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FULLY CLOSED MAPS AND NON-METRIZABLE HIGHER-DIMENSIONAL ANDERSON–CHOQUET CONTINUA

BY

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Abstract. Fedorchuk's fully closed (continuous) maps and resolutions are applied in constructions of non-metrizable higher-dimensional analogues of Anderson, Choquet, and Cook's rigid continua. Certain theorems on dimension-lowering maps are proved for inductive dimensions and fully closed maps from spaces that need not be hereditarily normal, and some of the examples of continua we construct have non-coinciding dimensions.

Fully closed (continuous) maps and resolutions appear in numerous constructions (see S. Watson [48], V. V. Fedorchuk [26] for surveys), in particular, in constructions of homogeneous spaces with non-coinciding dimensions. In this paper we apply such maps in order to obtain examples of continua with strong hereditary rigidity properties.

A non-degenerate continuum X is called

- an *Anderson–Choquet continuum* if every non-degenerate subcontinuum P of X has exactly one embedding $P \to X$, the identity id_P;
- a *Cook continuum* if every non-degenerate subcontinuum P of X has exactly one non-constant map $P \to X$, the identity id_P.

Examples were constructed by R. D. Anderson and G. Choquet [2] (a plane hereditarily decomposable continuum), and H. Cook [14] (a metric, one-dimensional, hereditarily indecomposable continuum). T. Maćkowiak [37] constructed a metric, chainable, hereditarily decomposable Cook continuum.

All known examples of such continua are one-dimensional. A metric Cook continuum must have dimension ≤ 2 (Maćkowiak [38]), and if it is hereditarily indecomposable, then it must be one-dimensional (Krzempek [32]). On the other hand, several authors investigated rigidity properties of higher-dimensional continua (J. J. Charatonik [8], M. Reńska [45], E. Pol

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[40–43, 34]; see [32] for more references). In [32] the present author constructed a metric, n-dimensional (for arbitrary $n > 1$), hereditarily indecomposable continuum no two of whose disjoint n-dimensional subcontinua are homeomorphic, but he was not able to ensure that the continuum be Anderson–Choquet.

In this paper we achieve better results for the non-metric case: we construct examples of non-metrizable higher-dimensional (with respect to the dimensions dim, ind, and Ind) Anderson–Choquet continua and Cook continua. Another aim is to study the behavior of the Charalambous–Filippov–Ivanov inductive dimension Ind_0 under fully closed maps.

In Sections 1–3 we gather some facts about fully closed maps, ring-like maps, and dimensions. Application of Ind_0 simplifies estimating inductive dimensions in some well-known examples. We show that *if f is a fully closed map from a non-empty compact space X to a first countable space, then* $\mathrm{Ind}_0 X \leq \mathrm{Ind}_0 fX + \mathrm{Ind}_0 f$. This enables us to prove that (1) Fedorchuk's first countable compact spaces [20] with $\dim = n < 2n-1 \leq \mathrm{ind} \leq 2n$ also have $\mathrm{Ind} \leq \mathrm{Ind}_0 = 2n$, and (2) V. A. Chatyrko's chainable continua and homogeneous continua [11] with $\dim = 1$ and $\mathrm{ind} = n$ also have $\mathrm{Ind} = \mathrm{Ind}_0 = n$. In Section 3 we slightly modify the Fedorchuk–Emeryk–Chatyrko resolution theorem, and that is our main tool for constructions.

Section 4 contains a construction of hereditarily indecomposable Anderson–Choquet continua with $\dim = n$ (arbitrary $n > 1$) and a construction of a Cook continuum with $\dim = 2$. In Section 5 we obtain chainable (hence, $\dim = 1$), hereditarily decomposable Cook continua with $n \leq \mathrm{ind} \leq \mathrm{Ind} \leq \mathrm{Ind}_0 = n+1$. All the continua are separable and first countable, and some have $\dim < \mathrm{ind}$.

1. Preliminaries: continua, maps, and covering dimension. By a *space* we mean a regular T_1 topological space, and all *maps* considered are continuous and closed. A *continuum* is a non-empty connected compact space. A subcontinuum A of a space X is said to be *terminal* if for every continuum $B \subset X$, either $A \cap B = \emptyset$, $A \subset B$, or $B \subset A$. A continuum is said to be

- *decomposable* if it is the union of two proper subcontinua;
- *hereditarily decomposable* (abbrev. HD) if each of its non-degenerate subcontinua is decomposable;
- *hereditarily indecomposable* (abbrev. HI) if none of its subcontinua is decomposable (equivalently, each of its subcontinua is terminal);
- *chainable* if for every open cover, there exists a natural number n and a closed refinement $F_1, \ldots, F_n$ of the cover such that $F_i \cap F_j \neq \emptyset$ iff $|i-j| \leq 1$.

A (closed continuous) map $f : X \to Y$ is said to be

- *irreducible* if for every closed proper subset $F \subsetneq X$, also $fF \subsetneq fX$;
- *monotone* [*atomic*] if for every point $y \in Y$, the pre-image $f^{-1}y$ is a continuum [respectively, a terminal continuum];
- *ring-like* if for every point $x \in X$ and every pair of open sets $U \ni x$ and $V \ni fx$, there is an open set $V' \ni fx$ such that $V' \subset V$ and $f^{-1}\operatorname{bd} V' \subset U$;
- *fully closed* if for every pair of disjoint closed subsets $F, G \subset X$, the intersection $fF \cap fG$ is a discrete subspace of Y ([1]).

We shall frequently use the simple fact that *if f is a fully closed* [*ring-like*] *map from a space X and $X' \subset X$ is closed, then also the restriction $f|X'$ is fully closed* [*respectively, ring-like*].

The following proposition is well-known (cf. [26, Proposition II.3.10]).

1.1. Proposition. *Suppose X and Y are compact spaces, and $f: X \to Y$ is a map with every point-inverse metrizable. If the set $C_2f = \{y \in Y : \operatorname{card} f^{-1}y > 1\}$ is countable and Y is metrizable, then X is metrizable. The converse is true if moreover f is a fully closed map.*

Proof. If $C_2f = \{y_i : i = 1, 2, \dots\}$ and Y is metrizable, then let $\mathcal{B}_i$ and $\mathcal{B}$ be countable bases for $f^{-1}y_i$ and Y, respectively. As readily seen, the family $\{f^{-1}B : B \in \mathcal{B}\} \cup \bigcup_i \mathcal{B}_i$ is a countable network for X. It follows that X has a countable base (cf. R. Engelking [18, Theorem 3.1.19]).

Assume that f is fully closed and there is a metric on X. Using sequential compactness of X, one easily checks that the set $\{y \in Y : \operatorname{diam} f^{-1}y \geq 1/n\}$ is finite for every n. Thus, C_2f is countable. ∎

1.2. Proposition (A. Emeryk and Z. Horbanowicz [17, Theorem 1]). *A map f from a continuum X is atomic iff $A = f^{-1}fA$ for every continuum $A \subset X$ such that fA is not a single point.* ∎

The point of the foregoing proposition is that if the irreducibility condition $A = f^{-1}fA$ is satisfied for all subcontinua $A \subset X$ with non-degenerate images, then f is a monotone map.

1.3. Remark. It is easily seen that any ring-like map $f : X \to Y$ has an even stronger property: it is *connected irreducible,* i.e. $A = f^{-1}fA$ for every closed subspace $A \subset X$ such that fA is connected and contains more than one point (see [26, II.1.15]). In particular, f is irreducible whenever $Y = fX$ is connected and contains more than one point.

([1]) An extensive survey [26] by Fedorchuk is devoted to fully closed maps, ring-like maps, and their applications. See [26, Section II.1] for equivalent definitions of fully closed maps. For terms not explicitly defined herein the reader is directed to the monographs by R. Engelking [18, 19] and K. Kuratowski [35].

1.4. Proposition. *If f is a ring-like map from a compact space X onto a non-degenerate continuum, then X is a continuum and f is an atomic map.*

Proof. Assume that $f : X \to Y$ is a ring-like map, X is compact, and fX is a non-degenerate continuum. Suppose that $X = A \cup B$, where A and B are non-empty, closed, and disjoint. By Remark 1.3, the complement $fX \setminus fA$ is non-empty. Consider any component M of $fX \setminus fA$. By Janiszewski's boundary bumping theorem (see K. Kuratowski [35, §47, III, Theorem 2]), the closure $\operatorname{cl} M$ meets fA, and is not a single point. Since $\operatorname{cl} M \subset fB$ and f is connected irreducible, we obtain $f^{-1} \operatorname{cl} M \subset B$, a contradiction. Therefore, X is a continuum. Finally, f is an atomic map by Proposition 1.2 and Remark 1.3. ∎

The next useful proposition belongs to folklore (see Fedorchuk [26, proof of Lemma III.3.3] and [27, Proposition 1.3]).

1.5. Proposition. *Suppose that f is a* (*closed*) *monotone map from a space X, and $A, B \subset fX$ are disjoint closed sets. If L is a partition* (2) *in X between $f^{-1}A$ and $f^{-1}B$, then fL is a partition in fX between A and B.* ∎

1.6. Proposition. *Suppose that $f : X \to Y$ and $g : Y \to Z$ are surjective ring-like maps between compact spaces X, Y, and Z. If g is a monotone map, then the composition gf is ring-like.*

Proof. Take $x \in X$ and open sets $U \ni x$ and $W \ni z = gfx$. We can assume that $gfU \subset W$. Since f is ring-like, there is an open set $V' \ni fx$ such that $\operatorname{cl} V' \subset g^{-1}W$ and $f^{-1} \operatorname{bd} V' \subset U$. If $g^{-1}z$ is not a singleton, we can moreover have $g^{-1}z \not\subset \operatorname{cl} V'$. Clearly, $\operatorname{bd} V' \subset Y \setminus f(X \setminus U)$. There are two cases.

(1) If $g^{-1}z$ is a non-degenerate continuum, then there is a point $y \in \operatorname{bd} V'$ with $gy = z$. The set $Y \setminus f(X \setminus U)$ is an open neighborhood of y, and there is an open set $W' \ni z$ such that $W' \subset W$ and $g^{-1} \operatorname{bd} W' \subset Y \setminus f(X \setminus U)$. Thus, $(gf)^{-1} \operatorname{bd} W' \subset U$.

(2) If $g^{-1}z$ is a single point, then $\operatorname{bd} V'$ is a partition in Y between fx and $g^{-1}(Z \setminus W)$. By Proposition 1.5, $g \operatorname{bd} V'$ is a partition in Z between z and $Z \setminus W$. Hence, there is an open set $T \subset \operatorname{cl} T \subset W$ such that $z \in T$ and $\operatorname{bd} T \subset g \operatorname{bd} V'$. As g is ring-like and $\operatorname{bd} T \subset W \cap g[Y \setminus f(X \setminus U)]$, for each $t \in \operatorname{bd} T$ there is an open set $W_t \subset W$ such that $g^{-1} \operatorname{bd} W_t \subset Y \setminus f(X \setminus U)$. Then $\operatorname{bd} T \subset W_{t_1} \cup \cdots \cup W_{t_n}$, where $t_1, \ldots, t_n \in \operatorname{bd} T$. We put $W' = T \cup W_{t_1} \cup \cdots \cup W_{t_n} \subset W$, and have $\operatorname{bd} W' \subset \operatorname{bd} W_{t_1} \cup \cdots \cup \operatorname{bd} W_{t_n}$. Thus, we obtain $g^{-1} \operatorname{bd} W' \subset Y \setminus f(X \setminus U)$ and $(gf)^{-1} \operatorname{bd} W' \subset U$. ∎

(2) We say that a closed set $L \subset X$ is a *partition in X* if the complement $X \setminus L$ is not connected. Moreover, L is a *partition in X between disjoint sets* $A, B \subset X$ if there are disjoint open sets $U, V \subset X$ such that $X \setminus L = U \cup V$, $A \subset U$, and $B \subset V$.

1.7. Proposition (cf. Chatyrko [11, Proposition 2]). *Suppose* $f : X \to Y$ *is a surjective ring-like map from a compact space* X, *and* $g : Y \to Z$ *is a surjective monotone map without degenerate point-inverses. If every partition in* Y *contains a point-inverse of* g, *then every partition in* X *contains a point-inverse of the composition* gf.

Proof. Assume that $X \neq \emptyset$ and every partition in Y contains a point-inverse of g. Since the empty set is not a partition in Y, Y is a non-degenerate continuum. Hence, f is irreducible by Remark 1.3, and monotone by Proposition 1.4. Take a partition L in X. The irreducibility of f implies that L is a partition between some point-inverses $f^{-1}a$ and $f^{-1}b$, where $a, b \in Y$. By Proposition 1.5, fL is a partition in Y between a and b. Then $g^{-1}z \subset fL$ for some $z \in Z$, and again by Remark 1.3, $f^{-1}g^{-1}z \subset L$. ∎

1.8. Proposition (implicit in Chatyrko [10]). *If* f *is a ring-like map from a compact space* X, *then* $\dim fX \leq \dim X$.

Proof (cf. Chatyrko [10, p. 124]). We shall prove that for every natural number n, the inequality $n \leq \dim fX$ implies $n \leq \dim X$. For $n = 0$, this is obvious. For $n = 1$, fX contains a non-degenerate continuum Y. Then, by Proposition 1.4, $f^{-1}Y$ is a non-degenerate continuum, and hence $1 \leq \dim X$.

Let us recall that a normal space Y has $\dim Y \geq n$ iff there exists an essential family $(A_1, B_1), \ldots, (A_n, B_n)$ in Y, i.e. $A_i, B_i \subset Y$ are disjoint closed subsets for each i, and for any partitions L_i between A_i and B_i, the intersection $\bigcap_{i=1}^n L_i$ is non-empty (cf. Engelking [19, Theorem 3.2.6]).

Let $2 \leq n \leq \dim fX$. Since fX contains a component of dimension $\geq n$, we can assume that fX is a continuum. Then f is a monotone map by Proposition 1.4. Take an essential family $(A_1, B_1), \ldots, (A_n, B_n)$ in fX. We shall show that the pre-images $(f^{-1}A_1, f^{-1}B_1), \ldots, (f^{-1}A_n, f^{-1}B_n)$ form an essential family in X. If L_i are partitions in X between $f^{-1}A_i$ and $f^{-1}B_i$, then fL_i are partitions in fX between A_i and B_i (Proposition 1.5). By Lemma 5.2 in [47], the intersection $\bigcap_{i=2}^n fL_i$ contains a continuum P which meets both A_1 and B_1. Since $f(L_i \cap f^{-1}P) = P$ for $i = 2, \ldots, n$, Remark 1.3 implies that $f^{-1}P = L_i \cap f^{-1}P$ and $f^{-1}P \subset \bigcap_{i=2}^n L_i$. As f is monotone, $f^{-1}P$ is a continuum and meets L_1, and hence $\bigcap_{i=1}^n L_i$ is non-empty. Therefore, $n \leq \dim X$. ∎

The fiberwise covering dimension of a map $f : X \to Y$ is defined as

$$\dim f = \sup\{\dim f^{-1}y : y \in Y\}.$$

Other fiberwise dimension functions ind, Ind, etc. for maps are defined similarly.

1.9. Theorem (Fedorchuk, see [26, Theorem III.2.4]). *If* f *is a fully closed map from a normal space* X, *then* $\dim X \leq \max\{\dim fX, \dim f\}$. ∎

The following is a consequence of Theorem 1.9 and Proposition 1.8.

1.10. COROLLARY. *If f is a ring-like fully closed map from a compact space X, then* $\dim X = \max\{\dim fX, \dim f\}$. ∎

2. Maps that reduce inductive dimensions. Since the theory of Ind is unsatisfactory outside the class of hereditarily normal spaces, we shall use another inductive dimension function Ind_0, which was introduced by M. G. Charalambous [5, 6] and A. V. Ivanov [31].

2.1. DEFINITION. For normal spaces X, $\mathrm{Ind}_0 X \in \{-1, 0, 1, 2, \ldots, \infty\}$ is defined so that

(a) $\mathrm{Ind}_0 X = -1$ iff $X = \emptyset$;
(b) $\mathrm{Ind}_0 X \leq n \geq 0$ iff for every pair of disjoint closed sets $A, B \subset X$, there is a G_δ partition L between A and B such that $\mathrm{Ind}_0 L \leq n-1$;
(c) $\mathrm{Ind}_0 X = n$ iff $\mathrm{Ind}_0 X \leq n$ and it is not true that $\mathrm{Ind}_0 X \leq n-1$;
(d) $\mathrm{Ind}_0 X = \infty$ if for every $n \in \mathbb{N}$, it is not true that $\mathrm{Ind}_0 X \leq n$.

It is clear that $\mathrm{Ind}\, X \leq \mathrm{Ind}_0 X$ for every normal space X, and $\mathrm{Ind}\, X = \mathrm{Ind}_0 X$ if X is perfectly normal.

2.2. COUNTABLE SUM THEOREM FOR Ind_0 (Charalambous [6], Ivanov [31]). *Suppose that $X = \bigcup_{i=1}^{\infty} F_i$ is a normal space, and F_i are closed G_δ-subsets of X. If $\mathrm{Ind}_0 F_i \leq n$ for every i, then $\mathrm{Ind}_0 X \leq n$.* ∎

The assumption that F_i are G_δ-sets is necessary in Theorem 2.2 even if X is a hereditarily normal compact space (see [31]). Besides [6, 31], see Charalambous and Chatyrko [7] for more (also bibliographical) information on Ind_0.

The following theorem on dimension-lowering fully closed maps seems to be important because of its applications.

2.3. THEOREM. *If f is a fully closed map from a non-empty normal space X to a space each of whose discrete closed subspaces is a G_δ-set, then* $\mathrm{Ind}_0 X \leq \mathrm{Ind}_0 fX + \mathrm{Ind}_0 f$.

We shall modify the proof of Theorem III.2.8 in [26]. First, we need some standard preparation (see [26, pp. 4213–4216] for details). Let $f : X \to Y$ be a map, and $M \subset Y$ be an arbitrary set. Consider the decomposition

$$\mathcal{M} = \{f^{-1}y : y \in Y \setminus M\} \cup \{\{x\} : x \in f^{-1}M\}$$

of X. Let $Y^M = X/\mathcal{M}$ be the quotient space, $f^M : X \to Y^M$ the natural quotient projection, and $\pi^M : Y^M \to Y$ the only map such that $f = \pi^M f^M$. *If f is fully closed, then $\mathcal{M}$ is upper semicontinuous, Y^M is a regular space, and f^M, π^M are fully closed maps.*

A proof of this lemma (cf. [19, Lemma 1.2.9]) is routine:

2.4. Lemma. *Suppose that $M, A, B \subset X$ are closed subsets of a normal space X, $A \cap B = \emptyset$, and L is a partition in M between $M \cap A$ and $M \cap B$. If $X \setminus L$ is a normal space, then there are disjoint open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$, $M \setminus L = (U \cup V) \cap M$, and $\operatorname{cl} U \cap \operatorname{cl} V \subset L$.* ∎

Proof of Theorem 2.3. We start with some general construction for arbitrary X, f, $Y = fX$, and disjoint closed sets $A, B \subset X$. We can assume that $p = \operatorname{Ind}_0 Y < \infty$ and $q = \operatorname{Ind}_0 f < \infty$. Clearly $p, q \geq 0$. Since f is fully closed, $M = fA \cap fB$ is a discrete closed subspace of Y. Consider Y^M, $f^M : X \to Y^M$, and $\pi^M : Y^M \to Y$. The restriction $f^M|f^{-1}M$ is a homeomorphism onto $N = (\pi^M)^{-1}M$, and we shall construct a G_δ partition in Y^M between the disjoint sets $f^M A$ and $f^M B$. The pre-image $f^{-1}M$ is homeomorphic to the discrete sum of point-inverses $f^{-1}y$, $y \in M$, and hence $\operatorname{Ind}_0 N \leq q$. There is a G_δ partition L in N between $N \cap f^M A$ and $N \cap f^M B$, where $\operatorname{Ind}_0 L \leq q - 1$. As f^M is a closed map, Y^M is a normal space. Since $M \subset Y$ is a G_δ-set, $Y^M \setminus N$ and $Y^M \setminus L$ are F_σ-sets in Y^M, and hence they are also normal spaces (see [18, Exercise 2.1.E]). By Lemma 2.4, there are disjoint open sets $U, V \subset Y^M$ such that $f^M A \subset U$, $f^M B \subset V$, $N \setminus L = (U \cup V) \cap N$, and $\operatorname{cl} U \cap \operatorname{cl} V \subset L$. As $Y \setminus M$ is an open F_σ-subset of Y, it is a countable union of closed G_δ-subsets F_i of Y. Since $\pi^M|Y^M \setminus N$ is a homeomorphism onto $Y \setminus M$, we obtain $\operatorname{Ind}_0(Y^M \setminus N) \leq p$ by Theorem 2.2. Thus, there are disjoint open sets $U', V' \subset Y^M \setminus N$ such that $\operatorname{cl} U \setminus N \subset U'$, $\operatorname{cl} V \setminus N \subset V'$, and $L' = Y^M \setminus (N \cup U' \cup V')$ is a G_δ-set with $\operatorname{Ind}_0 L' \leq p - 1$. Observe that

$$L \cup L' = Y^M \setminus (U \cup U' \cup V \cup V')$$

is a G_δ partition in Y^M between $f^M A$ and $f^M B$. Therefore, $(f^M)^{-1}(L \cup L')$ *is a G_δ partition in X between A and B.*

We now proceed by induction on p. If $p = 0$, then L' is empty and $\operatorname{Ind}_0 (f^M)^{-1}(L \cup L') = \operatorname{Ind}_0 L \leq q - 1$. As we took arbitrary sets A and B, we have $\operatorname{Ind}_0 X \leq q = p + q$. Assume the conclusion of the theorem is true for fully closed maps whose images have $\operatorname{Ind}_0 < p > 0$. Then L' is the countable union of closed G_δ-sets $L_i = L' \cap (\pi^M)^{-1} F_i \subset Y^M$ with $\operatorname{Ind}_0 L_i \leq p - 1$. The restrictions $f^M|(f^M)^{-1}L_i : (f^M)^{-1}L_i \to L_i$ are fully closed, and by the induction hypothesis we obtain $\operatorname{Ind}_0 (f^M)^{-1}L_i \leq p + q - 1$. Since L and $(f^M)^{-1}L$ are homeomorphic, we have $\operatorname{Ind}_0 (f^M)^{-1}(L \cup L') \leq p + q - 1$ by Theorem 2.2. We have shown that $\operatorname{Ind}_0 X \leq p + q$ because $(f^M)^{-1}(L \cup L')$ is a G_δ partition between disjoint closed sets A and B, which were taken arbitrarily. ∎

2.5. Corollary. *Suppose that f is a fully closed map from a non-empty normal space X onto a perfectly normal space. If every point-inverse of f is perfectly normal, then* $\operatorname{Ind} X \leq \operatorname{Ind} fX + \operatorname{Ind} f$. ∎

The foregoing corollary may be considered as an Ind-analogue of Fedorchuk's Theorem 4 in [20], which was stated for ind and special, resolution fully closed maps f. In a recent paper [28, pp. 117–120] Fedorchuk proves the inequality for Ind and resolution maps f, where fX are (metric, compact) two-manifolds.

Theorem 2.3 and Corollary 2.5 enable estimating inductive dimensions in some well-known constructions (see [26] for a survey). In particular, Fedorchuk's continua B ([20])—let us write B_n instead—were the first examples of *separable and first countable* compact spaces with non-coinciding dimensions dim and ind. Fedorchuk proved that $\dim B_n = n$ and $2n-1 \le \operatorname{ind} B_n \le 2n$. Since each B_n has a fully closed map onto the n-dimensional sphere, and every point-inverse of the map is homeomorphic to the n-dimensional torus, we obtain

2.6. Corollary. *Fedorchuk's continua B_n also have*

$$\operatorname{Ind} B_n \le \operatorname{Ind}_0 B_n \le 2n. \ \blacksquare$$

In fact, we shall see that $\operatorname{Ind}_0 B_n = 2n$ by Theorem 2.12.

Chatyrko [11] constructed separable first countable continua I_n and $(S_1)_n$. He proved I_n are chainable, $(S_1)_n$ are homogeneous, $\dim I_n = \dim (S_1)_n = 1$, and $\operatorname{ind} I_n = \operatorname{ind} (S_1)_n = n$.

2.7. Corollary. *Chatyrko's continua I_n and $(S_1)_n$ have also*

$$\operatorname{Ind} I_n = \operatorname{Ind} (S_1)_n = \operatorname{Ind}_0 I_n = \operatorname{Ind}_0 (S_1)_n = n.$$

Proof. There is a sequence $\cdots \xrightarrow{\pi_n^{n+1}} I_n \xrightarrow{\pi_{n-1}^n} \cdots \xrightarrow{\pi_2^3} I_2 \xrightarrow{\pi_1^2} I_1 = [0,1]$ of fully closed onto maps π_n^{n+1} (see [11]). For each n and every $t \in I_n$, the pre-image $(\pi_n^{n+1})^{-1}t$ is homeomorphic to $[0,1]$. Using induction and Theorem 2.3, we obtain $\operatorname{Ind}_0 I_n \le n$.

In the case of $(S_1)_n$ there exists an analogous sequence of maps, whose point-inverses are homeomorphic to a circumference. $\blacksquare$

In Corollary 2.5 one can replace those perfectly normal spaces by another class of spaces in which $\operatorname{Ind} = \operatorname{Ind}_0$. This could be the class of hereditarily perfectly κ-normal spaces (Fedorchuk [24]); surely, every discrete closed subset of fX should be G_δ in fX.

The assumption that f is fully closed is necessary in Corollary 2.5. Under a set-theoretical assumption consistent with ZFC, Fedorchuk [23] constructed a perfect map $f_{\mathrm{F}} : X_{\mathrm{F}} \to Y_{\mathrm{F}}$, where X_{F} and Y_{F} are perfectly normal, locally compact, and countably compact spaces, $\dim X_{\mathrm{F}} = \operatorname{Ind} X_{\mathrm{F}} = 1$, $\operatorname{Ind} Y_{\mathrm{F}} = 0$, and $\operatorname{Ind} f_{\mathrm{F}}^{-1} y = 0$ for every $y \in Y_{\mathrm{F}}$. On the other hand, it is not sufficient to assume only that f is fully closed. Chatyrko [12] has constructed a certain fully closed map $f_{\mathrm{Ch}} : X_{\mathrm{Ch}} \to A_{\mathfrak{c}}$ from a compact space X_{Ch} with $\operatorname{Ind} X_{\mathrm{Ch}} = \operatorname{Ind}_0 X_{\mathrm{Ch}} = 2$ onto a compact space $A_{\mathfrak{c}}$ with a unique accumulation point y_0 and $\operatorname{card} A_{\mathfrak{c}} = \mathfrak{c}$. All point-inverses $f_{\mathrm{Ch}}^{-1}y$, where $y_0 \neq y \in A_{\mathfrak{c}}$, are single points, and $1 = \operatorname{Ind} f_{\mathrm{Ch}}^{-1}y_0 < \operatorname{Ind}_0 f_{\mathrm{Ch}}^{-1}y_0 = 2$. For some other maps $f : X \to Y$, even $\operatorname{Ind} X - \operatorname{Ind} fX - \operatorname{Ind} f > 1$. For every pair of natural numbers $m > n \geq 1$, the present author [33] has constructed a compact space $X_{m,n}$ such that $\operatorname{Ind} X_{m,n} = m$ and every component of $X_{m,n}$ is homeomorphic to the n-dimensional cube $[0,1]^n$. Consequently, if $\mathcal{D}$ stands for the decomposition of $X_{m,n}$ into components, and $f_{\mathrm{K}} : X_{m,n} \to X_{m,n}/\mathcal{D}$ is the natural quotient map (it is not fully closed), then $\operatorname{Ind} X_{m,n} - \operatorname{Ind} X_{m,n}/\mathcal{D} - \operatorname{Ind} f_{\mathrm{K}} = m - n$.

2.8. Lemma. *Suppose that f is a fully closed map from a normal space X, $L \subset X$ is a closed G_δ-set, and $A, B \subset fX$ are disjoint closed sets. Then*

(a) *$fL \cap f(X \setminus L)$ is the countable union of discrete closed subspaces of fX* (cf. [25, Definition 3 and Lemma 2]).

If moreover every discrete closed subspace of fX is a G_δ-subset, then

(b) *fL is a G_δ-set in fX;*
(c) *whenever L is a partition between $f^{-1}A$ and $f^{-1}B$, there is a countable family of discrete closed sets $\Gamma_i \subset fX \setminus (A \cup B)$ such that the union $fL \cup \bigcup_i \Gamma_i$ is a G_δ partition in fX between A and B.*

Proof. (a) There is a sequence of closed sets $F_i \subset X$ such that $X \setminus L = \bigcup_i F_i$. The intersections $fF_i \cap fL$ are discrete and closed, and $f(X\setminus L) \cap fL = \bigcup_i (fF_i \cap fL)$.

(b) If $fF_i \cap fL$ are G_δ in fX, then $fF_i \setminus fL$ are F_σ. Since $fX \setminus fL = \bigcup_i (fF_i \setminus fL)$, fL is G_δ.

(c) appears to be implicitly shown in the proof of [26, Theorem III.2.6] if one can use (b). The new point is to apply Lemma 2.8(b) and prove that if $P = L$ in [26, p. 4247] is G_δ, then fP, $U_i \cup f^{-1}fP$, $f(U_i \cup f^{-1}fP)$, and $K = f(U_1 \cup f^{-1}fP) \cap f(U_2 \cup f^{-1}fP) = fP \cup \bigcup_{j,k} \Gamma_{jk}$ are G_δ-sets. ∎

Applying induction, Lemma 2.8(c), Theorem 2.2, and Proposition 1.4, we obtain the following two theorems. (The first one is an Ind_0-analogue of Theorem III.2.6 on Ind in [26].)

2.9. Theorem. *If f is a fully closed map from a normal space X to a space each of whose discrete closed subspaces is a G_δ-set, then* $\operatorname{Ind}_0 fX \leq \operatorname{Ind}_0 X + 1$. ∎

2.10. Theorem. *If f is a ring-like fully closed map from a compact space X to a first countable space, then* $\operatorname{Ind}_0 fX \leq \operatorname{Ind}_0 X$. ∎

2.11. Lemma. *If f is a ring-like fully closed map from a compact space X onto a non-degenerate continuum, then every G_δ partition in X contains a point-inverse of f.*

Proof. Take a G_δ partition L in X. There are two cases.

(1) If fL is uncountable, then by Lemma 2.8(a), $fL \cap f(X \setminus L)$ is countable, $fL \setminus f(X \setminus L) \ni y$ for some $y \in fX$, and $f^{-1}y \subset L$.

(2) Suppose that fL is countable. Then L contains a *thin* partition F, i.e. there are disjoint non-empty open sets $U_1, U_2 \subset X$ such that $X \setminus F = U_1 \cup U_2$ and $F = \operatorname{bd} U_1 = \operatorname{bd} U_2$. By Remark 1.3 and Proposition 1.4, f is a monotone irreducible map, and X is a continuum. By the same argument as in Fedorchuk [25, proof of Lemma 4, p. 167], we infer that $f^{-1}x \subset F$ for every point $x \in fX$ isolated in fF. ∎

2.12. THEOREM. *If f is a ring-like fully closed map from a non-empty compact space X onto a first countable space Y, and $\operatorname{Ind}_0 f^{-1}y = \operatorname{Ind}_0 f$ for every $y \in Y$, then $\operatorname{Ind}_0 X = \operatorname{Ind}_0 Y + \operatorname{Ind}_0 f$.*

Proof. Theorem 2.3 yields the inequality "$\le$". Fix $m = \operatorname{Ind}_0 f$. The inequality "$\ge$" will be proved by induction on $n = \operatorname{Ind}_0 X < \infty$. Clearly, $n \ge m$. Let $n = m$, and suppose that $\operatorname{Ind}_0 Y > 0$. Then, Y has a non-degenerate component Y'. By Lemma 2.11, every G_δ partition L in $X' = f^{-1}Y'$ contains a point-inverse of f and has $\operatorname{Ind}_0 L \ge m$. Hence, $\operatorname{Ind}_0 X' > n$, a contradiction. Thus, $\operatorname{Ind}_0 Y = 0$ and the inequality "$\ge$" is true.

Assume that $n > m$. In order to show that $\operatorname{Ind}_0 Y \le n - m$, take a pair of disjoint closed sets $A, B \subset Y$. There is a G_δ partition L in X between $f^{-1}A$ and $f^{-1}B$, with $\operatorname{Ind}_0 L \le n - 1$. By Lemma 2.8(c) there is a G_δ partition $K \supset fL$ in Y between A and B, where $K \setminus fL$ is countable. Lemma 2.8(a) implies that also $K \cap f(X \setminus L) = \{y_i : i = 1, 2, \ldots\}$. We have $f^{-1}K = L \cup \bigcup_i f^{-1}y_i$, and $\operatorname{Ind}_0 f^{-1}K \le n - 1$ by Theorem 2.2. By the induction hypothesis, $\operatorname{Ind}_0 K \le n - m - 1$. We have shown that $\operatorname{Ind}_0 Y \le n - m$. ∎

The following statement may be considered as a generalization of Theorem 3 in Fedorchuk [20].

2.13. THEOREM. *Suppose that $f : X \to Y$ is a surjective ring-like map between compact spaces X and Y, and $\dim Y \ge 1$. If $\operatorname{ind} f^{-1}y \ge m$ for every $y \in Y$, then $\operatorname{ind} X \ge \dim Y + m - 1$.*

Proof. If $\dim Y = \infty$, then $\operatorname{ind} X \ge \dim X = \infty$ by Proposition 1.8 (and [19, Theorem 3.1.29]). We can assume that $\dim Y < \infty$. It suffices to prove by induction that for every natural number $k \ge 1$, the inequality $\dim Y \ge k$ implies $\operatorname{ind} X \ge k + m - 1$. For $k = 1$ the implication is obvious.

We shall use the following classical notion. A compact space M with $\dim M = n$ is called an *n-dimensional Cantor manifold* provided that every partition L in M has $\dim L \ge n - 1$. P. S. Aleksandrov [1] proved that every compact space Z with $1 \le n = \dim Z < \infty$ contains an n-dimensional Cantor manifold (see also [19, Theorem 1.9.9 and Exercise 3.2.F]).

Let $n = \dim Y \geq k \geq 2$. We can assume that Y is an n-dimensional Cantor manifold. Then X is a continuum and f is an irreducible monotone map by Remark 1.3 and Proposition 1.4. If L is a partition in X, then fL is a partition in Y (Proposition 1.5), and $\dim fL \geq n-1 \geq k-1$. Now, fL contains a component P with $\dim P \geq n-1$. Since $P = f(L \cap f^{-1}P)$, Remark 1.3 implies $f^{-1}P \subset L$. By the obvious induction hypothesis, $\operatorname{ind} L \geq \operatorname{ind} f^{-1}P \geq k+m-2$. Therefore, $\operatorname{ind} X \geq k+m-1$. ■

3. Main tools for constructions. For our constructions we need ingredients of two types. The following resolution theorem (the first type) is a modification of well-known results.

3.1. Theorem (cf. Chatyrko [10], Emeryk [15], Fedorchuk [20]). *Suppose that X is a first countable continuum, and for every $x \in X$, Y_x is a metrizable continuum. Then there exists a first countable continuum $Z = Z(X, Y_x)$ with a map $\pi : Z \to X$ such that*

(a) *for every $x \in X$, the pre-image $\pi^{-1}x$ is homeomorphic to Y_x;*
(b) *π is ring-like and fully closed.*

Moreover, the conjunction of (a) *and* (b) *implies that*

(c) $\dim Z = \max\{\dim X, \dim \pi\}$*;*
(d) *if X is perfectly normal, then* $\operatorname{Ind} Z \leq \operatorname{Ind} X + \operatorname{Ind} \pi$*;*
(e) *if X is separable, then so is Z;*
(f) *if all Y_x are non-degenerate continua, and $P \subset Z$ is a metrizable continuum, then the image πP is a single point;*
(g) *if X and all Y_x are hereditarily indecomposable [hereditarily decomposable], then so is Z.*

Sketch of proof. Since Chatyrko's paper [10] has not been translated into English, we sketch his construction for the convenience of the reader ([3]). We can assume that each Y_x is a subspace of the Hilbert cube $[0,1]^\infty$. Using the local connectedness of $[0,1]^\infty$, one constructs a map $g_x : (0,1] \to [0,1]^\infty$ such that for each natural number n, $Y_x \subset \operatorname{cl} g_x(0, 1/n] \subset \mathrm{B}(Y_x, 1/n)$, where B stands for a ball. One takes a map $f_x : X \to [0,1]$ with $f^{-1}0 = \{x\}$, and writes $h_x : X \setminus \{x\} \to [0,1]^\infty$ for the composition $g_x(f_x | X \setminus \{x\})$. Set $Z = \bigcup \{\{x\} \times Y_x : x \in X\} \subset X \times [0,1]^\infty$, and $\pi : Z \to X$, $\pi(x,y) = x$. The topology on Z is generated by a base of neighborhoods at any point $(x,y) \in Z$; the base consists of all sets

$$W(V,U) = [\{x\} \times (V \cap Y_x)] \cup \pi^{-1}(U \cap h_x^{-1} V),$$

([3]) In a forthcoming paper, joint with M. G. Charalambous, we describe a generalization of Chatyrko's construction in more detail.

where $U \subset X$ and $V \subset [0,1]^\infty$ are neighborhoods of $x \in X$ and $y \in Y_x$ respectively. The above is a generalization of Fedorchuk's construction [20] (cf. also Fedorchuk [25, the proof of Lemma 1] and [26, Section III.1]). One checks that Z is a first countable continuum, and π satisfies (a) and (b).

Corollary 1.10 yields (c), and Corollary 2.5 yields (d). Statement (e) is an easy consequence of the fact that π is an irreducible map (see Remark 1.3). Statement (f) follows from Remark 1.3 and Proposition 1.1, and (g) is a simple property of atomic maps. ∎

3.2. Remarks. (1) If a subcontinuum P of $Z = Z(X, Y_x)$ in Theorem 3.1 is non-metrizable, then $P = \pi^{-1}\pi P$ by Remark 1.3. Hence, *if all Y_x are non-degenerate continua, then for every non-degenerate continuum $P \subset Z$ and any point $z \in P$, there is a non-degenerate metrizable continuum Q such that $z \in Q \subset P$.*

(2) One can combine the proofs by Chatyrko [10] and Fedorchuk [25, Lemma 1] in order to obtain a map π that also satisfies assertion (2) of Lemma 1 in [25]. This enables one to construct the continuum $Z = Z(X, Y_x)$ under the continuum hypothesis (see [25, pp. 166–167]) so that

(†) *if* CH *is true, and the continuum X in Theorem* 3.1 *is perfectly normal and hereditarily separable, then Z is perfectly normal and hereditarily separable.*

We shall also need Cook continua (the second type of ingredients), whose subcontinua will be taken as the Y_x's of Theorem 3.1.

3.3. Example (Cook [14]; see A. Pultr and V. Trnková [44, Appendix A] for a detailed construction). *There exists a metrizable, one-dimensional, hereditarily indecomposable Cook continuum M_1 that does not contain non-degenerate continuous images of plane continua.*

Proof. J. W. Rogers, Jr. [46] observed that Cook's continuum [14] does not contain non-degenerate continuous images of plane continua. ∎

3.4. Example (Maćkowiak [37]). *There exists a metrizable, chainable, hereditarily decomposable Cook continuum.* ∎

4. Anderson–Choquet (and similar) continua with dim > 1. The following continua are neither Anderson–Choquet nor Cook, but they are HI analogues of Fedorchuk's spaces with non-coinciding dimensions ([20], cf. also our Corollary 2.6).

4.1. Theorem. *For every natural number $n \geq 1$, there exists a non-metrizable, separable, first countable, hereditarily indecomposable continuum Z such that* $\dim Z = n$ *and* $2n - 1 \leq \operatorname{ind} Z \leq \operatorname{Ind} Z \leq \operatorname{Ind}_0 Z = 2n$.

Proof. By a theorem of R. H. Bing [3], there exists a metric HI continuum X with $\dim X = n$. Let $Y_x = X$ for $x \in X$, apply Theorem 3.1, and put $Z = Z(X, Y_x)$.

The properties of Z are consequences of 2.12, 2.13, and 3.1(c–g). ■

Let us notice that the first examples of non-metrizable HI continua were constructed by Emeryk [16].

4.2. Theorem. *For every natural number $n \geq 1$, there exists a non-metrizable, separable, first countable, hereditarily indecomposable Anderson–Choquet continuum Z with $n = \dim Z \leq \operatorname{ind} Z \leq \operatorname{Ind} Z \leq \operatorname{Ind}_0 Z = n + 1$.*

Proof. Let X be a metric n-dimensional HI continuum (Bing [3]). Take a family $\{C_x : x \in X\}$ of pairwise disjoint non-degenerate subcontinua of Cook's continuum M_1 (Example 3.3), apply Theorem 3.1, and put $Z = Z(X, C_x)$.

Most of the desired properties of Z follow from Theorems 2.12 and 3.1(c–g). It remains to prove that Z is an Anderson–Choquet continuum. Let $\pi : Z \to X$ be the map of Theorem 3.1, and choose any non-degenerate continuum $P \subset Z$ and any embedding $\varphi : P \to Z$. Take an arbitrary point $z \in P$. By Remark 3.2(1), P contains a non-degenerate metrizable continuum $Q \ni z$. Theorem 3.1(f) guarantees that $Q \subset \pi^{-1}x$ and $\varphi Q \subset \pi^{-1}x'$ for some $x, x' \in X$. Since $\pi^{-1}x$ and $\pi^{-1}x'$ are homeomorphic to C_x and $C_{x'}$, respectively, we obtain $x = x'$, $\varphi|Q = \mathrm{id}_Q$, and $\varphi z = z$. We have shown that $\varphi = \mathrm{id}_P$. ■

We shall adapt the foregoing construction and proof in order to obtain hereditarily rigid finite-group actions. We start with some terminology. Let X be a space, and G a finite group. We write $H(X)$ for the group of all homeomorphisms $X \to X$. Every homomorphism $\xi : G \to H(X)$ is called a *G-action on X*; the value of ξ at $g \in G$ will be denoted by $g^\xi \in H(X)$. This G-action is said to be *fixed-point-free* if for each $g \in G \setminus \{e\}$, the homeomorphism $g^\xi : X \to X$ does not have a fixed point. Let $\zeta : G \to H(Y)$ be a G-action on a space Y. A map $f : X \to Y$ is said to be *equivariant* if $g^\zeta f = f g^\xi$ for each $g \in G$.

4.3. Theorem. *Suppose that X is a first countable continuum, G is a finite group, and ξ is a fixed-point-free G-action on X. Then there exists a first countable continuum Z with a fixed-point-free isomorphic G-action $\zeta : G \to H(Z)$ and an equivariant map $\pi : Z \to X$ such that*

(a) *π is a ring-like fully closed map, and all point-inverses of π are metrizable one-dimensional continua;*
(b) *for every non-degenerate continuum $P \subset Z$ and embedding $\varphi : P \to Z$, there is a homeomorphism $g^\zeta \in H(Z)$ such that $g^\zeta|P = \varphi$.*

An important point in this theorem is that $\dim Z = \dim X$ by Corollary 1.10, and $\operatorname{Ind}_0 Z = \operatorname{Ind}_0 X + 1$ by Theorem 2.12. In the proof it will be seen that we can ensure that all point-inverses of π are HI (if we use subcontinua of Example 3.3 in our construction), or alternatively, that they are HD (if we use Example 3.4).

We may apply this theorem to some standard examples of group actions. It is well-known that for every finite group G and every $n \geq 2$, there exists a (compact metric) n-manifold without boundary with a fixed-point-free G-action (see J. de Groot and R. J. Wille [30, p. 444]). In case $n = 1$, there exists a connected finite graph with a fixed-point-free G-action (G acts on its Cayley graph). Two simpler examples: Using Anderson and Choquet's original HD continuum [2], a Cayley graph of the group G, and the method from [30], one easily constructs a metric one-dimensional HD continuum Z with a fixed-point-free G-action ζ that is an isomorphism $G \to H(Z)$ and satisfies assertion (b) of Theorem 4.3. Using our Anderson–Choquet continuum (of Theorem 4.2) instead (and the same Cayley graph method), one constructs a non-metrizable, separable, and first countable continuum Z with $\dim Z = n$ and a similar G-action ζ on Z.

Let us notice that there are numerous papers on group representations in topology (see de Groot [29] for example) and in the more general context of category theory (see the bibliography in Pultr and Trnková [44]).

Proof of Theorem 4.3. (I) Consider the family $\mathcal{D}$ of all orbits $\{g^\xi x : g \in G\}$, $x \in X$, and the quotient space $X' = X/\mathcal{D}$. The quotient projection $q_\xi : X \to X'$ is a covering map. Take a family $\{C_{x'} : x' \in X'\}$ of pairwise disjoint non-degenerate subcontinua of Cook's continuum M_1 (Example 3.3). Use Theorem 3.1, and take $Z' = Z(X', C_{x'})$ and $\pi' : Z' \to X'$, a map that satisfies statements (a–g) of Theorem 3.1. Consider the set $Z = \bigcup_{x' \in X'} (q_\xi^{-1} x' \times \pi'^{-1} x') \subset X \times Z'$. We define $\pi(x,t) = x$, $q_\zeta(x,t) = t$, and $g^\zeta(x,t) = (g^\xi x, t)$ for $x \in X$, $t \in \pi'^{-1} q_\xi x$, and $g \in G$. The following diagram commutes:

$$\begin{tikzcd}
 & X \arrow[r, "q_\xi"] \arrow[dl, "g^\xi"'] & X' \\
X \arrow[urr, "q_\xi"'] & & \\
 & Z \arrow[uu, "\pi"'] \arrow[r, "q_\zeta"] \arrow[dl, "g^\zeta"'] & Z' \arrow[uu, "\pi'"'] \\
Z \arrow[uu, "\pi"] \arrow[urr, "q_\zeta"'] & &
\end{tikzcd}$$

and ζ, $g \mapsto g^\zeta$, is a monomorphism from G to the group of permutations of Z. Finally, we equip Z with the smallest topology such that π and q_ζ are continuous. Observe that if $U \subset X$ is an open set such that $q_\xi | U : U \to q_\xi U$

is a homeomorphism, then $q_\zeta|\pi^{-1}U : \pi^{-1}U \to \pi'^{-1}q_\xi U$ is also a homeomorphism. Indeed, $q_\zeta|\pi^{-1}U$ is one-to-one, all open subsets of $\pi^{-1}U$ have the form $q_\zeta^{-1}V \cap \pi^{-1}U$, where $V \subset Z'$ are open, and $q_\zeta(q_\zeta^{-1}V \cap \pi^{-1}U) = V \cap q_\zeta\pi^{-1}U = V \cap \pi'^{-1}q_\xi U$. It follows that q_ζ is a closed covering map, and hence Z is a compact space by [18, Theorem 3.7.2]. Similarly, g^ζ are homeomorphisms of Z. In view of the above diagram, π is an equivariant map.

(II) Every point-inverse $\pi^{-1}x$ is homeomorphic to $q_\zeta\pi^{-1}x = \pi'^{-1}q_\xi x$ and $C_{q_\xi x}$. As π is a monotone map, the connectedness of X implies the connectedness of Z. Using the fact that π' is ring-like, one easily checks that also π is ring-like.

In order to prove that π is fully closed take disjoint closed sets $A, B \subset Z$. By the compactness of X, it is sufficient to show that *if* $\operatorname{cl} U \subset X$ *and* $q_\xi|\operatorname{cl} U$ *is one-to-one, then* $\operatorname{cl} U$ *contains a finite number of points in* $\pi A \cap \pi B$. Indeed, consider the set $F = \pi^{-1}\operatorname{cl} U$ and the restriction $\pi|F$. The sets $g^\zeta F$, $g \in G$, are pairwise disjoint, and $q_\zeta|F$ is a one-to-one function. Thus, $q_\zeta(F\cap A)$ and $q_\zeta(F\cap B)$ are disjoint closed subsets of Z', and the intersection

$$\pi' q_\zeta(F \cap A) \cap \pi' q_\zeta(F \cap B) = q_\xi\pi(F \cap A) \cap q_\xi\pi(F \cap B) = q_\xi[\operatorname{cl} U \cap \pi A \cap \pi B]$$

is finite since π' is a fully closed map. Therefore, $\operatorname{cl} U \cap \pi A \cap \pi B$ is finite, and π is fully closed. Moreover, π satisfies statements analogous to Theorem 3.1(c–g) and Remark 3.2(1).

(III) Let $P \subset Z$ be a non-degenerate continuum, and $\varphi : P \to Z$ an embedding. We claim that *for every* $z \in P$ *there is a* $g \in G$ *such that* $\varphi z = g^\zeta z$. Indeed, by Remark 3.2(1), P contains a non-degenerate metrizable continuum $Q \ni z$. By a statement analogous to Theorem 3.1(f), there are $x, x' \in X$ such that $Q \subset \pi^{-1}x$ and $\varphi Q \subset \pi^{-1}x'$. The restrictions $q_\zeta|Q$ and $q_\zeta\varphi|Q$ are embeddings into $\pi'^{-1}q_\xi x$ and $\pi'^{-1}q_\xi x'$, respectively. Since these point-inverses are homeomorphic to $C_{q_\xi x}$ and $C_{q_\xi x'}$, respectively, we have $q_\xi x = q_\xi x'$, $q_\zeta|Q = q_\zeta\varphi|Q$, and $q_\zeta z = q_\zeta\varphi z$. Hence, there is a $g \in G$ such that $\varphi z = g^\zeta z$. The foregoing claim implies that $P = \bigcup_{g\in G} F_g$, where $F_g = \{z \in P : \varphi z = g^\zeta z\}$. The sets F_g are closed, pairwise disjoint, and P is connected. Hence, only one F_g is non-empty. Thus, there is a $g \in G$ such that $P = F_g$ and $\varphi = g^\zeta|P$.

When we apply (b) to $P = Z$, we infer that ζ is an isomorphism onto $H(Z)$. ■

A non-degenerate continuum X will be called a *weak Cook continuum* (4) if for every subcontinuum P of X, every map $f : P \to X$ with $P \cap fP = \emptyset$ is constant.

(4) There is some difference in terminology: in [36–38] our weak Cook continua are just called Cook continua.

4.4. Proposition (Maćkowiak [36, Proposition 29] (⁵)). *Suppose that X is a weak Cook continuum. If P is a subcontinuum of X and $f : P \to X$ is a non-constant map, then $fP \subset P$ and f is a monotone retraction.* ∎

4.5. Theorem. *There exists a non-metrizable, separable, first countable Cook continuum Z with* $2 = \dim Z \leq \operatorname{ind} Z \leq \operatorname{Ind} Z \leq \operatorname{Ind}_0 Z = 3$.

Proof. Take the square $[0,1]^2$ and a family $\{C_x : x \in [0,1]^2\}$ of pairwise disjoint non-degenerate subcontinua of Cook's continuum M_1. Put $Z = Z([0,1]^2, C_x)$, and let $\pi : Z \to [0,1]^2$ be the map of Theorem 3.1.

Most of the desired properties of Z follow from Theorems 2.12 and 3.1(c–g). We shall prove that Z is a weak Cook continuum. Assume that P is a subcontinuum of Z, and $f : P \to Z$ is a map with $P \cap fP = \emptyset$. We claim that *for every metrizable continuum $Q \subset P$, the restriction $f|Q$ is constant.* Indeed, fQ is a metrizable subcontinuum of Z. Theorem 3.1(f) implies that Q and fQ are homeomorphic to disjoint subcontinua of Cook's M_1. Hence, fQ is a single point. Thus, if P is metrizable, we are done. If not, $P = \pi^{-1}\pi P$ by Remark 1.3, and the above claim implies that there is a factorization $f = g(\pi|P)$, where $g : \pi P \to Z$ is continuous. Hence, fP is a metrizable continuum contained in a point-inverse of π. Since M_1 contains only degenerate images of plane continua, g and f are constant maps.

Now, choose a continuum $P \subset Z$ and a non-constant map $f : P \to Z$. Suppose *a contrario* that $f \neq \mathrm{id}_P$. Take a point $z \in P$ with $fz \neq z$. It is a consequence of Proposition 4.4 that $A = f^{-1}fz$ and $B = fP$ are non-degenerate continua with $A \cap B = \{fz\}$. Since A is a retract of $A \cup B \ni z$ and $\pi^{-1}\pi z$ is a Cook continuum, $\pi(A \cup B)$ is not a single point. Hence, $A \cup B = \pi^{-1}\pi(A \cup B) = \pi^{-1}\pi A \cup \pi^{-1}\pi B$ by Remark 1.3, $\pi A \not\subset \pi B$, $\pi B \not\subset \pi A$, $A = \pi^{-1}\pi A$, and $B = \pi^{-1}\pi B$. Thus, $\pi^{-1}\pi fz \subset A \cap B$, a contradiction. Therefore, $f = \mathrm{id}_P$, and Z is a Cook continuum. ∎

5. Chainable Cook continua with ind > 1. Let X be a chainable continuum. Elements $a \neq b$ of X are called *opposite end points* if every open cover of X has a finite closed refinement $F_1, \ldots, F_k$ such that $a \in F_1$, $b \in F_k$, and $F_i \cap F_j \neq \emptyset$ iff $|i-j| \leq 1$. Bing [4, Theorem 15] proved that *every non-degenerate metric chainable continuum contains a chainable continuum with a pair of opposite end points.*

The third ingredient for our construction in this section is the following series of chainable continua, which were mentioned in Corollary 2.7 (now, we need more detail).

5.1. Example (Chatyrko [11]; see also [9, 13] for $n = 2, 3$). *There exists an inverse sequence $(I_n, \pi^n_m)^\infty_{n,m=1}$, where I_n are separable, first countable, hereditarily decomposable chainable continua, and $\pi^n_m : I_n \to I_m$ are surjec-*

(⁵) The proof of Proposition 29(i) in [36] works for arbitrary Hausdorff continua.

tive maps such that

(a) I_1 *is homeomorphic to* $[0,1]$*;*
(b) *each* π_n^{n+1} *is atomic and fully closed, and each* π_1^n *is ring-like;*
(c) *for each* $n > 1$, $m \in \{1, n-1\}$, *and every* $t \in I_m$, *the pre-image* $(\pi_m^n)^{-1}t$ *is homeomorphic to* I_{n-m}*;*
(d) *for each* n *and every pair* $0 \le s < t \le 1$, *the pre-image* $(\pi_1^n)^{-1}[s,t]$ *is a chainable continuum with opposite end points* $a_{s,t} \in (\pi_1^n)^{-1}s$, $b_{s,t} \in (\pi_1^n)^{-1}t$*;*
(e) *for each* n, *every partition in* I_n *contains some point-inverse* $(\pi_1^n)^{-1}t$, *where* $t \in I_1$*;*
(f) $\operatorname{ind} I_n = \operatorname{Ind} I_n = \operatorname{Ind}_0 I_n = n$ *for each* n (see our Corollary 2.7). ■

5.2. Lemma (Chatyrko [11, Lemma 1]). *Suppose that* X, Y *are compact spaces, and* $f : X \to Y$, $g : Y \to [0,1]$ *are surjective maps such that*

(a) f *is fully closed, and both* g *and* gf *are ring-like;*
(b) *for every* $t \in [0,1]$, *the pre-image* $(gf)^{-1}t$ *is a chainable continuum with a pair of opposite end points;*
(c) *for every pair* $0 \le s < t \le 1$, *the pre-image* $g^{-1}[s,t]$ *is a chainable continuum with opposite end points* $a_{s,t} \in g^{-1}s, b_{s,t} \in g^{-1}t$.

Then X *is a chainable continuum with a pair of opposite end points* $a \in (gf)^{-1}0$, $b \in (gf)^{-1}1$. ■

5.3. Theorem. *For every natural number* $n \ge 1$, *there exists a non-metrizable, separable, first countable, chainable, hereditarily decomposable Cook continuum* Z *such that* $n \le \operatorname{ind} Z \le \operatorname{Ind} Z \le \operatorname{Ind}_0 Z = n+1$, *and every partition* L *in* Z *has* $\operatorname{ind} L \ge n-1$.

Proof. Let M be Maćkowiak's Cook continuum of Example 3.4, and I_n Chatyrko's continuum of Example 5.1. We claim that M *contains an uncountable family of pairwise disjoint non-degenerate subcontinua* M_x, $x \in I_n$. Indeed, M is HD, and by a theorem of Bing [4, Theorem 8], there is a monotone surjective map $f : M \to [0,1]$. If $0 < t < 1$ and $f^{-1}t$ were a single point, $f^{-1}[0,t]$ would be a retract of M, and M would not be a Cook continuum. Hence, $f^{-1}t$ is a non-degenerate continuum if $0 < t < 1$. As $\operatorname{card} I_n = 2^{\aleph_0}$, the claim is proved (6). By [4, Theorem 15], we can assume that every M_x

(6) Recall that a continuum X is said to be *Suslinian* if every family of pairwise disjoint non-degenerate subcontinua of X is countable. By a similar argument, we infer that *no metric Cook continuum* X *is Suslinian.* If X contains an indecomposable continuum, then it is obviously not Suslinian (see for instance [32, Lemma 5.5]). If X is HD, then we can assume that it is irreducible by [35, §48 I, Theorem 1], and again there is a surjective monotone map $f : X \to [0,1]$ by Kuratowski's theorems [35, pp. 200 and 216].

On the other hand, Maćkowiak [36, Theorem 30] constructed an example of a metrizable, Suslinian, chainable, weak Cook continuum.

has a pair of opposite end points. Finally, we apply Theorem 3.1, and take $Z = Z(I_n, M_x)$ with $\pi : Z \to I_n$.

Note that above we described a class of examples $Z = Z_n$ which have surjective, ring-like, fully closed maps $\pi : Z_n \to I_n$ whose point-inverses are homeomorphic to pairwise disjoint non-degenerate subcontinua of M, and each of the continua has a pair of opposite end points. Thus, statement 5.1(c) implies that for each $t \in [0,1]$, $(\pi_1^n \pi)^{-1} t$ belongs to the class of examples Z_{n-1}.

By Proposition 1.6, the composition $\pi_1^n \pi$ is ring-like. Using induction on n, Lemma 5.2 and assertions (a–d) of Example 5.1, we infer that Z is a chainable continuum with a pair of opposite end points. Using induction, Proposition 1.7, and assertions (c, e) of Example 5.1, we infer that every partition L in Z has $\operatorname{ind} L \geq n-1$, and hence $\operatorname{ind} Z \geq n$. By Theorem 2.12 applied to π, $\operatorname{Ind}_0 Z = n+1$.

Similarly to the proof of Theorem 4.5, we shall show that Z is a weak Cook continuum. Take a continuum $P \subset Z$ and a map $f : P \to Z$ with $P \cap fP = \emptyset$. Suppose that f is not constant. Our first claim is that *πP is not a single point and f has a factorization $f = g(\pi|P)$, where $g : \pi P \to Z$ is continuous.* Indeed, if $Q \subset P$ is a metrizable continuum, then fQ is metrizable, and is contained in some point-inverse $\pi^{-1} x$, where $x \in I_n$. Hence, Q and fQ are homeomorphic to disjoint subcontinua of M, and the restriction $f|Q$ must be constant. Thus, P is not metrizable, πP is not a single point by Theorem 3.1(f), and moreover, $f|\pi^{-1} t$ is constant for every $t \in \pi P$. The map $g : \pi P \to Z$, $gt = f\pi^{-1} t$, is well defined and continuous. Our claim ensures that the set below is non-empty, and we define

$$k_0 = \min\{k : f \text{ has a factorization } f = g_k(\pi_k^n \pi|P), \\ \text{where } g_k : \pi_k^n \pi P \to Z\}.$$

Now, observe that *if $Q \subset \pi_{k_0}^n \pi P \subset I_{k_0}$ is a metrizable continuum, then $g_{k_0}|Q$ is constant.* Indeed, Q is an arc, $g_{k_0} Q \subset Z$ is metrizable, and is contained in some point-inverse $\pi^{-1} x$, where $x \in I_n$. As M_x does not contain arcs, $g_{k_0}|Q$ must be constant. This shows that $\pi_{k_0}^n \pi P$ is non-metrizable, $k_0 > 1$, and $\pi_{k_0-1}^n \pi P$ is not a single point. As $\pi_{k_0-1}^{k_0}$ is atomic, every point-inverse $(\pi_{k_0-1}^{k_0})^{-1} t$, $t \in \pi_{k_0-1}^n \pi P$, is a terminal continuum, and hence $(\pi_{k_0-1}^{k_0})^{-1} t \subset \pi_{k_0}^n \pi P$. By the observation emphasized above, the restriction $g_{k_0}|(\pi_{k_0-1}^{k_0})^{-1} t$ is constant for every $t \in \pi_{k_0-1}^n \pi P$. Thus, the map $g_{k_0-1} : \pi_{k_0-1}^n \pi P \to Z$, $g_{k_0-1} t = g_{k_0}(\pi_{k_0-1}^{k_0})^{-1} t$, is well defined and continuous. We have $f = g_{k_0-1}(\pi_{k_0-1}^n \pi|P)$, and this contradicts the definition of k_0. Therefore, f must be a constant map.

In the same way as in the proof of Theorem 4.5, one shows that Z is a Cook continuum. ∎

6. Remarks and open problems. Proposition 1.7 or alternatively Theorem 2.13 allow us to iterate the constructions in Section 4 in order to obtain continua Z with arbitrarily large difference $\operatorname{ind} Z - \dim Z > 0$. For example, one can take $Y_x = [0,1]^2$ for every $x \in [0,1]^2 = Z_1$, apply Theorem 3.1, and have $Z_2 = Z([0,1]^2, Y_x)$, $\pi_1^2 : Z_2 \to [0,1]^2$, $\dim Z_2 = 2$. Then one takes pairwise disjoint non-degenerate subcontinua C_x of Cook's M_1 for $x \in Z_2$, and puts $Z = Z(Z_2, C_x)$ with $\pi : Z \to Z_2$. It follows from 1.6 and 2.13 that $\operatorname{ind} Z \geq 3 > 2 = \dim Z$. Moreover, Z is a Cook continuum by the same argument as in the proof of Theorem 5.3.

It follows from Remark 1.8(2) that *if* CH *is true, then all the examples of continua Z constructed in Sections* 4–5 *can be perfectly normal, hereditarily separable, and have* $\operatorname{ind} Z = \operatorname{Ind} Z = \operatorname{Ind}_0 Z$. In Section 5, instead of Chatyrko's continua I_n one should use perfectly normal chainable continua given by A. A. Odintsov [39].

We suggest the following open problems.

In most of our examples of continua Z we have got the annoying difference between a lower bound of $\operatorname{ind} Z$ and the exact value of $\operatorname{Ind}_0 Z$.

6.1. Question. *Suppose that $f : X \to Y$ is a surjective, fully closed ring-like map from a continuum X, Y is the interval* $[0,1]$ *or another metrizable one-dimensional continuum, and every point-inverse of f is a metrizable one-dimensional* [*n-dimensional*] *continuum. Can it happen that* $\operatorname{ind} X = 1$ [*respectively,* $\operatorname{ind} X = n$ *or even* $\operatorname{Ind} X = n$]*?*

By Theorem 2.12, the above continuum X must have $\operatorname{Ind}_0 X = 2$ [respectively, $\operatorname{Ind}_0 X = n + 1$].

6.2. Question. *Do there exist* [*hereditarily indecomposable*] *Cook continua whose* $\dim = n$ *for* $n \geq 3$ [*respectively,* $n \geq 2$]*?*

Such continua do not exist in the metric case: see [32] and [38].

6.3. Question. *Does there exist a continuum* [*a dense-in-itself zero-dimensional compact space*] *no two of whose disjoint infinite closed subsets are homeomorphic?*

Fedorchuk [21, 22, 25] and Chatyrko [10] constructed examples of hereditarily n-dimensional (with respect to the covering dimension, $n > 0$ arbitrary) continua. The example in [22] (with CH assumed) does not have infinite zero-dimensional closed subspaces, and if it could be Anderson–Choquet, it would not contain a pair of disjoint homeomorphic infinite closed subsets.

REFERENCES

[1] P. S. Alexandroff [P. S. Aleksandrov], *On the dimension of normal spaces*, Proc. Roy. Soc. London 189 (1947), 11–39.

[2] R. D. Anderson and G. Choquet, *A plane continuum no two of whose nondegenerate subcontinua are homeomorphic: an application of inverse limits*, Proc. Amer. Math. Soc. 10 (1959), 347–353.

[3] R. H. Bing, *Higher-dimensional hereditarily indecomposable continua*, Trans. Amer. Math. Soc. 71 (1951), 267–273.

[4] —, *Snake-like continua*, Duke Math. J. 18 (1951), 653–663.

[5] M. G. Charalambous, *Uniform dimension functions*, Ph.D. Thesis, Univ. of London, 1971.

[6] —, *Two new inductive dimension functions for topological spaces*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 18 (1975), 15–25.

[7] M. G. Charalambous and V. A. Chatyrko, *Some estimates of the inductive dimensions of the union of two sets*, Topology Appl. 146-147 (2005), 227–238.

[8] J. J. Charatonik, *On chaotic curves*, Colloq. Math. 41 (1979), 219–236.

[9] V. A. Chatyrko, *On snakelike compact spaces*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1984, no. 4, 15–18 (in Russian); English transl.: Moscow Univ. Math. Bull. 39 (1984), no. 4, 19–23.

[10] —, *On hereditarily indecomposable nonmetrizable continua*, Mat. Zametki 46 (1989), no. 3, 122–125 (in Russian).

[11] —, *On bicompacta with noncoinciding dimensions*, Trudy Moskov. Mat. Obshch. 53 (1990), 192–228, 261 (in Russian); English transl.: Trans. Moscow Math. Soc. 1991, 199–236.

[12] —, *On properties of subsets of* $[0, \omega_{\mathfrak{c}}] \times I$, Questions Answers Gen. Topology 26 (2008), 97–104.

[13] V. A. Chatyrko and V. V. Fedorchuk, *On the Brouwer dimension of one-dimensional compact Hausdorff spaces*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 2005, no. 2, 22–27, 71 (in Russian); English transl.: Moscow Univ. Math. Bull. 60 (2005), no. 2, 22–27.

[14] H. Cook, *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. 60 (1967), 241–249.

[15] A. Emeryk, *An atomic map onto an arbitrary metric continuum*, Fund. Math. 77 (1972), 145–149.

[16] —, *A hereditarily indecomposable non-metric Hausdorff continuum*, ibid. 92 (1976), 63–64.

[17] A. Emeryk and Z. Horbanowicz, *On atomic mappings*, Colloq. Math. 27 (1973), 49–55.

[18] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

[19] —, *Theory of Dimensions, Finite and Infinite*, Heldermann, Lemgo, 1995.

[20] V. V. Fedorchuk, *On bicompacta with noncoinciding dimensions*, Dokl. Akad. Nauk SSSR 182 (1968), 275–277 (in Russian); English transl.: Soviet Math. Dokl. 9 (1968), 1148–1150.

[21] —, *Bicompacta without intermediate dimensions*, Dokl. Akad. Nauk SSSR 213 (1973), 795–797 (in Russian); English transl.: Soviet Math. Dokl. 14 (1973), 1808–1811.

[22] —, *A bicompactum all of whose infinite closed subsets are n-dimensional*, Mat. Sb. 96 (1975), no. 1, 41–62 (in Russian); English transl.: Math. USSR-Sb. 25 (1975), 37–57.

[23] V. V. Fedorchuk, *Fully closed mappings and the consistency of some theorems of general topology with the axioms of set theory*, Mat. Sb. 99 (1976), no. 1, 3–33 (in Russian); English transl.: Math. USSR-Sb. 28 (1976), 1–26.

[24] —, *On the dimension of κ-metrizable bicompacta, in particular Dugundji spaces*, Dokl. Akad. Nauk SSSR 234 (1977), 30–33 (in Russian); English transl.: Soviet Math. Dokl. 18 (1977), 605–609.

[25] —, *On the dimension of hereditarily normal spaces*, Proc. London Math. Soc. 34 (1978), 163–175.

[26] —, *Fully closed maps and their applications*, J. Math. Sci. (N. Y.) 136 (2006), 4201–4292; Russian original: Fundam. Prikl. Mat. 9 (2003), 105–235.

[27] —, *On some problems in topological dimension theory*, Uspekhi Mat. Nauk 57 (2002), no. 2, 139–178; English transl.: Russian Math. Surveys 57 (2002), 361–398.

[28] —, *An example of a compact space with different Lebesgue, Brouwer, and inductive dimensions*, Mat. Sb. 195 (2004), no. 12, 109–122 (in Russian); English transl.: Sb. Math. 195 (2004), 1809–1822.

[29] J. de Groot, *Groups represented by homeomorphism groups I*, Math. Ann. 138 (1959), 80–102.

[30] J. de Groot and R. J. Wille, *Rigid continua and topological group-pictures*, Arch. Math. (Basel) 9 (1958), 441–446.

[31] A. V. Ivanov, *The dimension of not perfectly normal spaces*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 31 (1976), no. 4, 21–27 (in Russian); English transl.: Moscow Univ. Math. Bull. 31 (1976), no. 3-4, 64–69.

[32] J. Krzempek, *Some examples of higher-dimensional rigid continua*, Houston J. Math. 35 (2009), 485–500.

[33] —, *Components and inductive dimensions of compact spaces*, Czechoslovak Math. J. 60 (2010), 445–456.

[34] J. Krzempek and E. Pol, *The non-existence of common models for some classes of higher-dimensional hereditarily indecomposable continua*, in: Continuum Theory: in Honor of Professor David P. Bellamy on the Occasion of His 60th Birthday (W. Lewis et al., eds.), Aportaciones Mat. Investig. 19, Soc. Mat. Mexicana, México, 2007, 71–84.

[35] K. Kuratowski, *Topology, Vol. II*, Academic Press, New York, and PWN, Warszawa, 1968.

[36] T. Maćkowiak, *The condensation of singularities in arc-like continua*, Houston J. Math. 11 (1985), 535–558.

[37] —, *Singular arc-like continua*, Dissertationes Math. 257 (1986), 40 pp.

[38] —, *Terminal continua and the homogeneity*, Fund. Math. 127 (1987), 178–186.

[39] A. A. Odintsov, *Snake-like bicompacta and some problems in dimension theory*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1986, no. 1, 59–62, 93 (in Russian); English transl.: Moscow Univ. Math. Bull. 41 (1986), no. 1, 59–62.

[40] E. Pol, *Hereditarily indecomposable continua with exactly n autohomeomorphisms*, Colloq. Math. 94 (2002), 225–234.

[41] —, *Collections of higher-dimensional hereditarily indecomposable continua with incomparable Fréchet types*, Topology Appl. 146-147 (2005), 547–561.

[42] —, *Hereditarily indecomposable continua with finitely many continuous surjections*, Bol. Soc. Mat. Mexicana (3) 11 (2005), 139–147.

[43] —, *An m-dimensional hereditarily indecomposable continuum with exactly n continuous mappings onto itself*, Topology Proc. 31 (2007), 243–256.

[44] A. Pultr and V. Trnková, *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories*, North-Holland, Amsterdam, 1980.

[45] M. Reńska, *Rigid hereditarily indecomposable continua*, Topology Appl. 126 (2002), 145–152.

[46] J. W. Rogers, Jr., *Continua that contain only degenerate continuous images of plane continua*, Duke Math. J. 37 (1970), 479–483.

[47] L. R. Rubin, R. M. Schori and J. J. Walsh, *New dimension-theory techniques for constructing infinite-dimensional examples*, General Topology Appl. 10 (1979), 93–102.

[48] S. Watson, *The construction of topological spaces: planks and resolutions*, in: Recent Progress in General Topology, M. Hušek and J. van Mill (eds.), North-Holland, Amsterdam, 1992, 673–757.

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