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MINKOWSKIAN RHOMBI AND SQUARES INSCRIBED IN CONVEX JORDAN CURVES

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Abstract. We show that any convex Jordan curve in a normed plane admits an inscribed Minkowskian square. In addition we prove that no two different Minkowskian rhombi with the same direction of one diagonal can be inscribed in the same strictly convex Jordan curve.

1. Introduction. The classical inscribed square problem in the Euclidean plane asks whether for every *Jordan curve* (i.e., for any simple closed curve) there exists at least one inscribed square, where a square is said to be *inscribed* in a curve if all its vertices lie on that curve. See [4] and [5] for detailed discussions and references for this problem. In general, the problem is still open, but positive answers have been given in several special cases. For example, the case of *convex Jordan curves* (i.e., of Jordan curves bounding a convex region) is considered in [2], [3], [13], and [1]; W. Stromquist [11] showed that each sufficiently smooth Jordan curve admits one inscribed square; M. Nielsen and S. Wright proved in [9] that every centrally symmetric Jordan curve admits an inscribed square. Another result, which is closely related to the inscribed square problem and proved in [8], says that any Jordan curve C contains the four vertices of some rhombus with two sides parallel to any given line l in $\mathbb{R}^2$. The aim of the present paper is to extend the inscribed square problem to Minkowski planes, and to prove that every convex Jordan curve admits an inscribed Minkowskian square. The type of methods used for this also yields a related result on Minkowskian rhombi.

By X we denote a *normed* or *Minkowski plane* (i.e., a two-dimensional real Banach space) with *origin* o, *norm* $\|\cdot\|$, *unit disc* $B_X := \{x \in X : \|x\| \leq 1\}$ (which is a compact, convex region centered at its interior point o), *unit circle* $S_X := \{x \in X : \|x\| = 1\}$, and a positive *orientation* ω. A Minkowski plane is said to be *strictly convex* if there is no non-trivial segment contained in S_X. Basic references for the geometry of Minkowski spaces are [7], [6], and the monograph [12]. For any two distinct points $x, y \in X$ we denote by

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$[x, y]$ the *segment* with endpoints x and y, by $\langle x, y\rangle$ the *line* passing through x and y, and by $[x, y\rangle$ the *ray* with starting point x passing through y. A convex quadrilateral $uvst$ is said to be a *parallelogram* if $v - u = s - t$ and $u - t = v - s$. A parallelogram $uvst$ is said to be a *Minkowskian rhombus* if $\|u-v\| = \|v-s\|$, and a Minkowskian rhombus $uvst$ is called a *Minkowskian square* if $\|u-s\| = \|v-t\|$. We note that, unlike the Euclidean case, general Minkowskian rhombi cannot be characterized as quadrilaterals having four sides of equal lengths. Namely, the convex quadrilateral formed by the points $(0, 2)$, $(1, 0)$, $(-1, -2)$, and $(-1, 0)$ and shown in Figure 1 has four sides of equal lengths, but it is clearly not a Minkowskian rhombus.

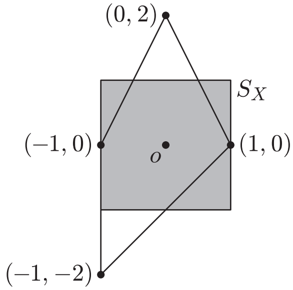

Fig. 1. A quadrilateral having four sides of equal lengths may not be a Minkowskian rhombus.

The *convex hull* of a set S is denoted by $\operatorname{conv} S$.

Let C be a convex Jordan curve. In particular, C is said to be *strictly convex* provided it contains no non-trivial line segment. Clearly, if C is strictly convex, then a line can intersect C at most twice.

The following theorem, proved in Section 3 below, is our main result.

Theorem 1.1. *For any convex Jordan curve C in a Minkowski plane X there exists at least one Minkowskian square inscribed in C.*

Additionally we prove in Section 2 that in a strictly convex Minkowski plane no two different Minkowskian rhombi with the same prescribed direction of one diagonal can be inscribed in the same strictly convex Jordan curve. This is completed by some results on parallelograms inscribed in strictly convex Jordan curves. These are, regarding the proof techniques used, strongly related to the results on inscribed Minkowskian squares and rhombi, but possibly also of independent interest.

2. Parallelograms and rhombi inscribed in strictly convex Jordan curves. The proof of Theorem 1.1, provided by A. Emch [2] for the case that C is a strictly convex and smooth Jordan curve in the Euclidean plane, is based on the following fact: two distinct rhombi with corresponding parallel sides or parallel diagonals can never be inscribed in the same strictly convex Jordan curve (see Theorem V in [2]). Since there is only one (non-zero) direction perpendicular to a given (non-zero) direction in the Euclidean plane, this means that there exists at most one Euclidean rhombus with one diagonal parallel to a given direction inscribed in a strictly convex Jordan curve. This is not true in general Minkowski planes; see Figure 2.

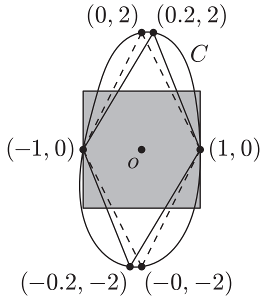

Fig. 2. Two Minkowskian rhombi with one diagonal in common are inscribed in the same strictly convex Jordan curve, where the plane is endowed with the maximum norm.

In this section we first prove a proposition and a lemma concerning parallelograms inscribed in strictly convex Jordan curves, and then we show, as announced, the following: if the underlying normed plane is strictly convex, then two distinct Minkowskian rhombi, each of which has a diagonal parallel to a given direction, cannot be inscribed in the same strictly convex Jordan curve.

Proposition 2.1. *Two distinct parallelograms with correspondingly parallel diagonals cannot be inscribed in the same strictly convex Jordan curve.*

Proof. Suppose that two distinct parallelograms $uvst$ and $u'v's't'$ are inscribed in the same strictly convex Jordan curve C so that $\langle u, s\rangle$ is parallel to $\langle u', s'\rangle$, and $\langle v, t\rangle$ is parallel to $\langle v', t'\rangle$. We can assume that the underlying Minkowski plane is equipped with a Euclidean background structure. Then there exists an affine transformation T such that $T(u - s)$ is orthogonal to $T(v-t)$ in the Euclidean sense. Therefore the image $T(uvst)$ is a rhombus in the Euclidean sense. On the other hand, since T is an affine transformation, $T(u'-s')$ is parallel to $T(u-s)$, and $T(v'-t')$ is parallel to $T(v-t)$. Therefore $T(u'v's't')$ is also a rhombus in the Euclidean sense. Note that $T(uvst)$ and $T(u'v's't')$ are both inscribed in $T(C)$, which contradicts Theorem V in [2] (note that the proof of that theorem in [2] does not need the assumption that the curve is smooth). ∎

Lemma 2.2. *If the convex regions bounded by two parallelograms do not intersect each other, then the parallelograms cannot be inscribed in the same strictly convex Jordan curve.*

Proof. Let $uvst$ and $u'v's't'$ be two parallelograms such that

$$\operatorname{conv}\{u,v,s,t\} \cap \operatorname{conv}\{u',v',s',t'\} = \emptyset. \tag{2.1}$$

Let o_1 and o_2 be their respective centers of symmetry. Suppose that $uvst$ and $u'v's't'$ are both inscribed in one strictly convex Jordan curve C.

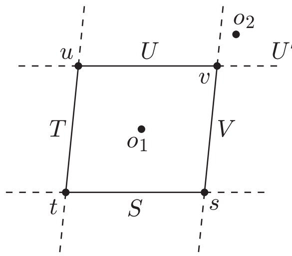

Fig. 3. The location of o_1 and o_2.

Clearly, the lines $\langle u,v\rangle$, $\langle v,s\rangle$, $\langle s,t\rangle$, and $\langle t,u\rangle$ divide the whole plane into nine parts as shown in Figure 3. We show that o_2 has to lie in one of the four open regions U, V, S, and T (see again Figure 3). Otherwise we may assume, without loss of generality, that o_2 lies in the closed region U'. It is clear that $o_2 \notin [v, 2v-s\rangle \cup [v, 2v-u\rangle$. Then for any relatively interior point x of the segment $[u,v]$, the segment $[o_2,x]$ has to intersect the set $\langle v,s\rangle \setminus [v,s]$ in a point y. Since C is strictly convex, o_2 and x are both interior points of $\operatorname{conv} C$, which implies that y is also an interior point of $\operatorname{conv} C$. Again the convexity of $\operatorname{conv} C$ implies that $v \in [y,s]$ has to be an interior point of $\operatorname{conv} C$, a contradiction. Thus o_2 has to lie in the strip bounded by $\langle u,v\rangle$ and $\langle s,t\rangle$, or in the strip bounded by $\langle v,s\rangle$ and $\langle u,t\rangle$.

Without loss of generality, we may now assume that o_2 lies between the parallel lines $\langle u,v\rangle$ and $\langle s,t\rangle$. Then all four points u', v', s', and t' have to lie between the lines $\langle u,v\rangle$ and $\langle s,t\rangle$. Otherwise we may assume that u' and o_2 lie on different sides of the line $\langle u,v\rangle$. Then the segment $[o_2,u']$ has to intersect $\langle u,v\rangle$ in a point x, which is an interior point of $\operatorname{conv} C$. From (2.1) it follows that $x \notin [u,v]$. Then one of the two points u and v has to be an interior point of $\operatorname{conv} C$, a contradiction. Similarly we may assume that o_1 lies between the parallel lines $\langle u',v'\rangle$ and $\langle s',t'\rangle$. Then all four points u, v, s, and t have to lie between the lines $\langle u',v'\rangle$ and $\langle s',t'\rangle$.

Next we show that $\langle u,v\rangle$ is parallel to $\langle u',v'\rangle$, which in turn will imply that $\langle u,v\rangle$ and $\langle s,t\rangle$ bound the same region as $\langle u',v'\rangle$ and $\langle s',t'\rangle$ do. If $\langle u,v\rangle$ is not parallel to $\langle u',v'\rangle$, then the four lines $\langle u,v\rangle$, $\langle u',v'\rangle$, $\langle s,t\rangle$, and $\langle s',t'\rangle$ bound a new parallelogram $u''v''s''t''$. We may assume that $\langle u'',v''\rangle = \langle u,v\rangle$ and $\langle u'',t''\rangle = \langle u',v'\rangle$; see Figure 4. Then $uvst$ and $u'v's't'$ are contained in $\operatorname{conv}\{u'',v'',s'',t''\}$. It follows that $u \in [u'',v'']$, $s \in [s'',t'']$, $u' \in [u'',t'']$, and $s' \in [v'',s'']$. Clearly, $[u,s]$ has to intersect $[u',s']$, contrary to (2.1).

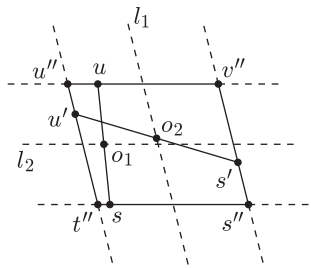

Fig. 4. The new parallelogram $u''v''s''t''$.

Now, as we have mentioned, the lines $\langle u,v\rangle$ and $\langle s,t\rangle$ bound the same region as $\langle u',v'\rangle$ and $\langle s',t'\rangle$ do. We may therefore assume that $\langle u,v\rangle = \langle u',v'\rangle$ and $\langle s,t\rangle = \langle s',t'\rangle$. It follows from (2.1) that $[u,v] \subset \langle u',v'\rangle \setminus [u',v']$. This means that $\langle u',v'\rangle$ intersects C in at least four distinct points, contrary to the strict convexity of C. ∎

Lemma 2.3. *Let x, y, z, and p be four distinct points in a strictly convex Minkowski plane X such that both p and z are equidistant to x and y, and that $p \in \operatorname{conv}\{x,y,z\}$. Then $\left\|p - \frac{1}{2}(x+y)\right\| < \left\|z - \frac{1}{2}(x+y)\right\|$.*

Proof. It follows from [7, Proposition 14 and Lemma 25] that $\|p-y\| < \|z-y\|$. If $p \in \left[z, \frac{1}{2}(x+y)\right]$, then there is nothing to prove. Thus we may assume that $p \in \operatorname{conv}\left\{x, z, \frac{1}{2}(x+y)\right\} \setminus \left[z, \frac{1}{2}(x+y)\right]$. Then, by [7, Proposition 7],

$$\|p-y\| + \left\|z - \frac{1}{2}(x+y)\right\| > \left\|p - \frac{1}{2}(x+y)\right\| + \|z-y\|,$$

which implies that $\left\|z - \frac{1}{2}(x+y)\right\| > \left\|p - \frac{1}{2}(x+y)\right\|$. ∎

Theorem 2.4. *Let X be a strictly convex Minkowski plane. Then two different Minkowskian rhombi $uvst$ and $u'v's't'$ with*

$$\frac{u-s}{\|u-s\|} = \frac{u'-s'}{\|u'-s'\|}$$

cannot be inscribed in the same strictly convex Jordan curve C.

Proof. Suppose that $uvst$ and $u'v's't'$ are both inscribed in C. First we show that $[u',s']$ has to intersect the interior of $\operatorname{conv}\{u,v,s,t\}$. Otherwise, an argument similar to that in the proof of Lemma 2.2 would show that $[u',s']$ has to lie either in the strip bounded by $\langle u,v\rangle$ and $\langle s,t\rangle$, or in the strip bounded by $\langle u,t\rangle$ and $\langle v,s\rangle$. We may assume that $[u',s'] \subset [v,s] + \{\lambda(v-u) : \lambda \geq 0\}$. Let z be a point such that $z - v = s - u$, and p and q be two points such that

$$\frac{p-u'}{\|p-u'\|} = \frac{v-u}{\|v-u\|} = -\frac{q-s'}{\|q-s'\|}, \qquad \frac{p-s'}{\|p-s'\|} = \frac{v-s}{\|v-s\|} = -\frac{q-u'}{\|q-u'\|}.$$

It is not difficult to verify that $q \in \operatorname{conv}\{s, u', s'\}$. By [7, Proposition 17],

$$v' \in \{p + \lambda(u' - p) + \mu(s' - p) : \lambda\mu \geq 0\},$$
$$t' \in \{q + \lambda(u' - q) + \mu(s' - q) : \lambda\mu \geq 0\}.$$

We distinguish three cases:

Case I: The line $\langle v, z\rangle$ lies strictly between $\langle u, s\rangle$ and $\langle u', s'\rangle$; see Figure 5.

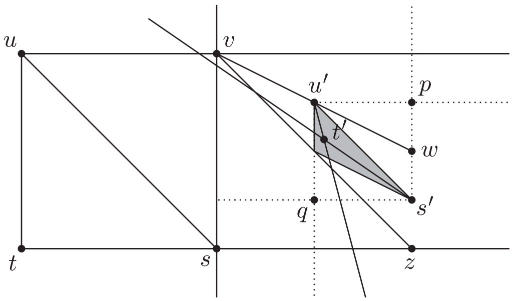

Fig. 5. Proof of Theorem 2.4: $\langle v, z\rangle$ lies strictly between $\langle u, s\rangle$ and $\langle u', s'\rangle$.

Since

$$u' \in v + \{\lambda(v - u) + \mu(z - v) : \lambda, \mu \geq 0\},$$

we have

$$[v, u'\rangle \subset v + \{\lambda(v - u) + \mu(z - v) : \lambda, \mu \geq 0\}.$$

This implies that

$$[v, u'\rangle \setminus [v, u'] \subset u' + \{\lambda(s' - u') + \mu(p - u') : \lambda, \mu \geq 0\}.$$

Thus the ray $[v, u'\rangle$ has to intersect the segment $[p, s']$ in a point w. Then $[u', w] \setminus \{u'\} \cap \operatorname{conv} C = \emptyset$, which implies that $v' \in \operatorname{conv}\{u', w, s'\}$. Let w' be a point in $[u', q]$ such that $\|u' - w'\| = \|s' - w\|$. Then t' has to lie in $\operatorname{conv}\{u', w', s'\}$ (the shaded region in Figure 5).

Now one can see that

$$t' \in \operatorname{conv}\{q, u', s'\} \subset \operatorname{conv}\{s, u', s'\},$$

a contradiction.

Case II: The line $\langle u', s'\rangle$ lies strictly between $\langle u, s\rangle$ and $\langle v, z\rangle$; see Figure 6.

In a similar way to Case I it can be shown that the line $\langle v, u'\rangle$ has to intersect the segment $[q, s']$ in a point w. Then $[u', w] \setminus \{u'\} \cap \operatorname{conv} C = \emptyset$, which implies that $t' \in \operatorname{conv}\{u', w, s'\}$. Again we would obtain $t' \in \operatorname{conv}\{s, u', s'\}$, a contradiction.

Case III: The segment $[u', s']$ is contained in $[v, z]$; see Figure 7.

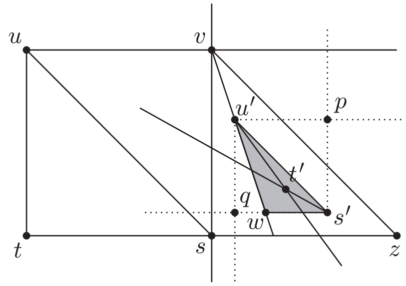

Fig. 6. Proof of Theorem 2.4: $\langle u', s' \rangle$ lies strictly between $\langle u, s \rangle$ and $\langle v, z \rangle$.

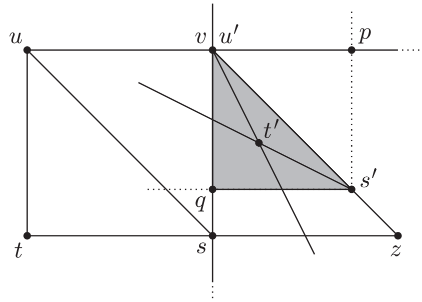

Fig. 7. Proof of Theorem 2.4: $\langle u', s' \rangle$ is contained in $[v, z]$.

Since $z \notin \operatorname{conv} C$, we see that s' lies strictly between u' and z, and therefore u' has to coincide with v. Now t' has to lie in $\operatorname{conv}\{u', q, s'\} \subset \operatorname{conv}\{u', s, s'\}$, again a contradiction.

With similar arguments we can also show that $[u, s]$ intersects the interior of $\operatorname{conv}\{u', v', s', t'\}$.

Now we may assume that $\langle u', s' \rangle$ lies between v and $\langle u, s \rangle$. Then $v \in \operatorname{conv}\{p, u', s'\}$; see Figure 8 (otherwise, we could interchange v and t). One can easily verify that v' has to lie in the shaded region in Figure 8. More

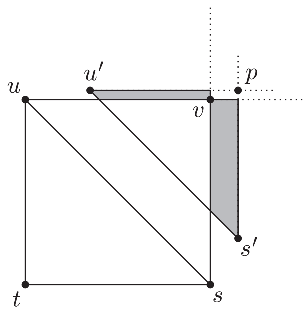

Fig. 8. Proof of Theorem 2.4: v' lies in the shaded region.

precisely,

$$v' \in \{p + \lambda(u' - p) + \mu(s' - p) : \lambda, \mu \geq 0,\ \lambda + \mu \leq 1\} \setminus \{v + \lambda(u - v) + \mu(s - v) : \lambda\mu \geq 0\}.$$

Since $u' \notin \langle u, v\rangle$ and $s' \notin \langle s, v\rangle$, we have $v' \neq p$. Thus, by Lemma 2.3,

$$\frac{\left\|v' - \frac{1}{2}(u' + s')\right\|}{\|u' - s'\|} < \frac{\left\|p - \frac{1}{2}(u' + s')\right\|}{\|u' - s'\|} = \frac{\left\|v - \frac{1}{2}(u + s)\right\|}{\|u - s\|}.$$

On the other hand, interchanging $uvst$ and $u'v's't'$ yields

$$\frac{\left\|v' - \frac{1}{2}(u' + s')\right\|}{\|u' - s'\|} > \frac{\left\|v - \frac{1}{2}(u + s)\right\|}{\|u - s\|},$$

a contradiction. ∎

3. The proof of Theorem 1.1. Let C be a convex Jordan curve in a Minkowski plane X. A chord $[x, y]$ of C is said to be an *affine diameter* of C if there exist different parallel supporting lines of $\operatorname{conv} C$, say H_1 and H_2, such that $x \in H_1$ and $y \in H_2$. Equivalently, $[x, y]$ is an affine diameter of C if it is a longest chord of $\operatorname{conv} C$ in the direction $(x - y)/\|x - y\|$ (see [10, 3.1]). It is obvious that, when C is strictly convex, there exists a unique affine diameter in any given direction.

From now on, let C be strictly convex. For any $x \in S_X$, let $[x_W, x_E]$ be the affine diameter of C such that

$$\frac{x_E - x_W}{\|x_E - x_W\|} = x;$$

let x_N (x_S, resp.) be the point of C such that there exists a line l_N (l_S, resp.) which is parallel to $\langle -x, x\rangle$ and supports $\operatorname{conv} C$ at x_N (x_S, resp.) such that, in addition, the orientation from x_E to x_N (x_S, resp.) is ω ($-\omega$, resp.). Let x_0 be the point of intersection of $[x_N, x_S]$ and $[x_W, x_E]$; see Figure 9.

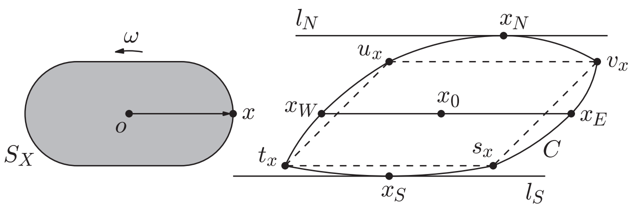

Fig. 9. The points $x_N, x_S, \ldots$.

Lemma 3.1. *Let C be a strictly convex Jordan curve in a Minkowski plane X. Then, for any point $x \in S_X$, there exists a unique Minkowskian rhombus inscribed in C with one side parallel to the line $\langle -x, x\rangle$.*

Proof. First we show the existence of the desired Minkowskian rhombus. Let $x \in S_X$ be given, and λ_0 be the number in $[0,1]$ such that

$$(1-\lambda_0)x_N + \lambda_0 x_S = x_0.$$

For any number $\lambda \in [0,\lambda_0]$ there exist precisely two distinct chords of C, say $[u(x,\lambda), v(x,\lambda)]$ and $[s(x,\lambda), t(x,\lambda)]$, satisfying

$$(1-\lambda)x_N + \lambda x_S \in [u(x,\lambda), v(x,\lambda)]$$

and

$$v(x,\lambda) - u(x,\lambda) = s(x,\lambda) - t(x,\lambda) = \|v(x,\lambda) - u(x,\lambda)\|x.$$

Let

$$f(\lambda) = \|u(x,\lambda) - v(x,\lambda)\| - \|u(x,\lambda) - t(x,\lambda)\|.$$

Then

$$\lim_{\lambda\to 0} f(\lambda) = -\|x_N - x_S\| < 0 \quad \text{and} \quad \lim_{\lambda\to\lambda_0} f(\lambda) = \|x_W - x_E\| > 0.$$

From the continuity of f it follows that there exists a number $\lambda(x) \in [0,\lambda_0]$ such that $f(\lambda(x)) = 0$. Let

$$u = u(x,\lambda(x)), \qquad v = v(x,\lambda(x)), \qquad s = s(x,\lambda(x)), \qquad t = t(x,\lambda(x)).$$

Then it is clear that the parallelogram $uvst$ is a Minkowskian rhombus inscribed in C with one side parallel to $\langle -x, x\rangle$.

Next we show that this Minkowskian rhombus is unique. Suppose the contrary, namely, that there exists another Minkowskian rhombus $u'v's't'$ with

$$\frac{u-v}{\|u-v\|} = \frac{u'-v'}{\|u'-v'\|}$$

which is inscribed in C. Since C is strictly convex, we may assume that $\langle u,v\rangle$ lies strictly between $\langle u',v'\rangle$ and $\langle s,t\rangle$, and $\langle s,t\rangle$ lies strictly between $\langle u,v\rangle$ and $\langle s',t'\rangle$; see Figure 10.

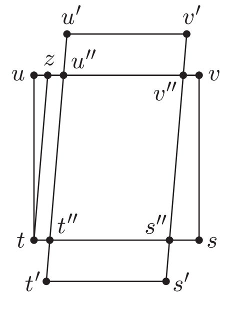

Fig. 10. The uniqueness of the inscribed Minkowskian rhombus.

Clearly, we have

$$\|u' - v'\| = \|s' - t'\| < \|u - v\| = \|s - t\|.$$

Let u'', v'', s'', and t'' be four points such that

$$\{u''\} = [u', t'] \cap [u, v], \quad \{v''\} = [v', s'] \cap [u, v],$$
$$\{s''\} = [v', s'] \cap [s, t], \quad \{t''\} = [u', t'] \cap [s, t].$$

If $\langle u', t' \rangle$ is parallel to $[u, t]$, then

$$\|u' - t'\| > \|u - t\| = \|u - v\| > \|u' - v'\|,$$

a contradiction. Thus, we may assume that $\|u - u''\| > \|t - t''\|$. Let $z \in [u, u'']$ be such that $\|z - u''\| = \|t - t''\|$. Then

$$\|u' - t'\| > \|u'' - t''\| = \|t - z\| \geq \|u - t\| - \|u - z\| = \|z - v\| > \|u' - v'\|,$$

a contradiction. ∎

Remark 3.2. The existence part of the proof of Lemma 3.1 is essentially the same as the proof of Theorem 1 in [8], simplified to deal with only convex Jordan curves.

Now, for any strictly convex Jordan curve C, we can relate each point $x \in S_X$ to four points u_x, v_x, s_x, and t_x in a unique way such that $u_x v_x s_x t_x$ is a Minkowskian rhombus inscribed in C, and $\langle u_x, v_x \rangle$ lies between l_N and $\langle x_W, x_E \rangle$ with

$$\frac{v_x - u_x}{\|v_x - u_x\|} = x.$$

Lemma 3.3. *Let C be a strictly convex Jordan curve. Then the mappings*

$$U(x) = u_x, \quad V(x) = v_x, \quad S(x) = s_x, \quad \text{and} \quad T(x) = t_x$$

from S_X to C are all continuous.

Proof. We only show that $U(x)$ is continuous; the continuity of the other mappings can be proved in a similar way.

Suppose that there exists a point $x_0 \in S_X$ and a sequence $\{x_n\} \subset S_X$ with $\lim_{n\to\infty} x_n = x_0$ such that $\lim_{n\to\infty} U(x_n)$ does not exist or $\lim_{n\to\infty} U(x_n) \neq U(x_0)$. In each case we obtain a subsequence $\{U(x_{n_k})\}$ such that $\lim_{k\to\infty} U(x_{n_k}) \neq U(x_0)$. By taking a further subsequence, we may also assume that there exist points u_0, v_0, s_0, and t_0 such that

$$u_0 = \lim_{k\to\infty} U(x_{n_k}) = \lim_{k\to\infty} u_{x_{n_k}}, \quad v_0 = \lim_{k\to\infty} V(x_{n_k}) = \lim_{k\to\infty} v_{x_{n_k}},$$
$$s_0 = \lim_{k\to\infty} S(x_{n_k}) = \lim_{k\to\infty} s_{x_{n_k}}, \quad t_0 = \lim_{k\to\infty} T(x_{n_k}) = \lim_{k\to\infty} t_{x_{n_k}}.$$

Since $u_{x_{n_k}} v_{x_{n_k}} s_{x_{n_k}} t_{x_{n_k}}$ is a Minkowskian rhombus inscribed in C and the curve C is a closed set, $u_0 v_0 s_0 t_0$ is also a Minkowskian rhombus inscribed in C. Furthermore,

$$\frac{u_0 - v_0}{\|u_0 - v_0\|} = \lim_{k\to\infty} \frac{u_{x_{n_k}} - v_{x_{n_k}}}{\|u_{x_{n_k}} - v_{x_{n_k}}\|} = -\lim_{k\to\infty} x_{n_k} = -x_0.$$

Since $u_0 \neq u_{x_0}$, we see that $u_0v_0s_0t_0$ is a Minkowskian rhombus different from $u_{x_0}v_{x_0}s_{x_0}t_{x_0}$, inscribed in C, and having one side parallel to $\langle -x, x\rangle$. By Lemma 3.1, this is a contradiction. ■

Remark 3.4. In [8], by applying a more complicated method, M. Nielsen proved that, given any simple closed curve C and any line l in $\mathbb{R}^2$, the curve C contains the four vertices of some rhombus with two sides parallel to l; see [8, Theorem 1]. However, for C not strictly convex, we have no idea whether the Minkowskian rhombus related to a given direction is unique.

Lemma 3.5. *Any strictly convex Jordan curve C admits an inscribed Minkowskian square.*

Proof. First we introduce the following function for any point $x \in S_X$:

$$f(x) = \|u_x - s_x\| - \|v_x - t_x\|.$$

Let x be an arbitrary point in S_X. If $f(x) = 0$, then the proof is complete. Otherwise we may assume that, without loss of generality, $f(x) < 0$. Let

$$x_0 = \frac{s_x - v_x}{\|s_x - v_x\|}.$$

Then

$$u_{x_0} = v_x, \quad v_{x_0} = s_x, \quad s_{x_0} = t_x, \quad t_{x_0} = u_x.$$

Thus

$$f(x_0) = \|v_x - t_x\| - \|u_x - s_x\| = -f(x) > 0.$$

Since $U(x)$, $V(x)$, $S(x)$, and $T(x)$ are all continuous with respect to x, $f(x)$ is continuous. Hence there exists a point $z_0 \in S_X$ such that $f(z_0) = 0$. It is clear that the parallelogram $u_{z_0}v_{z_0}s_{z_0}t_{z_0}$ is a Minkowskian square inscribed in C. ■

Proof of Theorem 1.1. Using the Euclidean background structure of X we denote by E a largest Euclidean circle contained in the bounded region enclosed by C. We may also assume that E is centered at the origin.

For all $n \in \mathbb{N}$, $n \geq 1$, let $C_n = \big(1 - \frac{1}{n}\big)C + \frac{1}{n}E$. Then, for each n, the curve C_n is strictly convex and contained in the bounded region enclosed by C, and therefore, by Lemma 3.5, it admits an inscribed square $u_nv_ns_nt_n$. Since each curve C_n is strictly convex, the segments $[u_n, v_n]$ and $[s_n, t_n]$ are separated by the affine diameter of C_n parallel to the line $\langle u_n, v_n\rangle$.

Similarly, the segments $[u_n, t_n]$ and $[v_n, s_n]$ are separated by the affine diameter of C_n parallel to the line $\langle u_n, t_n\rangle$. Therefore both $\|u_n - v_n\|$ and $\|u_n - t_n\|$ cannot tend to zero. By compactness, we can choose convergent subsequences $\{u_{n_k}\}$, $\{v_{n_k}\}$, $\{s_{n_k}\}$, and $\{t_{n_k}\}$. Let $u = \lim_{k\to\infty} u_{n_k}$, $v = \lim_{k\to\infty} v_{n_k}$, $s = \lim_{k\to\infty} s_{n_k}$, and $t = \lim_{k\to\infty} t_{n_k}$. Then it is clear that $uvst$ is a Minkowskian square inscribed in C, which completes the proof. ∎

REMARK 3.6. There is another way to prove Theorem 1.1. Namely, first one can show that for any points x and y in S_X a unique parallelogram inscribed in a strictly convex Jordan curve can be found, whose two diagonals are parallel to $\langle -x, x\rangle$ and $\langle -y, y\rangle$, respectively. By using a technique similar to that in the proof of Lemma 3.3 one can prove that such a parallelogram varies continuously when x is fixed and y varies continuously. Then the intermediate value theorem implies the existence of a Minkowskian rhombus inscribed in C with one diagonal parallel to $\langle -x, x\rangle$. Now, if the underlying normed plane is strictly convex, Theorem 2.4 can be applied to show that such a Minkowskian rhombus is unique. Also one can show that this Minkowskian rhombus varies continuously when x does, and so finally a Minkowskian square inscribed in C would be obtained. The case when C is convex and the underlying normed plane is strictly convex can be solved by applying the technique used in the proof of Theorem 1.1. When the Minkowski plane is not strictly convex, one can approximate the norm $\|\cdot\|$ of this plane by strictly convex norms $\{\|\cdot\|_n\}_{n=1}^{\infty}$. For each n a Minkowskian square P_n with respect to $\|\cdot\|_n$ can be found which is inscribed in C. A subsequence of $\{P_n\}_{n=1}^{\infty}$ will converge to a parallelogram inscribed in C, which is a Minkowskian square with respect to $\|\cdot\|$. This proof is more complicated than the one above, but almost all necessary techniques are illustrated in our current proof.

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