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AN OVERDETERMINED ELLIPTIC PROBLEM IN A DOMAIN WITH COUNTABLY RECTIFIABLE BOUNDARY

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Abstract. We examine an elliptic equation in a domain Ω whose boundary $\partial\Omega$ is countably $(m-1)$-rectifiable. We also assume that $\partial\Omega$ satisfies a geometrical condition. We are interested in an overdetermined boundary value problem (examined by Serrin [Arch. Ration. Mech. Anal. 43 (1971)] for classical solutions on domains with smooth boundary). We show that existence of a solution of this problem implies that Ω is an m-dimensional Euclidean ball.

1. Introduction. We shall study the following boundary value problem:

$$\begin{aligned} \Delta u &= -1 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega,\\ \frac{\partial u}{\partial \nu} &= -cr^{\beta} && \text{on } \partial\Omega, \end{aligned}$$

where $r = \sqrt{x_1^2 + \cdots + x_m^2}$, c, β are constants and $\beta = 0$ or 1. Our goal is to show that if Ω is an open bounded subset of $\mathbb{R}^m$ and there exists $u \in \widetilde{H}^2(\Omega)$ which satisfies the above system then Ω must be a Euclidean ball. This problem has been studied by many authors (Serrin, Prajapat, Amdeberhan and others). Our contribution is a weakening of the assumptions on the boundary $\partial\Omega$ as well as on the solution.

We now introduce our hypothesis on $\partial\Omega$. We assume that $\partial\Omega$ is *countably $(m-1)$-rectifiable*, namely $\partial\Omega$ is a union of countably many Lipschitz manifolds plus an exceptional set of $\mathcal{H}^{m-1}$ measure zero. In addition, we assume that the measure $\mathcal{H}^{m-1}$ restricted to $\partial\Omega$ has a special behavior, namely $\mathcal{H}^{m-1}(\partial\Omega \cap B(x,r)) \sim r^{m-1}$.

The definition of the Sobolev spaces $\widetilde{H}^2(\Omega)$ will be recalled later. Here we explain the meaning of the normal derivative $\partial u/\partial\nu$. The expression $\partial u/\partial\nu$ may be understood as the trace on Lipschitz manifolds; it is well defined $\mathcal{H}^{m-1}$ almost everywhere (this is a corollary of Rademacher's theorem, see [23], [5]).

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Let us recall the history of this problem. The first fundamental contribution is due to Serrin [15]. He obtained the above result for $\beta = 0$ assuming that $u \in C^2(\Omega)$ and $\partial\Omega$ is smooth (see [15, Theorem 1]). In the same volume of the Archive for Rational Mechanics and Analysis, Weinberger published a short proof of Serrin's result (see [22]). In fact, Serrin showed this result for more general elliptic equations. In 1998, the assumption on $\partial\Omega$ was weakened by Prajapat (see [14]). He assumed that $\partial\Omega$ is Lipschitz with possibly one corner or cusp. Later, Amdeberhan [2] considered $\beta = 1$, Ω with smooth boundary and $u \in C^2(\Omega)$. Kawohl and others (see [6]) examined overdetermined boundary value problems for degenerate elliptic equations on star-shaped or simply connected ($m = 2$) domains under the assumption that $\partial\Omega$ is of class $C^{2,\alpha}$. In particular, this includes equations with the p-Laplacian.

Our method of proof relies on the integration by parts formula on domains with geometrically admissible boundaries for functions from Sobolev spaces. The definition of geometrically admissible set will be provided below. Roughly speaking, we compute the trace using a result of Triebel (see [21, Corollary 9.8]). Then we show the main theorem. Our method of proof is similar to that used in [2] (case $\beta = 1$) and [22] (case $\beta = 0$) for functions from Sobolev spaces. It is worth noticing that Amdeberhan [2] and Weinberger [22] applied elementary arguments. Serrin used the so-called "moving planes method" and Aleksandrov's theorem (see [1]): every embedded surface in $\mathbb{R}^m$ with constant mean curvature must be a sphere.

Before going to the next section we discuss physical motivations for the problem. Following [15] we present a few examples. Let us consider a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross sectional form Ω. If we fix rectangular coordinates in space with the z-axis directed along the pipe, it is well known that the flow velocity u is then a function of (x, y) alone satisfying the Poisson differential equation (for $m = 2$)

$$\Delta u = -A \quad \text{in } \Omega,$$

where A is a constant related to the viscosity and density of the fluid and to the rate of change of pressure per unit length along the pipe. Supplementary to the differential equation one has the adherence condition

$$u = 0 \quad \text{on } \partial\Omega.$$

Finally, the tangential stress per unit area on the pipe wall is given by the quantity $\nu\partial u/\partial n$, where ν is the viscosity. Our result states that the tangential stress on the pipe wall is the same at all points of the wall if and only if the pipe has circular cross section.

Notice that our result can be applied to weaken the assumptions in Proposition 5.4 of [7]. Indeed, the authors of that paper used Serrin's result under the assumption that the boundary is smooth. But the assumption that the boundary is not smooth is more natural from the crystalline geometry point of view considered there.

Exactly the same differential equation and boundary condition arise in the linear theory of torsion of a solid straight bar of cross section Ω (see [17]). Our theorem states that, when a solid straight bar is subject to torsion, the magnitude of the resulting traction at the surface of the bar is independent of the position if and only if the bar has circular cross section. In our case, i.e. for a countably rectifiable set, we can interpret this result in the following manner. In the class of bars whose boundaries are not regular (countably rectifiable) there exists exactly one bar such that the traction at the surface of the bar is independent of the position.

Before we present the main result we recall known definitions and make some comments. We use the standard notation $\mathcal{H}^m$ for the m-dimensional Hausdorff measure. We recall (see [3]) that a Borel set $S \subset \mathbb{R}^l$ is *countably m-rectifiable* if there is a sequence of Lipschitz maps

$$f_i : E_i \subset \mathbb{R}^m \to \mathbb{R}^l,$$

such that

$$S = \bigcup_{i=1}^{\infty} f_i(E_i) \cup B$$

and $\mathcal{H}^m(B) = 0$.

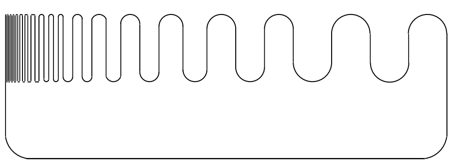

Fig. 1

Figure 1 represents an example of a countably 1-rectifiable set (sometimes called the Warsaw circle). Notice that from the McShane lemma (any Lipschitz map on a closed subset can be extended to a Lipschitz map on the whole space) we can take $E_i = \mathbb{R}^m$ (see [4], [10], [11]).

Remark 1. It is well known that the above definition is equivalent to the definition where we replace Lipschitz maps f_i by maps of class C^1 (see [5], [16]).

However, it turns out that the class of countably rectifiable sets is too broad. We will consider sets with an additional property. Namely we shall call a countably $(m-1)$-rectifiable set $S \subset \mathbb{R}^l$ $(m-1)$-*geometrically admissible* if there exists $C > 0$ such that for any $x \in S$ and $r \in (0, 1/2)$,

$$C^{-1} r^{m-1} \le \mathcal{H}^{m-1}(B(x,r) \cap S) \le C r^{m-1},$$

where $B(x,r)$ is the m-dimensional ball. This condition will be denoted by $\mathcal{H}^{m-1}(B(x,r)\cap S)\sim r^{m-1}$.

Remark 2. One can show that if $S\subset\mathbb{R}^m$ is bounded and ∂S is $(m-1)$-geometrically admissible, then $\mathcal{H}^{m-1}(\partial S)<\infty$.

Remark 3. It is easy to notice that not every countably rectifiable set is geometrically admissible. A good example is the Warsaw circle (see Figure 1). Another example shown in Figure 3. It is taken from Nikodym's paper [13]. The Warsaw circle and Nikodym's example are similar in some sense. Nice examples can be found in the book of Maz'ya [12, Chapter 1, Example 2]. From this monograph we have taken an example of a set which is geometrically admissible (see Figure 2).

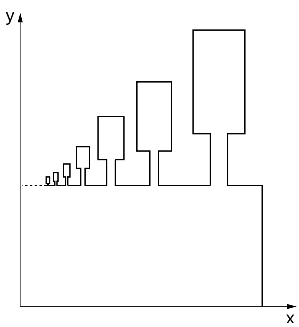

Fig. 2

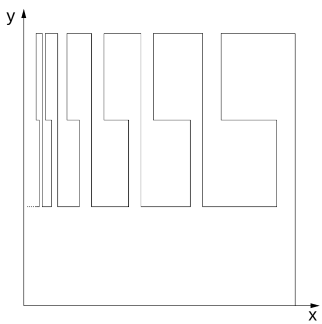

Fig. 3

Let us recall the definition of Sobolev spaces $\widetilde{H}^s(\Omega)$ (see [9]). For every positive s we denote by $\widetilde{H}^s(\Omega)$ the space of all u defined in Ω such that $\widetilde{u}\in H^s(\mathbb{R}^m)$ where $\widetilde{u}$ is the continuation of u by zero outside Ω. We define a Hilbert norm on $\widetilde{H}^s(\Omega)$ by

$$\|u\|_{\widetilde{H}^s(\Omega)}=\|\widetilde{u}\|_{H^s(\mathbb{R}^m)}.$$

2. The main result. First we formulate and prove a version of the integration by parts formula. The main point is to weaken the assumptions on $\partial\Omega$. This result is our basic tool. The difficulty is in the proof of the integration by parts formula. The geometric admissibility condition $(\mathcal{H}^{m-1}(B(x,r)\cap S)\sim r^{m-1})$ is essential in order to compute the trace and use the result from [21].

Theorem 1. *Suppose $u\in\widetilde{H}^2(\Omega)$ and $v\in\widetilde{H}^1(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^m$. Assume that $\partial\Omega$ is $(m-1)$-geometrically admissible. Then*

$$\int\limits_{\Omega} v\Delta u\,d\mathcal{H}^m(x)=-\int\limits_{\Omega}\nabla u\nabla v\,d\mathcal{H}^m(x)+\int\limits_{\partial\Omega} v\,\frac{\partial u}{\partial\nu}\,d\mathcal{H}^{m-1}(x).$$

Proof. First, note that if $u \in C^\infty(\Omega) \cap \widetilde{H}^2(\Omega)$ and $v \in C^\infty(\Omega) \cap \widetilde{H}^1(\Omega)$, then the formula holds. Indeed, the exceptional set has measure zero and we are dealing with smooth maps (see [23] and [5]).

Next, we prove the following lemma.

LEMMA 1. *If $u \in \widetilde{H}^2(\Omega)$ and $v \in \widetilde{H}^1(\Omega)$, then there exists a constant c such that*

$$\int_{\partial\Omega} \left| v \frac{\partial u}{\partial \nu} \right| d\mathcal{H}^{m-1}(x) \le c\|v\|_{\widetilde{H}^1(\Omega)} \|u\|_{\widetilde{H}^2(\Omega)}.$$

Proof. We apply a result of Triebel [21, Corollary 9.8]. In order to explain it, we define a Radon measure ν on $\mathbb{R}^m$ by

$$\nu(A) = \mathcal{H}^{m-1}(\partial\Omega \cap A).$$

It is easy to see that $\operatorname{supp} \nu = \partial\Omega$, and indeed from Remark 2 we infer that ν is a Radon measure.

Now we have to check that the assumptions of Corollary 9.8 from [21] are satisfied. Indeed, by taking $s = 1, p = 2, r = 2, d = m - 1$ in [21, Corollary 9.8], it is easy to check that

$$\nu(B(x,r)) \sim r^d, \qquad \nu(2Q_{\nu l}) \le c2^{-\nu(m-1)}, \qquad s - m/p > -d/r,$$

where $Q_{\nu l}$ is the cube in $\mathbb{R}^m$ with sides parallel to the axes, centered at $2^{-\nu} l$, and with side length $2^{-\nu}$. Here $l \in \mathbb{Z}^m$ and $\nu \in \mathbb{N}_0$.

Triebel's result already mentioned ([21, Corollary 9.8]) says that if the above conditions are satisfied then there exists a trace operator

$$\mathrm{Tr}_{\partial\Omega} : F^s_{p,q}(\mathbb{R}^m) \to L^r(\partial\Omega).$$

Recall that $W^{s,p}(\mathbb{R}^m) = F^s_{p,2}(\mathbb{R}^m)$, where $F^s_{p,2}$ are the Lizorkin–Triebel spaces. Hence, we obtain a sequence of inequalities

$$\begin{aligned}\int_{\partial\Omega} \left| v \frac{\partial u}{\partial \nu} \right| d\mathcal{H}^{m-1}(x) &\le c\|\widetilde{v}\|_{F^1_{2,2}(\mathbb{R}^m)} \|\nabla \widetilde{u}\|_{F^1_{2,2}(\mathbb{R}^m)} \\ &\le c\|\widetilde{v}\|_{H^1(\mathbb{R}^m)} \|\nabla \widetilde{u}\|_{H^1(\mathbb{R}^m)} \le c\|v\|_{\widetilde{H}^1(\Omega)} \|u\|_{\widetilde{H}^2(\Omega)},\end{aligned}$$

where we applied the Schwarz inequality. From this the lemma follows. ∎

Now, we can return to the proof of the theorem. Recall from [9] that

$$\overline{C^\infty(\Omega) \cap \widetilde{H}^2(\Omega)}^{\widetilde{H}^2(\Omega)} = \widetilde{H}^2(\Omega), \qquad \overline{C^\infty(\Omega) \cap \widetilde{H}^1(\Omega)}^{\widetilde{H}^1(\Omega)} = \widetilde{H}^1(\Omega).$$

Take any $u \in \widetilde{H}^2(\Omega)$ and $v \in \widetilde{H}^1(\Omega)$. Next, fix $\varepsilon > 0$ and choose $v_\varepsilon \in C^\infty(\Omega) \cap \widetilde{H}^1(\Omega)$ and $u_\varepsilon \in C^\infty(\Omega) \cap \widetilde{H}^2(\Omega)$ such that

$$\|v_\varepsilon - v\|_{\widetilde{H}^1(\Omega)} \le \varepsilon, \qquad \|u_\varepsilon - u\|_{\widetilde{H}^2(\Omega)} \le \varepsilon.$$

These functions satisfy

$$\int_{\Omega} v_\varepsilon \Delta u_\varepsilon \, d\mathcal{H}^m(x) = -\int_{\Omega} \nabla u_\varepsilon \nabla v_\varepsilon \, d\mathcal{H}^m(x) + \int_{\partial\Omega} v_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} \, d\mathcal{H}^{m-1}(x). \tag{1}$$

Applying the Schwarz inequality and the above lemma one can show the following inequalities:

$$\Big| \int_{\Omega} v_\varepsilon \Delta u_\varepsilon \, d\mathcal{H}^m(x) - \int_{\Omega} v \Delta u \, d\mathcal{H}^m(x) \Big| \le M\varepsilon,$$

$$\Big| \int_{\Omega} \nabla u_\varepsilon \nabla v_\varepsilon \, d\mathcal{H}^m(x) - \int_{\Omega} \nabla u \nabla v \, d\mathcal{H}^m(x) \Big| \le M\varepsilon,$$

$$\Big| \int_{\partial\Omega} v_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} \, d\mathcal{H}^{m-1}(x) - \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{m-1}(x) \Big| \le cM\varepsilon.$$

Finally, we can let $\varepsilon \to 0$ under the integrals in (1) to obtain

$$\int_{\Omega} v \Delta u \, d\mathcal{H}^m(x) = -\int_{\Omega} \nabla u \nabla v \, d\mathcal{H}^m(x) + \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{m-1}(x).$$

This ends the proof of the theorem. ∎

REMARK 4. The conclusion of the above theorem is true if we assume that $v \in \widetilde{H}^1(\Omega)$ and $u \in \widetilde{E}(\Delta, L^2(\Omega))$, where $\widetilde{E}(\Delta, L^2(\Omega)) = \{u \in \widetilde{H}^1(\Omega) : \Delta\widetilde{u} \in L^2(\mathbb{R}^m)\}$. This follows from the fact that $\partial^2 \widetilde{u}/\partial x_i \partial x_k = -\mathcal{R}_i \mathcal{R}_k \Delta \widetilde{u}$ (see [18], [19]). But in the next theorem we need H^2 regularity in order to apply the maximum principle.

Now, we can formulate the main result of this paper. We may view the theorem below as a generalization of Prajapat's result [14] for the overdetermined problem for Lipschitz domains with cusps. Indeed, countably rectifiable sets have countably many cusps.

THEOREM 2. *Suppose that Ω is a bounded open subset of $\mathbb{R}^m$ such that $\partial\Omega$ is $(m-1)$-geometrically admissible. If there exists a solution $u \in \widetilde{H}^2(\Omega)$ of the problem*

$$\begin{aligned} \Delta u &= -1 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} &= -cr^\beta && \text{on } \partial\Omega, \end{aligned}$$

where $r = \sqrt{x_1^2 + \cdots + x_m^2}$ and c, β are constants and $\beta = 0$ or 1, then Ω is an m-dimensional Euclidean ball.

Proof. Our method of proof is similar to [2] (case $\beta = 1$) and [22] (case $\beta = 0$). We refer the reader to those papers for details.

It is easy to see that the assumptions of Theorem 2 imply that ([2, Lemma 1] for $\beta = 1$)

$$\int_{\Omega} u \, d\mathcal{H}^m(x) = c^2 \int_{\Omega} r^2 \, d\mathcal{H}^m(x), \tag{2}$$

and also that ([22] for $\beta = 0$)

$$(m+2) \int_{\Omega} u \, d\mathcal{H}^m(x) = mc^2 \mathcal{H}^m(\Omega), \tag{3}$$

where we applied Theorem 1.

It is not hard to see that the expressions

$$\left(\frac{\partial u}{\partial \nu}\right)^2 - c^2 r^2 \quad \text{for } \beta = 1,$$

$$\left(\frac{\partial u}{\partial \nu}\right)^2 + \frac{2}{m} u \quad \text{for } \beta = 0$$

are constants on $\partial\Omega$, which is a consequence of the boundary conditions. Next from the weak maximum principle (see [8, notes in Chapter 8] or [20, Appendix B]) and identity (2) (respectively (3)), we deduce that these expressions are constants in Ω. From this we obtain

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = -\frac{1}{m} \delta_{ij}.$$

So the solution of our equation takes the form

$$u = c - \frac{r^2}{2m}.$$

Since u vanishes on $\partial\Omega$ and has radial symmetry we conclude that Ω is a ball. ∎

Finally, let us state some open questions. Is it possible to weaken further the assumptions on $\partial\Omega$? Is geometric admissibility really necessary?

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