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# ALGEBRAS STANDARDLY STRATIFIED IN ALL ORDERS

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**Abstract.** The aim of this work is to characterize the algebras which are standardly stratified with respect to any order of the simple modules. We show that such algebras are exactly the algebras with all idempotent ideals projective. We also deduce as a corollary a characterization of hereditary algebras, originally due to Dlab and Ringel.

**Preliminaries.** In this paper all algebras are finite-dimensional $K$-algebras, basic and indecomposable, where $K$ is an algebraically closed field. Using a fundamental theorem of Gabriel all such algebras $\Lambda$ are, up to isomorphism, of the form $\Lambda = KQ/I$ where $Q$ is a finite quiver and $I$ an admissible ideal.

All modules are finitely generated left modules and the category of such modules is denoted by $\operatorname{mod} \Lambda$.

Let $v_1, \dots, v_n$ be the vertices of $Q$ in a fixed order, and let $e_i$ denote the idempotent associated with the vertex $v_i$, for each $i$ with $1 \le i \le n$. We also denote by $S_1, \dots, S_n$ the corresponding simple modules, and $P_i$ the projective cover of $S_i$, denoted also by $P_i(\Lambda)$.

We define the standard module $\Delta_i$ as a maximal quotient of $P_i$ with composition factors $S_j,\ j \le i$ ([R]).

Given a module $A$ and a set $B$ of modules define the *trace* of $B$ in $A$, denoted by $\tau_B(A)$, as the sum of all images of morphisms of modules of $B$ in $A$, that is,

$$\tau_B(A) = \sum_{f \in \operatorname{Hom}(A,B)} \operatorname{Im} f.$$

An alternative definition of the standard modules is the following: $\Delta_i = P_i/U_i$, where $U_i$ is the trace of the set of all projectives $P_j$ with $j > i$ in $P_i$, that is, $U_i = \tau_{\amalg_{j>i} P_j}(P_i)$.

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Let $\varepsilon_k = e_k + \cdots + e_n$ for $1 \le k \le n$ and $\varepsilon_{n+1} = 0$. Using the fact that $U_i = \Lambda\varepsilon_{i+1}\Lambda e_i$, we see that another description is $\Delta_i = \Lambda e_i / \Lambda\varepsilon_{i+1}\Lambda e_i$.

Given $\Delta = \{\Delta_1, \ldots, \Delta_n\}$, consider $F(\Delta)$, the full subcategory of $\operatorname{mod}\Lambda$ whose objects are the modules $M \in \operatorname{mod}\Lambda$ such that $M$ has a filtration with factors in $\Delta$, that is, there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ with $M_i/M_{i-1} \simeq \Delta_k$ for some $k$.

An algebra $\Lambda$ is called *standardly stratified* if $\Lambda \in F(\Delta)$. If a standardly stratified algebra is such that the endomorphism ring of each standard module is a division ring then $\Lambda$ is called a *quasi-hereditary algebra* (see for example [R] and [X]).

We say that $\Lambda$ is **iip** if all idempotent ideals are projective [P]. This means that all ideals $I$ such that $I^2 = I$ are projective modules.

A useful characterization of **iip** algebras was given by M. I. Platzeck [P, Proposition 1.2]: $\Lambda$ is **iip** if and only if $\tau_{\widehat{P}_i}(\Lambda)$ is projective for all $i$, where $\widehat{P}_i = \bigoplus_{j \ne i} P_j$.

The main result of this note, Theorem 3, asserts that these algebras are exactly the algebras which are standardly stratified with respect to every order.

**1. Preparatory lemmas.** In this section, we give several lemmas that will permit a better comprehension of our main results. Some of them are known results, important in themselves. For completeness we give their proofs.

LEMMA 1. *If $\Lambda$ is standardly stratified in the order $e_1, \ldots, e_n$ then $\Lambda/\Lambda\varepsilon_i\Lambda$ is standardly stratified in the order $e_1, \ldots, e_{i-1}$.*

*Proof.* This is similar in nature to Lemma A.3.5 of [D]. Denote by $B$ the algebra $\Lambda/\Lambda\varepsilon_i\Lambda$. Moreover, let $\Delta_k(\Lambda)$ and $\Delta_k(B)$ denote the $k$th $\Delta$-modules of $\Lambda$ and $B$, respectively.

Observe first that $\Delta_j(\Lambda)$, $j = 1, \ldots, i-1$, are $B$-modules, because $\Delta_j(\Lambda) = \Lambda e_j/\Lambda\varepsilon_i\Lambda e_j$ for $j < i$ is a $\Lambda$-module annihilated by $\Lambda\varepsilon_i\Lambda$.

Observe that $\operatorname{Top}(\Delta_j(\Lambda)) = S_j$ and $\Delta_r(\Lambda)$ has all composition factors in the set $\{S_k : k < r\}$.

We show that $\Delta_j(\Lambda) = \Delta_j(B)$. Observe that $\Delta_r(\Lambda)$ is a $B$-module and it is a quotient of the $r$th projective $B$-module $P_r(B)$; moreover, $\Delta_r(\Lambda)$ has all its composition factors in $\{S_k : k < r\}$, and since $\Delta_r(B)$ is the maximal quotient of $P_r(B)$ with this property, we obtain an epimorphism from $\Delta_r(B)$ onto $\Delta_r(\Lambda)$.

We also have an epimorphism from $P_r(\Lambda)$ onto $P_r(B)$ and another from $P_r(B)$ onto $\Delta_r(B)$, thus we have an epimorphism from $P_r(\Lambda)$ onto $\Delta_r(B)$, so $\Delta_r(B)$ is a quotient of $P_r(\Lambda)$. Observe also that $\Delta_r(B)$ has all composition factors in $\{S_k : k < r\}$. Since $\Delta_r(\Lambda)$ is the maximal quotient of $P_r(\Lambda)$ with this property, we have an epimorphism of $\Delta_r(\Lambda)$ onto $\Delta_r(B)$. Therefore $\Delta_r(\Lambda) \simeq \Delta_r(B)$ as $B$-modules.

Let us show now that $P_r(B) \in F_B(\Delta)$ for all $r$. Recall that $P_r(B) = P_r(\Lambda)/(\Lambda\varepsilon_i\Lambda)P_r(\Lambda)$ and if $M$ is a $\Lambda$-module then $M/(\Lambda\varepsilon_i\Lambda)M$ is a $B$-module. Now because $\Lambda$ is standardly stratified, we have $P_r(\Lambda) \in F_\Lambda(\Delta)$, so that we have a filtration $Q_r^s \subset \cdots \subset Q_r^2 \subset Q_r^1 \subset P_r(\Lambda)$ with factors $\Delta_k(\Lambda)$, $k < r$. Passing to factors we get

$$\frac{Q_r^s}{(\Lambda\varepsilon_i\Lambda)Q_r^s} \subset \cdots \subset \frac{Q_r^2}{(\Lambda\varepsilon_i\Lambda)Q_r^2} \subset \frac{Q_r^1}{(\Lambda\varepsilon_i\Lambda)Q_r^1} \subset P_r(B) = \frac{P_r(\Lambda)}{(\Lambda\varepsilon_i\Lambda)P_r(\Lambda)}.$$

Thus $P_r(B) \in F_B(\Delta)$. ∎

LEMMA 2. *If $\Lambda$ is an* **iip** *algebra, then $\tau_P(Q)$ is projective for $P$ indecomposable projective and $Q$ projective. Moreover, $\tau_P(\Lambda)$ is projective for any projective module $P$.*

*Proof.* Note that $\tau_P(Q)$ is a summand of $\tau_P(\Lambda^s)$, so it suffices to prove that $\tau_P(\Lambda)$ is projective. If $P = \Lambda e$ for some idempotent $e$ then $\tau_P(\Lambda) = \Lambda e\Lambda$, which is an idempotent ideal and therefore is projective.

On the other hand, if $P = P_1^{n_1} \amalg \cdots \amalg P_t^{n_t}$, with $P_i \not\simeq P_j$ indecomposable then $\tau_P(\Lambda) = \tau_{P_1 \amalg \cdots \amalg P_t}(\Lambda)$. Clearly $P_1 \amalg \cdots \amalg P_t = \Lambda e_1 + \cdots + \Lambda e_t = \Lambda(e_1 + \cdots + e_t)$, so $\tau_{P_1 \amalg \cdots \amalg P_t}(\Lambda) = \Lambda(e_1 + \cdots + e_t)\Lambda$, which is an idempotent ideal and therefore projective. ∎

**2. The main result.** We are now in a position to prove the main result of this note.

THEOREM 3. *The algebra $\Lambda$ is standardly stratified in all orders if and only if it is an algebra with all idempotent ideals projective.*

*Proof.* First we prove the "only if" part. Let $\Lambda$ be an algebra which is standardly stratified in all orders. We select an order such that $l(P_i) \le l(P_{i+1})$ for all $i$. We claim that in this case $P_k = \Delta_k$ for all $k$.

By definition we have $P_n = \Delta_n$; we also have an exact sequence

$$0 \to \tau_{P_n}(P_{n-1}) \to P_{n-1} \to \Delta_{n-1} \to 0.$$

If $\tau_{P_n}(P_{n-1}) \neq 0$, then $\tau_{P_n}(P_{n-1}) \simeq P_n^{k_n}$ for some $k_n$; but since $\Lambda$ is standardly stratified, this cannot happen because of our hypothesis on the lengths of the projective modules, therefore $\tau_{P_n}(P_{n-1}) = 0$.

We continue by induction. Assume that $\Delta_n, \Delta_{n-1}, \ldots, \Delta_{n-j+1}$ are projective $\Lambda$-modules. If $\tau_{\coprod_{r>n-j} P_r}(P_{n-j}) \neq 0$ then

$$\tau_{\coprod_{r>n-j} P_r}(P_{n-j}) \simeq \coprod_{r>n-j} P_r^{k_r};$$

but this cannot happen by the hypothesis on the lengths, so $P_j = \Delta_j$.

Thus we have an order where $P_k = \Delta_k$ and therefore in this case $F(\Delta)$ coincides with Proj, the full subcategory of $\operatorname{mod} \Lambda$ whose objects are the projective finitely generated modules.

Observe also that $\operatorname{Hom}(P_j, P_i) = \operatorname{Hom}(\Delta_j, \Delta_i) = 0$ for $j > i$. An algebra $\Lambda$ with this property is usually called *quasi-triangular*. In this case the vertex $v_n$ is a source in $Q$, if we forget the loops in it. That is, there is no arrow ending at $v_n$ and starting at another vertex.

Let $\Lambda = KQ/I$. Consider the quiver $\overline{Q}$ obtained from $Q$ by elimination of $v_n$ and the ideal $\bar{I}$ generated by the relations of $I$ that remain after eliminating the arrows that start at $v_n$.

As a consequence of Lemma 1, and the fact that $\operatorname{Hom}(P(n), P(j)) = 0$ if $n \neq j$, the algebra $\Lambda/\Lambda e_n \Lambda = K\overline{Q}/\bar{I}$ is also standardly stratified in all orders.

Let $\widehat{e}_n = 1 - e_n$. Then

$$\Lambda = (e_n + \widehat{e}_n)\Lambda(e_n + \widehat{e}_n) = e_n \Lambda e_n + e_n \Lambda \widehat{e}_n + \widehat{e}_n \Lambda e_n + \widehat{e}_n \Lambda \widehat{e}_n.$$

In this sum, $L = e_n \Lambda e_n$ is a local algebra, so it is standardly stratified, and

$$e_n \Lambda \widehat{e}_n = 0 = \operatorname{Hom}(P_n, \widehat{P}_n),$$
$$M = \widehat{e}_n \Lambda e_n = \tau_{\widehat{P}_n}(P_n), \qquad U = \widehat{e}_n \Lambda \widehat{e}_n = \Lambda/\Lambda e_n \Lambda = K\overline{Q}/\bar{I}.$$

Thus $\Lambda$ has a matrix presentation of the form

$$\Lambda \simeq \begin{pmatrix} L & 0 \\ M & U \end{pmatrix}.$$

We can describe the projective modules of the algebra $\Lambda$ as triples in the following way.

There is the projective $P_n = (L, M \otimes L, \mathrm{id})$ and $P_i$ is a projective $U$-module for $i = 1, \dots, n-1$. By Proposition 16 of [MMS], $M = \widehat{e}_n \Lambda e_n = \tau_{\widehat{P}_n}(P_n) \in F_{\widehat{e}_n \Lambda \widehat{e}_n}(\Delta)$.

Because $\Lambda$ is standardly stratified in all orders, we can choose an order so that $e_n$ comes first.

We show, by induction on the number of simple modules, that $\Lambda$ is **iip**.

It is clear that a local algebra is **iip**.

To see that $\Lambda$ is an algebra whose idempotent ideals are projective, we use Proposition 1.2 of [P], and we show that $\tau_{\widehat{P}_i}(\Lambda)$ is projective for all $i$.

We also see that $U$ is **iip** by induction hypothesis, because it has fewer non-isomorphic simple modules than $\Lambda$.

Note that $\tau_{\widehat{P}_i}(\Lambda) \cong \tau_{\widehat{P}_i}(P_n) \oplus \tau_{\widehat{P}_i}(\widehat{P}_n)$ is isomorphic to $P_n \oplus \tau_{\widehat{P}_i}(U)$ if $i \neq n$ and to $M \oplus U$ if $i = n$. Moreover, $\tau_{\widehat{P}_i}(U)$ is projective since $U$ is **iip**.

It remains to prove that $M$ is a projective $U$-module. Since $U = \widehat{e}_n \Lambda \widehat{e}_n = \Lambda/\Lambda e_n \Lambda = K\overline{Q}/\bar{I}$ is standardly stratified in all orders, we can choose an order such that $P_k = \Delta_k$ and therefore $F_{\widehat{e}_n \Lambda \widehat{e}_n}(\Delta) = \mathrm{Proj}$; since $M = \widehat{e}_n \Lambda e_n = \tau_{\widehat{P}_n}(P_n) \in F_{\widehat{e}_n \Lambda \widehat{e}_n}(\Delta)$, we see that $M$ is projective.

We now prove the "if" part. Let $\Lambda$ be an algebra with all idempotent ideals projective and $e_1, \ldots, e_n$ an order of idempotents; we will show that $\Lambda$ is standardly stratified in this order.

For this we show that $P_k \in F(\Delta)$ for all $k$. First $P_n = \Delta_n \in F(\Delta)$.

We now suppose that $P_n, P_{n-1}, \ldots, P_{n-k+1} \in F(\Delta)$ and we prove that $P_{n-k} \in F(\Delta)$.

If $P_{n-k} = \Delta_{n-k}$, then it is clear that $P_{n-k} \in F(\Delta)$. If not we have a short exact sequence $0 \to \tau_{\coprod_{r>n-k} P_r}(P_{n-k}) \to P_{n-k} \to \Delta_{n-k} \to 0$. By Lemma 2 the module $\tau_{\coprod_{r>n-k} P_r}(P_{n-k})$ is projective and thus it is a direct sum of copies of $P_{n-k+1}, \ldots, P_n$. Hence $P_{n-k} \in F(\Delta)$. ∎

**3. Some applications.** In this section we get various consequences of Theorem 3.

COROLLARY 4. *If $\Lambda$ is standardly stratified in all orders then there exists an order such that $F(\Delta) = \mathrm{Proj}$.*

*Proof.* If $\Lambda$ is standardly stratified in all orders, then, as in the proof of Theorem 3, we can choose an order such that $l(P_i) \le l(P_{i+1})$ for all $i$, and in that case all $P_k = \Delta_k$ and therefore $F(\Delta) = \mathrm{Proj}$. ∎

The next corollary relates to modules of finite projective dimension.

COROLLARY 5. *If $\Lambda$ is standardly stratified in all orders then there is an order such that $F(\Delta) = P^{<\infty}$, where $P^{<\infty}$ is the subcategory of modules of finite projective dimension.*

*Proof.* If $\Lambda$ is standardly stratified in all orders then $\Lambda$ is an algebra with all idempotent ideals projective and we can take an order such that $\mathrm{Hom}(P_j, P_i) = 0$ for $j < i$, thus $\Delta_i = P_i/\tau_{\widehat{P}_i}(P_i)$ and it follows that $F(\Delta) = P^{<\infty}$ (see [P, CMMP]). ∎

We have the following remark from the proof.

REMARK 6. If $\Lambda$ is standardly stratified in all orders and $P_i, P_j$ are indecomposable projective modules such that $l(P_i) \le l(P_j)$ then $\mathrm{Hom}(P_j, P_i) = 0$. Therefore after deleting all loops the quiver $Q$ contains no oriented cycle.

We can obtain a characterization of hereditary algebras that generalizes a result of Dlab and Ringel [D, Theorem A.2.9].

COROLLARY 7. *The following conditions are equivalent:*

(i) $\operatorname{gl\,dim} \Lambda < \infty$ *and $\Lambda$ is standardly stratified in all orders.*

(ii) *$Q$ does not have oriented cycles and $\Lambda$ is quasi-hereditary in all orders.*
(iii) *$\Lambda$ is quasi-hereditary in all orders.*
(iv) *$\Lambda$ is hereditary.*

*Proof.* (ii)⇒(i). Clear.

(i)⇒(ii). If $\Lambda$ is standardly stratified in all orders then by Theorem 3, $\Lambda$ is an algebra with idempotent projectives. If $\Lambda$ is an algebra with all idempotent ideals projective and with finite global dimension, then the quiver of $\Lambda$ has no oriented cycles (see [P, CMMP]). Because $\Lambda$ is standardly stratified in all orders and $\operatorname{gl\,dim} \Lambda < \infty$ we see that it is quasi-hereditary in all orders.

(ii)⇒(iii). Evident.

(iii)⇒(i). It is clear that if $\Lambda$ is quasi-hereditary in all orders then $\Lambda$ is standardly stratified in all orders. The quiver $Q$ of $\Lambda$ has no loops since any standard module (with respect to any order of the simple modules) has trivial endomorphism algebra. Thus, by the remark above, $Q$ is triangular.

(i)⇒(iv). If $\Lambda$ is standardly stratified in all orders then by the previous corollary there exists an order such that $F(\Delta) = P^{<\infty}$ and using also the hypothesis $\operatorname{gl\,dim} \Lambda < \infty$ we get $F(\Delta) = P^{<\infty} = \operatorname{mod} \Lambda$.

Theorem 2.5 of [P] says that if $\Lambda$ is an algebra with all idempotent ideals projective then $\operatorname{fd} \Lambda \leq 1$, so $\operatorname{gl\,dim} \Lambda = \operatorname{fd} \Lambda \leq 1$, therefore $\Lambda$ is hereditary.

(iv)⇒(i). If $\Lambda$ is a hereditary algebra, it is clear that $\operatorname{gl\,dim} \Lambda < \infty$. It is also clear that $\Lambda$ has all idempotent ideals projective, so it is standardly stratified in any order of the simple modules. ■

## 4. Remarks and examples

Remark 8. If $A$ is a local algebra then $F(\Delta) = \operatorname{Proj} = P^{<\infty}$.

Remark 9. There exist orders such that $F(\Delta) = \operatorname{Proj} = P^{<\infty}$, but $A$ is not **iip**.

1. Let $A = KQ/I$ where $Q$ is the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[loop right, "\beta"] \end{tikzcd}$$

and $I$ is the ideal generated by the relations $\beta\alpha = 0$ and $\beta^2 = 0$. In the order $1, 2$ this algebra is not standardly stratified because $P_1 \notin F(\Delta)$, although in the order $2, 1$, $F(\Delta) = \operatorname{Proj}$, thus $A$ is not **iip**. But if we analyze the modules of finite projective dimension, they are exactly the projective modules, because if a module has finite projective dimension then it has even dimension as $K$-space and the only indecomposable modules with even dimension are the projective modules.

2. Even if there exist distinct orders such that $F(\Delta) = \mathrm{Proj}$ and $F(\Delta) = P^{<\infty}$, $A$ is not necessarily **iip**, as shown by the following example.

Let $A = KQ/I$ where $Q$ is the quiver

$$\overset{n}{\bullet} \xrightarrow{\alpha_{n-1}} \overset{n-1}{\bullet} \quad \cdots \quad \overset{2}{\bullet} \xrightarrow{\alpha_1} \overset{1}{\bullet}$$

and $I$ is the ideal generated by the relations $\alpha_{i+1}\alpha_i$ for $i = 1, \ldots, n-2$. In this case $F(\Delta) = \mathrm{Proj}$ in the order $1, 2, \ldots, n$, while $F(\Delta) = P^{<\infty}$ in the order $n, n-1, \ldots, 2, 1$. But this algebra is not standardly stratified in all orders because otherwise, since it is of finite global dimension, it would be hereditary by Corollary 7. We can also show directly that this algebra is not standardly stratified in the order $n, 1, 2, \ldots, n-1$.

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