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THE STANLEY–FÉRAY–ŚNIADY FORMULA FOR THE GENERALIZED CHARACTERS OF THE SYMMETRIC GROUP

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Abstract. We show that the explicit formula of Stanley–Féray–Śniady for the characters of the symmetric group has a natural extension to the generalized characters. These are the spherical functions of the unbalanced Gel'fand pair $(S_n \times S_{n-1}, \operatorname{diag} S_{n-1})$.

1. Introduction. Recently, two different explicit formulas have been found for the characters of the symmetric group: the Stanley–Féray formula, conjectured by R. P. Stanley [10] and proved by V. Féray [5], and the formula of M. Lassalle [8] (see [3] for an account). Actually, these are formulas for *spherical functions* rather than characters. Indeed, these formulas give the normalized characters obtained by dividing each of them by the dimension of the corresponding representation. These are the spherical functions of the Gel'fand pair $(S_n \times S_n, \operatorname{diag} S_n)$, where $\operatorname{diag} S_n = \{(\pi,\pi) : \pi \in S_n\}$. In [11] E. Strahov showed that some of the classical results for the characters of the symmetric group may be extended to the spherical functions of the unbalanced Gel'fand pair $(S_n \times S_{n-1}, \operatorname{diag} S_{n-1})$, where $\operatorname{diag} S_{n-1} = \{(\pi,\pi) : \pi \in S_{n-1}\}$. This amounts to considering the algebra of all S_{n-1}-conjugacy invariant functions on S_n rather than the S_n-conjugacy invariant functions. It is a natural problem to extend a result for the normalized characters to the generalized characters of the symmetric group. In the present paper we show that the Stanley–Féray formula, in the form proved by Féray and P. Śniady in [6], may be naturally extended to the generalized characters.

2. Preliminaries. We recall some basic facts on unbalanced Gel'fand pairs. We refer to [2, 3, 11, 12] for more details and proofs (but we follow the notation of our joint monographs with T. Ceccherini-Silberstein and F. Tolli). If X is a finite set, we denote by $L(X)$ the space of all complex-valued functions defined on X. Let G be a finite group. We say that $H \le G$ is a *multiplicity free subgroup* of G when $\operatorname{Res}^G_H \sigma$ is a multiplicity free rep-

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resentation of H for every irreducible representation σ of G. We recall that the action of $G \times H$ on $G \equiv (G \times H)/\operatorname{diag} H$ is $(g,h) \cdot g_0 = g g_0 h^{-1}$. The subgroup H is multiplicity free if and only if $(G \times H, \operatorname{diag} H)$ is a Gel'fand pair, if and only if the algebra of H-conjugacy invariant functions on G is commutative.

Let $\widehat{G}$ (resp. $\widehat{H}$) be a complete set of pairwise inequivalent (unitary) irreducible representations of G (resp. H). For $\sigma \in \widehat{G}$ we denote by σ' the adjoint of σ. If $\rho \in \widehat{H}$ and $\sigma \in \widehat{G}$, we write $\rho \leq \operatorname{Res}^G_H \sigma$ to indicate that ρ is contained in $\operatorname{Res}^G_H \sigma$; $\sigma \boxtimes \rho$ denotes the tensor product of σ and ρ; χ^σ and χ^ρ are the characters of σ and ρ (they are not normalized: $\chi^\rho(1_G)$ is equal to the dimension d_ρ of ρ). If H is multiplicity free, the decomposition of the permutation representation η of $G \times H$ on $L(G)$ is the following:

$$\eta \cong \bigoplus_{\sigma \in \widehat{G}} \bigoplus_{\substack{\rho \in \widehat{H}: \\ \rho \leq \operatorname{Res}^G_H \sigma'}} (\sigma \boxtimes \rho). \tag{2.1}$$

In particular, for $H = G$ the $G \times G$-irreducible representation $\sigma \boxtimes \sigma'$ coincides with the σ-isotypic component in $L(G)$, that is, the subspace of $L(G)$ spanned by the matrix coefficients of σ. The spherical function associated to $\sigma \boxtimes \rho$ has the following expression:

$$\phi_{\sigma,\rho}(g) = \frac{1}{|H|} \sum_{h \in H} \overline{\chi^\sigma(gh)\chi^\rho(h)}.$$

Following [11], we call $\phi_{\sigma,\rho}$ a *generalized character* of G.

Proposition 2.1. *Suppose that H is a multiplicity free subgroup of G. With the notation above we have:*

(i) *$\phi_{\sigma,\rho}(h) = (1/d_\rho)\chi^\rho(h)$ for all $h \in H$;*
(ii) *if $\psi \in L(G)$ is H-conjugacy invariant, it belongs to the σ-isotypic component of $L(G)$ and $\psi(h) = (1/d_\rho)\chi^\rho(h)$ for all $h \in H$, then $\psi = \phi_{\sigma,\rho}$.*

Proof. Suppose that $\operatorname{Res}^G_H \sigma' = \bigoplus_{i=1}^m \rho_i$ with $\rho_1, \dots, \rho_m \in \widehat{H}$ (pairwise inequivalent) and $\rho_1 = \rho$.

(i) For every $h \in H$ we have

$$\phi_{\sigma,\rho}(h) = \frac{1}{|H|} \sum_{t \in H} \overline{\chi^\sigma(th)} \chi^\rho(t^{-1}) = \frac{1}{|H|} \Big[\Big(\sum_{i=1}^m \chi^{\rho_i} \Big) * \chi^\rho \Big](h) = \frac{1}{d_\rho} \chi^\rho(h).$$

(ii) Since ψ is H-conjugacy invariant and belongs to the σ-isotypic component of $L(G)$, we see that $\psi = \sum_{i=1}^m c_i \phi_{\sigma,\rho_i}$ for suitable complex constants

$c_1, \dots, c_m$. Therefore

$$\frac{1}{d_\rho}\chi^\rho(h) = \psi(h) = \sum_{i=1}^m c_i \phi_{\sigma,\rho_i}(h) = \sum_{i=1}^m \frac{c_i}{d_{\rho_i}}\chi^{\rho_i}(h) \quad \text{for all } h \in H$$

implies that $c_i = \delta_{i,1}$, that is, $\psi = \phi_{\sigma,\rho}$. ∎

3. Brender's formula. In this section we give a short proof of the main result in [1]. It is a formula for the generalized character of the symmetric group analogous to (8) in [6]. Let S_n be the symmetric group of degree n. We think of it as the group of all permutations of the set $\{1, \dots, n\}$. We denote by $\widetilde{S}_{n-1}$ the stabilizer of 1 in S_n. Then (see [1, 2, 3, 11]) $\widetilde{S}_{n-1}$ is a multiplicity free subgroup of S_n. If $\lambda \vdash n$ we denote by S^λ the corresponding irreducible S_n-representation. We identify $\lambda \vdash n$ with its Young frame; if $\lambda \vdash n$ and $\mu \vdash n-1$ we write $\lambda \to \mu$ to indicate that μ may obtained from λ by removing one box (we denote this box by $\lambda \setminus \mu$); note that Strahov draws the arrow in the opposite direction. Then the branching rule for the symmetric group may be written in the form $\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda = \bigoplus_{\mu \vdash n-1:\, \lambda\to\mu} S^\mu$. Therefore, in the present setting, (2.1) is

$$L(S_n) = \bigoplus_{\lambda \vdash n} \bigoplus_{\substack{\mu \vdash n-1: \\ \lambda \to \mu}} (S^\lambda \boxtimes S^\mu).$$

We denote by $\phi_{\lambda,\mu}$ the generalized character associated to $S^\lambda \boxtimes S^\mu$.

We will use the following notation: a function $f \in L(S_n)$ will be identified with the formal sum $\sum_{\pi \in S_n} f(\pi)\pi$. If t is a λ-tableau (an injective filling of the Young frame of λ with the numbers $\{1, \dots, n\}$), we denote by R_t (resp. C_t) the row (resp. the column) stabilizer of t. It is well known that the element

$$E_t = \sum_{\gamma \in C_t} \sum_{\sigma \in R_t} \mathrm{sign}(\gamma)\gamma\sigma$$

belongs to the λ-isotypic component of $L(S_n)$ [7, 9]: it is a multiple of an idempotent that projects onto a minimal left ideal of $L(S_n)$ isomorphic to S^λ (see also Exercise 10.6.7 in [2] for a less standard proof). Denote by χ^λ the character of S^λ and by d_λ the dimension of S_n. We have

$$\chi^\lambda = \frac{d_\lambda}{n!} \sum_{\pi \in S_n} \pi E_t \pi^{-1}. \tag{3.1}$$

The proof is immediate: $f \mapsto \frac{1}{n!}\sum_{\pi \in S_n} \pi f \pi^{-1}$ is the orthogonal projection from $L(S_n)$ onto the subalgebra of S_n-conjugacy invariant functions and the value of E_t on 1_{S_n} is 1. See again Exercise 10.6.7 in [2] or (VI.6.1) in [9] (where $n!/d_{\mathcal{F}}$ must be replaced by $d_{\mathcal{F}}/n!$) or (8) in [6].

Proposition 3.1 ([1]). *If $\lambda \vdash n$, $\mu \vdash n-1$, $\lambda \to \mu$ and t is a λ-tableau with 1 in the box $\lambda \setminus \mu$ then*

$$\phi_{\lambda,\mu} = \frac{1}{(n-1)!} \sum_{\pi \in \widetilde{S}_{n-1}} \pi E_t \pi^{-1}. \tag{3.2}$$

Proof. The right hand side of (3.2) is $\widetilde{S}_{n-1}$-conjugacy invariant and belongs to the S^λ-isotypic component of $L(S_n)$. Moreover, following Brender [1] we may write $E_t = E_{t'} + \xi$, where t' is the μ-tableau obtained by removing the box containing 1 and

$$\xi = \sum_{\substack{\gamma \in C_t,\, \sigma \in R_t \\ \text{but } \gamma \notin \widetilde{S}_{n-1} \text{ or } \sigma \notin \widetilde{S}_{n-1}}} \operatorname{sign}(\gamma)\gamma\sigma.$$

Neither ξ nor $\frac{1}{(n-1)!}\sum_{\pi \in \widetilde{S}_{n-1}} \pi\xi\pi^{-1}$ contains elements of $\widetilde{S}_{n-1}$: if $\gamma \in C_t$, $\sigma \in R_t$ but $\gamma \notin \widetilde{S}_{n-1}$ or $\sigma \notin \widetilde{S}_{n-1}$ then $\gamma\sigma \notin \widetilde{S}_{n-1}$. Therefore from (3.1) applied to $\widetilde{S}_{n-1}$ we get

$$\frac{1}{(n-1)!} \sum_{\pi \in \widetilde{S}_{n-1}} \pi E_t \pi^{-1} = \frac{1}{d_\mu}\chi^\mu + \frac{1}{(n-1)!} \sum_{\pi \in \widetilde{S}_{n-1}} \pi\xi\pi^{-1}.$$

Now invoking Proposition 2.1(ii), we get the desired result. ∎

4. The Stanley–Féray–Śniady formula for the generalized characters. Let λ, μ and t be as in Proposition 3.1. For $\gamma, \sigma \in S_n$ we set

$\widetilde{N}^{\lambda,\mu}$ = the number of $\pi \in \widetilde{S}_{n-1}$ such that each cycle of γ is contained in a column of πt and each cycle of σ is contained in a row of πt.

As in [6], if $\square$ is a box of λ we denote by $r(\square)$ and $c(\square)$ respectively the row and the column to which $\square$ belongs. Note also that in our notation, S_{n-1} is the stabilizer of n; more generally, $S_l \le S_n$, $1 \le l \le n$, is the symmetric group on $\{1, \dots, l\}$ and we will need to consider elements π in S_l but not in $\widetilde{S}_{n-1}$, that is, permutations of $\{1, \dots, l\}$ that do not fix 1. Indeed, Proposition 2.1(i) tells us that the value of a generalized character $\phi_{\lambda,\mu}$ on an element $\pi \in \widetilde{S}_{n-1}$ is given by the formula for the classical characters. For $\gamma, \sigma \in S_l$, with $2 \le l \le n$, we set

$\widetilde{N}^{\lambda,\mu}_{S_l}(\gamma, \sigma)$ = the number of one-to-one maps $f : \{1, \dots, l\} \to \lambda$ such that $f(1) = \lambda \setminus \mu$, $c \circ f$ is constant on each cycle of γ and $r \circ f$ is constant on each cycle of σ, and (removing the injectivity)

$$\widehat{N}^{\lambda,\mu}(\gamma,\sigma) = \text{the number of functions } f : \{1,\dots,l\} \to \lambda \text{ such that } f(1) = \lambda\setminus\mu,\ c\circ f \text{ is constant on each cycle of } \gamma \text{ and } r\circ f \text{ is constant on each cycle of } \sigma.$$

Note that if $\gamma,\sigma \in S_l$ then

$$\widetilde{N}^{\lambda,\mu}(\gamma,\sigma) = (n-l)!\widetilde{N}^{\lambda,\mu}_{S_l}(\gamma,\sigma). \tag{4.1}$$

Indeed, when we compute $\widetilde{N}^{\lambda,\mu}(\gamma,\sigma)$ we need to determine the positions of $1,\dots,l$ in πt, while the positions of $l+1,\dots,n$ may be chosen arbitrarily. We recall that $(x)_k = x(x-1)\cdots(x-l+1)$.

Lemma 4.1. *If $\theta \in S_l$ then*

$$\phi_{\lambda,\mu}(\theta) = \frac{1}{(n-1)_{l-1}} \sum_{\substack{\gamma,\sigma\in S_l:\\ \gamma\sigma=\theta}} \operatorname{sign}(\gamma)\widehat{N}^{\lambda,\mu}(\gamma,\sigma).$$

Proof. We may rewrite (3.2) in the form

$$\phi_{\lambda,\mu} = \frac{1}{(n-1)!} \sum_{\pi\in\widetilde{S}_{n-1}} \sum_{\gamma\in C_{\pi t}} \sum_{\sigma\in R_{\pi t}} \operatorname{sign}(\gamma)\gamma\sigma.$$

Since

$$\gamma \in C_{\pi t} \iff \text{each cycle of } \gamma \text{ is contained in a column of } \pi t,$$
$$\sigma \in R_{\pi t} \iff \text{each cycle of } \sigma \text{ is contained in a row of } \pi t,$$

we deduce that

$$\phi_{\lambda,\mu}(\theta) = \frac{1}{(n-1)!} \sum_{\substack{\gamma,\sigma\in S_n:\\ \gamma\sigma=\theta}} \operatorname{sign}(\gamma)\widetilde{N}^{\lambda,\mu}(\gamma,\sigma).$$

Suppose that $\theta(i) = i$, $\sigma(i) = j$ and $\gamma(j) = i$, with $i \neq j$. If i,j are contained in a row of πt then they cannot be contained in a column of πt and vice versa. Therefore if $\widetilde{N}^{\lambda,\mu}(\gamma,\sigma) \neq 0$ and $\gamma\sigma = \theta$ then $\operatorname{supp}(\gamma)$, $\operatorname{supp}(\sigma) \subseteq \operatorname{supp}(\theta)$. In particular, if $\theta \in S_l$, the sum may be restricted to $\gamma,\sigma \in S_l$; keeping in mind (4.1) we get

$$\phi_{\lambda,\mu}(\theta) = \frac{1}{(n-1)_{l-1}} \sum_{\substack{\gamma,\sigma\in S_l:\\ \gamma\sigma=\theta}} \operatorname{sign}(\gamma)\widetilde{N}^{\lambda,\mu}_{S_l}(\gamma,\sigma).$$

One can end the proof using the identity

$$\sum_{\substack{\gamma,\sigma\in S_l:\\ \gamma\sigma=\theta}} \operatorname{sign}(\gamma)\widetilde{N}^{\lambda,\mu}_{S_l}(\gamma,\sigma) = \sum_{\substack{\gamma,\sigma\in S_l:\\ \gamma\sigma=\theta}} \operatorname{sign}(\gamma)\widehat{N}^{\lambda,\mu}(\gamma,\sigma),$$

which has the same proof of (10) in [6]. ∎

We denote by $C(\pi)$ the set of cycles of a permutation π. A *coloring* of the cycles of γ, σ is a function $h : C(\gamma) \sqcup C(\sigma) \to \mathbb{N}$. We set

$N^{\lambda,\mu}(\gamma,\sigma) =$ the number of colorings h of the cycles of γ and σ such that:

- the color of each cycle of γ is a column of λ;
- the color of each cycle of σ is a row of λ;
- the color of the cycle of γ containing 1 is $c(\lambda \setminus \mu)$;
- the color of the cycle of σ containing 1 is $r(\lambda \setminus \mu)$;
- if $c_1 \in C(\gamma),\ c_2 \in C(\sigma)$ and $c_1 \cap c_2 \neq \emptyset$ then $(h(c_1), h(c_2))$ are the coordinates of a box in λ.

Now we can prove the analogue of Theorem 2 in [6] for the generalized characters.

THEOREM 4.2. *If $\theta \in S_l$ then*

$$\phi_{\lambda,\mu}(\theta) = \frac{1}{(n-1)_{l-1}} \sum_{\substack{\gamma,\sigma \in S_l: \\ \gamma\sigma = \theta}} \operatorname{sign}(\gamma) N^{\lambda,\mu}(\gamma,\sigma).$$

Proof. The assertion follows from Lemma 4.1 and the identity $N^{\lambda,\mu}(\gamma,\sigma) = \widehat{N}^{\lambda,\mu}(\gamma,\sigma)$. This may be proved by means of the following natural bijection $h \mapsto f$ between the colorings h counted by $N^{\lambda,\mu}(\gamma,\sigma)$ and the functions f counted by $\widehat{N}^{\lambda,\mu}(\gamma,\sigma)$:

$$f(m) = (h(c_1), h(c_2)) \quad \text{if} \quad c_1 \in C(\gamma), c_2 \in C(\sigma) \text{ and } m \in c_1 \cap c_2. \ \blacksquare$$

EXAMPLE 4.3. Suppose that $r(\lambda \setminus \mu) = i$ and $c(\lambda \setminus \mu) = j$. If $\lambda = (\lambda_1, \ldots, \lambda_k)$ and the conjugate partition is $\lambda' = (\lambda'_1, \ldots, \lambda'_h)$ then $j = \lambda_i$ and $i = \lambda'_j$. Moreover,

$$N^{\lambda,\mu}((1)(2),(12)) = \lambda_i \quad \text{and} \quad N^{\lambda,\mu}((12),(1)(2)) = \lambda'_j.$$

Therefore

$$\phi_{\lambda,\mu}((12)) = \frac{\lambda_i - \lambda'_j}{n-1}.$$

This formula, in a slightly different form and by means of a completely different method, was found by P. Diaconis [4, (5.10)].

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