present an area formula for continuous mappings between metric spaces, under minimal regularity assumptions. In particular, we do not require any notion of differentiability. This is a consequence of a measure-theoretic notion of Jacobian, defined as the density of a suitable "pull-back measure". Finally, we give some applications and examples.

1. Introduction. The classical area formula relates the Hausdorff measure of Lipschitz subsets of the Euclidean space $\mathbb{E}_n$ to the differential of their parametrizations. Let $m \le n$, let $A \subset \mathbb{E}_m$ be a measurable set and let $f : A \to \mathbb{E}_n$ be a Lipschitz mapping. Then

$$\int_A Jf(x)\, d\mathcal{H}^m(x) = \int_{\mathbb{E}_n} N(f, A, y)\, d\mathcal{H}^m(y), \tag{1}$$

where $N(f, A, y)$ is the multiplicity function and

$$Jf(x) = \sqrt{\det(Df(x)^T Df(x))}$$

is the Jacobian of f, which is a.e. defined by Rademacher's theorem (see for instance [3]). This formula easily extends to Lipschitz mappings between Riemannian manifolds, where the Hausdorff measure $\mathcal{H}^m$ in each Riemannian manifold is constructed from the corresponding Riemannian distance and the Jacobian $Jf(x)$ is computed in a fixed orthonormal basis of the tangent space at x. The Lipschitz continuity of f can also be weakened to a suitable Sobolev regularity [10]. Minimal smoothness assumptions for the area formula have been found [4], and when f has a graph form the sharp Sobolev regularity has also been established [9].

Formula (1) has also been proved in the non-Riemannian setting of stratified groups [7, 12], where the Hausdorff measure is constructed with respect to the so-called Carnot–Carathéodory distance or any equivalent homogeneous distance.

Even in this framework, the core of the proof is related to a differentiability theorem, namely, the almost everywhere differentiability of [11]. In this

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case, the differentials are group homomorphisms, hence it is still possible to introduce a natural notion of Jacobian in terms of the differential, namely,

$$Jf(x) = \frac{\mathcal{H}^Q(Df(x)(B_1))}{\mathcal{H}^Q(B_1)} = C_{\mathbb{G},\mathbb{M}}\sqrt{\det(Df(x)^T Df(x))},$$

where $Df(x)$ is the so-called Pansu differential of f at x. This notion of Jacobian has been introduced in Subsection 3.2 of [7].

In a metric setting, the notion of *metric differentiability* goes back to the work of Kirchheim [5], who shows that the Hausdorff measure of a Lipschitz image of a Euclidean space only depends on the norm of the differential, without referring to a possible linear structure of the target. The *metric differential* of a Lipschitz mapping $f : \mathbb{E}_n \to (Y, \rho)$ at a point x is the seminorm s on $\mathbb{E}_n$ such that

$$\rho(f(y), f(x)) - s(y - x) = o(|y - x|)$$

as $y \to x$, where $|\cdot|$ is the Euclidean norm on $\mathbb{E}_n$. If Y is a normed linear space and f is differentiable at x, then $v \mapsto \|Df(x)(v)\|$ is the metric differential at x. The main point about this notion is that all of these Lipschitz mappings are almost everywhere metrically differentiable and for them an area formula holds [5] (see also [1]). In this formula the Jacobian of a seminorm s is

$$J(s) = \frac{\omega_n}{\mathcal{L}^n(\{v \in \mathbb{E}_n \mid s(v) \le 1\})},$$

where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure and $\omega_n = \mathcal{L}^n(\{v \in \mathbb{E}_n \mid |v| \le 1\}$. Clearly, when Y is another Euclidean space, this notion includes the classical notion of Jacobian. However, in the literature one can find notions of Jacobian that only use the Hausdorff measure, without involving any differential (see for instance [11, 12]). In these works one considers the limit of the quotient

$$\frac{\mathcal{H}^Q(f(B_{x,r}))}{\mathcal{H}^Q(B_{x,r})} \quad \text{as } r \to 0^+, \tag{2}$$

where Q is the Hausdorff dimension of the stratified group $\mathbb{G}$, the Lipschitz mapping f is defined in $\mathbb{G}$, and $B_{x,r} \subset \mathbb{G}$ is the metric ball of center x and radius $r > 0$. In Definition 2.21 of [12], the Jacobian is defined as a limit analogous to (2) and with this notion an area formula for Lipschitz mappings between stratified groups is proved.

The present note shows that the above notion of Jacobian introduces an abstract scheme to prove the area formula, which works in a pure metric setting, without relying on any notion of differentiability. In fact, we essentially regard the Jacobian (2) as the ratio between two measures, then we combine classical facts on differentiation of measures and other measure-theoretic results [3].

Our Theorems 1 and 2 point out the minimal requirements to prove an area formula for Lipschitz mappings between metric spaces. For instance, the less general conditions of Example 1 suffice to include all the above mentioned area formulae, as clarified in Example 2. In all geometric frameworks where the measure-theoretic area formula holds, one may look for new notions of differentiability (see Example 3). Even when the almost everywhere metric differentiability of Lipschitz mappings fails to hold, a metric area formula can still be written (see Example 4). This should somehow show that this way of thinking of the measure-theoretic area formula provides us with a unified approach to study this formula in both known and new geometric contexts.

2. Metric area formula. Throughout, let (X, d, μ) and (Y, ρ, ν) be two metric measure spaces, where X is complete and separable, μ is a Borel regular measure on X, and ν is a Borel measure on Y. We use the term "measure" for a countably subadditive nonnegative set function. We also assume that μ is finite on bounded sets and that there exists a μ-Vitali relation V.

According to 2.8.16 of [3], a subset V of $\{(x, S) \mid S \text{ is Borel and } x \in S\}$ is a *μ-Vitali relation* if for all $x \in X$ we have

$$\inf\{\text{diam}(S) \mid (x, S) \in V\} = 0$$

and for any $C \subset V$ and $A \subset X$ such that $\inf\{\text{diam}(S) \mid (y, S) \in C\} = 0$ for all $y \in A$, we can find a countable disjoint subfamily of $\{S \mid (x, S) \in C,\ x \in A\}$ whose union covers μ-almost all of A.

The next point is the notion of "pull-back measure" with respect to a continuous mapping. We appeal to a classical result of N. Lusin, concerning the universal measurability of analytic sets. A version of this result that can be found for instance in 2.2.13 of [3] is the following: *If X is a complete and separable metric space and $g : X \to Y$ is continuous, then for every Borel set $B \subset X$ the set $g(B)$ is ν-measurable.*

Let us fix a closed set $E \subset X$ and let $f : E \to Y$ be continuous. For each $S \subset E$, we set $\zeta(S) = \nu(f(S))$ and denote by $f^*\nu$ the measure arising from Carathéodory's construction with size function ζ, defined on the family of Borel sets (see 2.10.1 of [3]). We say that $f^*\nu$ is the *pull-back measure* of ν with respect to f. The measure $f^*\nu$ extends to the whole of X by setting $f^*\nu(A) = f^*\nu(A \cap E)$ for any $A \subset X$. In the following, $E \subset X$ will stand for any fixed closed set. Notice that $f^*\nu$ is a Borel regular measure on E, by the Carathéodory construction.

The *multiplicity function* of $f : E \to Y$ relative to A is defined as

$$N(f, A, y) = \#(A \cap f^{-1}(y)) \quad \text{for all } y \in Y.$$

For any Borel set $T \subset E$, Theorem 2.10.10 of [3] yields

$$f^*\nu(T) = \int_Y N(f,T,y)\,d\nu(y). \tag{3}$$

From 2.8.16 of [3], for each $\overline{\mathbb{R}}$-valued function φ defined on a subset of V, we define

$$(V)\limsup_{S\to x}\varphi(S) = \lim_{\varepsilon\to 0^+}\sup\{\varphi(S) \mid (x,S)\in V,\ S\in \mathrm{dmn}(\varphi),\ \mathrm{diam}(S)<\varepsilon\},$$

where $\mathrm{dmn}(\varphi)$ denotes the domain of φ. $(V)\lim$ and $(V)\liminf$ are introduced in a similar way. With these notions, we can give two possible notions of metric Jacobian.

Definition 1. Let $f : E \to Y$ be continuous and let $x \in E$. Then we introduce two *metric Jacobians* of f at x as follows:

$$J_f(x) = (V)\limsup_{S\to x}\frac{\nu(f(S\cap E))}{\mu(S)}, \qquad Jf(x) = (V)\limsup_{S\to x}\frac{f^*\nu(S)}{\mu(S)}. \tag{4}$$

Remark 1. It is important to notice that when $f^*\nu$ is absolutely continuous with respect to μ and finite on bounded sets, standard arguments show that

$$f^*\nu(A) = \int_Y N(f,A,y)\,d\nu(y) \tag{5}$$

for any μ-measurable set $A \subset E$, extending (3) to μ-measurable sets.

We will present in two distinct theorems the metric area formula under different assumptions, depending on the notion of metric Jacobian we use.

Theorem 1 (Area formula I). *Let $f : E \to Y$ be continuous and assume that the pull-back $f^*\nu$ is finite on bounded sets and absolutely continuous with respect to μ. Then Jf is μ-a.e. finite and for all μ-measurable sets $A \subset E$, we have*

$$\int_A Jf(x)\,d\mu(x) = \int_Y N(f,A,y)\,d\nu(y). \tag{6}$$

Proof. Under our assumptions, Theorem 2.9.7 of [3] shows that any μ-measurable set $A \subset X$ is also $f^*\nu$-measurable and the integral formula

$$f^*\nu(A) = \int_A \mathbf{D}(f^*\nu,\mu,V,x)\,d\mu(x)$$

holds, where $\mathbf{D}(f^*\nu,\mu,V,x)$ is the density of $f^*\nu$ with respect to μ and the Vitali relation V (see 2.9.1 of [3]). By the definition of metric Jacobian, for any μ-measurable set $A \subset E$, we have $f^*\nu(A) = \int_A Jf(x)\,d\mu(x)$. Thus, formula (5) concludes the proof. ■

It should be apparent how in the previous theorem the regularity requirements on the mapping f are transferred to the pull-back measure $f^*\nu$. The next lemma is a simple variant of Lemma 2.9.3 in [3], where we replace the Borel regularity of the measure σ with the absolute continuity with respect to μ.

LEMMA 1. *Let σ and μ be measures that are finite on bounded sets of X, where σ is absolutely continuous with respect to μ. Then for any $\alpha > 0$ and any μ-measurable set*

$$A \subset \left\{ x \in X \;\middle|\; (V) \liminf_{S \to x} \frac{\sigma(S)}{\mu(S)} < \alpha \right\},$$

we have $\sigma(A) \leq \alpha \mu(A)$.

The next version of the metric area formula uses the more manageable notion of metric Jacobian J_f, hence it requires some additional assumptions on f.

THEOREM 2 (Area formula II). *Let $f : E \to Y$ be continuous and assume that the pull-back $f^*\nu$ is finite on bounded sets and absolutely continuous with respect to μ. If $A \subset E$ is μ-measurable and there exist disjoint μ-measurable sets $\{E_i\}_{i \in \mathbb{N}}$ such that*

$$\mu\Big(E \setminus \bigcup_{i \in \mathbb{N}} E_i\Big) = 0,$$

$f_{|E_i}$ is injective for every $i \geq 1$ and $J_f(x) = 0$ for μ-a.e. $x \in E_0$, then

$$\int_A J_f(x)\, d\mu(x) = \int_Y N(f, A, y)\, d\nu(y). \tag{7}$$

Proof. We can assume that any E_i is contained in E. Let us fix $\varepsilon > 0$ and consider a sequence of closed sets $C_i \subset E_i$ such that $\mu(E_i \setminus C_i) \leq \varepsilon 2^{-i}$ for any $i \in \mathbb{N}$. Let us set $f_i = f_{|C_i}$ and notice that for all $x \in C_i$ we have

$$J_{f_i}(x) = (V) \limsup_{S \to x} \frac{\nu(f(S \cap C_i))}{\mu(S)} \leq (V) \limsup_{S \to x} \frac{\nu(f(S \cap E))}{\mu(S)} = J_f(x).$$

By Corollary 2.9.9 of [3] applied to both $\mathbf{1}_{C_i}$ and $\mathbf{1}_{C_i}\, Jf$, it follows that for μ-a.e. $x \in C_i$, we have

$$(V) \lim_{S \to x} \frac{1}{\mu(S)} \int_S \mathbf{1}_{C_i}(z)\, \mathbf{D}(f^*\nu, \mu, V, z)\, d\mu(z) = Jf(x), \tag{8}$$

$$(V) \lim_{S \to x} \frac{1}{\mu(S)} \int_S \mathbf{D}(f^*\nu, \mu, V, z)\, d\mu(z) = Jf(x). \tag{9}$$

Now, for all $x \in C_i$ such that (8) and (9) hold, we have

$$\begin{aligned} J_f(x) &= (V) \limsup_{S \to x} \frac{\nu(f(S \cap E))}{\mu(S)} \leq (V) \limsup_{S \to x} \frac{f^*\nu(S \cap E)}{\mu(S)} = Jf(x) \\ &\leq (V) \limsup_{S \to x} \frac{\nu(f(S \cap E \cap C_i))}{\mu(S)} + (V) \limsup_{S \to x} \frac{f^*\nu(S \cap E \setminus C_i)}{\mu(S)} \end{aligned}$$

$$\leq (V)\limsup_{S\to x}\frac{\nu(f_i(S\cap C_i))}{\mu(S)}+(V)\limsup_{S\to x}\frac{f^*\nu(S\setminus C_i)}{\mu(S)}$$
$$=(V)\limsup_{S\to x}\frac{\nu(f_i(S\cap C_i))}{\mu(S)}.$$

The last equality follows from (8) and (9), hence we get $J_f(x)=Jf(x)=J_{f_i}(x)$. These equalities hold a.e. in C_i for any $i\geq 1$. Let $B_1=\bigcup_{i=1}^\infty C_i$ and $A_1=\bigcup_{i=1}^\infty E_i$. Then $\mu(A_1\setminus B_1)\leq\varepsilon$, since we have shown that the previous equalities of metric Jacobians hold μ-a.e. in B_1. The arbitrary choice of ε allows constructing an increasing sequence of Borel sets $B_i\subset A_1$ such that $\mu(A_1\setminus B_n)\leq\varepsilon/n$ for all $n\geq 1$. In particular, setting $B_\infty=\bigcup_{n=1}^\infty B_n$, we have

$$\mu(A_1\setminus B_n)\searrow\mu(A_1\setminus B_\infty)$$

as $n\to\infty$ and this limit is zero. Thus, in view of formula (6), we get

$$f^*\nu(A\cap A_1)=\int\limits_{A\cap A_1}Jf(x)\,d\mu(x)=\int\limits_{A\cap A_1}J_f(x)\,d\mu(x).$$

We have obtained the formula

$$f^*\nu(A)=\int\limits_{A\cap A_1}J_f(x)\,d\mu(x)+f^*\nu(A\cap E_0). \tag{10}$$

We have to show that $f^*\nu(A\cap E_0)=0$. Let us consider for any $Z\subset X$ the "preimage measure" $f^\sharp\nu(Z)=\nu(f(Z))$ that is absolutely continuous with respect to μ. Since the set where $J_f>0$ in E_0 is μ-negligible and $f^*\nu$ is absolutely continuous with respect to μ, it is not restrictive to assume that J_f everywhere vanishes on E_0. Now, for every $\epsilon>0$ and every μ-measurable bounded set $F\subset E_0$, we get $f^\sharp\nu(F)\leq\epsilon\mu(F)$, due to Lemma 1 applied with $\sigma=f^\sharp\nu$. This clearly implies $f^\sharp\nu(E_0)=\nu(f(E_0))=0$, hence (5) gives $f^*\nu(E_0)=0$. Then (10) easily guides us to the conclusion. ∎

3. Examples

Example 1. Let (X,d) be a complete and separable metric space, let $\alpha>0$, let (Y,ρ) be a metric space and consider the metric measure spaces $(X,d,\mathcal{H}_d^\alpha)$ and $(Y,\rho,\mathcal{H}_\rho^\alpha)$. Let $E\subset X$ be closed and let $f:E\to Y$ be a Lipschitz mapping. We assume that

(1) $\mathcal{H}_d^\alpha$ is finite on bounded sets of X,
(2) for $\mathcal{H}_d^\alpha$-a.e. $x\in X$,

$$\liminf_{r\to0^+}\frac{\mathcal{H}_d^\alpha(D_{x,r})}{r^\alpha}>0.$$

These conditions imply that $V=\{(x,D_{x,r}):x\in X,\ r>0\}$ is an $\mathcal{H}_d^\alpha$-Vitali relation, that $f^*\mathcal{H}_\rho^\alpha$ is finite on bounded sets and absolutely continuous with respect to $\mathcal{H}_d^\alpha$. It follows that the conclusion of Theorem 1 holds, where $\mu = \mathcal{H}_d^\alpha$ and $\nu = \mathcal{H}_\rho^\alpha$. Furthermore, under these hypotheses for the metric measure space Y, the conclusion of Theorem 2 also holds. Thus, the metric area formula of [8] follows as a special case.

EXAMPLE 2. Let E be a closed subset of a separable metric space X, let (Y, ρ) be a metric space and let $f : E \to Y$ be Lipschitz. Then the conditions of Example 1 are satisfied in the following known cases:

(1) $(X, d, \mu) = (\mathbb{E}_n, |\cdot|, \mathcal{L}^n)$.
(2) $(X, d, \mu) = (M, d_g, v_g)$, where M is a complete n-dimensional Riemannian manifold, equipped with the Riemannian distance d_g and volume measure v_g.
(3) $(X, d, \mu) = (\mathbb{G}, \|\cdot\|, \mathcal{H}^Q)$, where $\mathbb{G}$ is a stratified group equipped with its homogeneous norm $\|\cdot\|$, $\mathcal{H}^Q$ is its Q-dimensional Hausdorff measure constructed with the distance induced by the homogeneous norm, and Y is any stratified group whose distance is induced by another homogeneous norm.

Thus, in all of these cases the conclusion of Theorem 1 holds. By [3, 5, 7, 11] in the same cases we have a Rademacher-type theorem with respect to a proper notion of differentiability, the corresponding area formulae hold and their Jacobian a.e. coincides with Jf, used in Theorem 1. Furthermore, the injective decomposition required in Theorem 2 is satisfied in all the above cases, hence its conclusion also holds and the metric Jacobian J_f of Theorem 2 in particular a.e. coincides with that of Theorem 1.

EXAMPLE 3. Let $K \subset \mathbb{R}^n$ be the invariant set associated to an Iterated Function System of contraction similarities $S_i : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the open set condition, such that $|S_i(x) - S_i(y)| = c_i|x - y|$ for all $x, y \in \mathbb{R}^n$, where $0 < c_i < 1$ and $i = 1, \dots, m$ (see for instance [2]). Let $s > 0$ be the unique real number such that $\sum_{i=1}^m c_i^s = 1$. Then one can check that there exist two geometric constants $c_1, c_2 > 0$ such that

$$c_1 \le \frac{\mathcal{H}^s(K \cap B(x, \rho))}{\rho^s} \le c_2$$

for all $0 < \rho < \min\{\operatorname{diam}(K), 1\}$. The open ball $B(x, \rho)$ is defined with respect to the Euclidean distance, and so is the Hausdorff measure $\mathcal{H}^s$. In view of the previous example, $V = \{(x, \overline{B(x, r)} \cap K) : x \in X,\ r > 0\}$ is an $\mathcal{H}^s$-Vitali relation on K. Therefore any Lipschitz mapping defined on the metric measure space $(K, |\cdot|, \mathcal{H}^s)$ with values in a metric measure space $(Y, \rho, \mathcal{H}_\rho^s)$ satisfies the measure-theoretic area formula (6) and, under the assumptions of Theorem 2, the measure-theoretic area formula (7). One may ask whether a suitable notion of differentiability is available to find a Jacobian that depends on this differential and that fits with our metric Jacobian.

Remark 2. The previous example may suggest regarding our approach also as a tool to investigate novel area formulae in settings where suitable metric differentiable structures are not available yet. In this sense, the conditions of Example 1 essentially represent minimal requirements to investigate the validity of an area formula.

The next example presents a special case where there is no metric differentiability. Nevertheless, our metric area formula (6) holds.

Example 4. Let us consider the homogeneous distance d of the first Heisenberg group $\mathbb{H}^1$ and the left invariant distance ρ on $\mathbb{H}^1$, constructed in [6]. Recall that ρ is not homogeneous and the inequality $\rho \leq d$ holds everywhere. In the above mentioned work, it is proved that I is *nowhere metrically differentiable*, according to the notion of [5] extended to the group setting. We have the maximal oscillations

$$\limsup_{t\to 0^+} \frac{\rho(I(x\delta_t z), I(x))}{d(x\delta_t z, x)} = 1 \quad \text{and} \quad \liminf_{t\to 0^+} \frac{\rho(I(x\delta_t z), I(x))}{d(x\delta_t z, x)} = 0. \tag{11}$$

Let us define the left invariant distance $\rho_\alpha = \rho + \alpha d$ on $\mathbb{H}^1$, where $\alpha \geq 0$. Then the identity mapping of $\mathbb{H}^1$ has Lipschitz constant $1+\alpha$. Let us equip $(\mathbb{H}^1, d)$ and $(\mathbb{H}^1, \rho_\alpha)$ with the Hausdorff measures $\mathcal{H}^4_d$ and $\mathcal{H}^4_{\rho_\alpha}$, respectively. Since $\mathcal{H}^4_d$ is doubling on $(\mathbb{H}^1, d)$, by Theorem 2.8.17 of [3], the *covering relation* of closed balls $\{(x, D_{x,r}) : x \in \mathbb{H}^1, r > 0\}$ forms an $\mathcal{H}^4_d$-Vitali relation in $(\mathbb{H}^1, d)$. Furthermore, the injectivity of I implies that $f^*\mathcal{H}^4_{\rho_\alpha} = \mathcal{H}^4_{\rho_\alpha} \leq (1+\alpha)^4\, \mathcal{H}^4_d$. Clearly, the identity mapping $I : (\mathbb{H}^1, d) \to (\mathbb{H}^1, \rho_\alpha)$ is nowhere metrically differentiable for any $\alpha \geq 0$ and $f^*\mathcal{H}^4_{\rho_\alpha}$ satisfies the assumptions of Theorem 1, hence for any $\mathcal{H}^4_d$-measurable set $A \subset \mathbb{H}^1$,

$$\mathcal{H}^4_{\rho_\alpha}(A) = \int_A JI(x)\, \mathcal{H}^4_d(x), \tag{12}$$

where for all $x \in \mathbb{H}^1$, we have

$$JI(x) = J_I(x) = \limsup_{r\to 0^+} \frac{\mathcal{H}^4_{\rho_\alpha}(D_{x,r})}{\mathcal{H}^4_d(D_{x,r})} = \limsup_{r\to 0^+} \frac{\mathcal{H}^4_{\rho_\alpha}(D_{0,r})}{\mathcal{H}^4_d(D_{0,r})} = c_\alpha < \infty.$$

Thus we have obtained $\mathcal{H}^4_{\rho_\alpha} = c_\alpha \mathcal{H}^4_d$ with $c_\alpha \geq 0$. As soon as $\mathcal{H}^4_{\rho_\alpha}$ is positive on open sets, which is the case for $\alpha > 0$, the previous equality also follows by uniqueness of the Haar measure on locally compact Lie groups. Notice that (12) does not refer to any notion of differentiability, although it turns out to be a simple change of variable formula for two different measures. This example also clarifies the dependence of the metric Jacobian on the fixed measures of the metric spaces.

Remark 3. By approximation with step functions, the metric area formulae (6) and (7) can be extended to all nonnegative measurable mappings

$u : A \to [0, \infty]$, hence

$$\int_A u(x) J_f(x)\, d\mu(x) = \int_Y \sum_{x \in f^{-1}(y)} u(x)\, d\nu(y). \tag{13}$$

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