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SUPERSYMMETRY CLASSES OF TENSORS

BY

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Abstract. We introduce the notion of a *supersymmetry class of tensors* which is the ordinary symmetry class of tensors with a natural $\mathbb{Z}_2$-gradation. We give the dimensions of *even* and *odd* parts of this gradation as well as their natural bases. Also we give a necessary and sufficient condition for the odd or even part of a supersymmetry class to be zero.

1. Motivation. In [4], a joint paper with A. Madadi, we introduced a concrete method to construct the irreducible representations of the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, using the notion of symmetry classes of tensors. A similar work can be done for the Lie superalgebra $\mathfrak{sl}(p|q)$, if we have a suitable notion of *supersymmetry classes of tensors*. On the other hand, recently the term *super linear algebra* is widely used in both mathematics and physics articles which study objects of linear algebra from the *super-structure* point of view (see [10]). Since in the last five decades, several articles have been published concerning symmetry classes of tensors, it is natural to have in hand the *super-version* of this notion. Although there are some articles concerning representations of Lie superalgebras which employ some special cases of this notion (see [1] or [8]), the present article is the first attempt toward introducing supersymmetry classes of tensors systematically and the author hopes the resulting notion will be interesting both from the multilinear algebra and from the representation of Lie superalgebras point of view.

2. Introduction. Let $V = V^0 \oplus V^1$ be an n-dimensional $\mathbb{Z}_2$-graded complex vector space with $\dim V^0 = r$. Suppose $e_1, \dots, e_r$ is a basis for V^0 and $e_{r+1}, \dots, e_n$ is a basis for V^1. For any homogeneous vector $v \in V$, we define $\deg v = i$ iff $v \in V^i$, for $i = 0, 1$. Let m be a positive integer and $V^{\otimes m}$ denote the tensor product of m copies of V. Then we have a $\mathbb{Z}_2$-gradation

$$V^{\otimes m} = \mathcal{L}^0 \oplus \mathcal{L}^1,$$

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where

$$\mathcal{L}^0 = \operatorname{span}_{\mathbb{C}}\Big\{v_1 \otimes \cdots \otimes v_m : \sum_i \deg v_i = \text{even}\Big\},$$
$$\mathcal{L}^1 = \operatorname{span}_{\mathbb{C}}\Big\{v_1 \otimes \cdots \otimes v_m : \sum_i \deg v_i = \text{odd}\Big\}.$$

Suppose Γ_n^m is the set of all m-tuples of integers $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $1 \leq \alpha_i \leq n$ for any i. If $\alpha \in \Gamma_n^m$, then we define the tensor

$$e_\alpha^\otimes = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m}.$$

Recall that the set $\{e_\alpha^\otimes : \alpha \in \Gamma_n^m\}$ is a basis for $V^{\otimes m}$. Now, let E_n^m be the set of all $\alpha \in \Gamma_n^m$ in which the number of entries with $\alpha_i > r$ is even. We define similarly the set O_n^m using the word *odd* instead of *even*. Then it is clear that the set $\{e_\alpha^\otimes : \alpha \in E_n^m\}$ is a basis for $\mathcal{L}^0$. Similarly the set $\{e_\alpha^\otimes : \alpha \in O_n^m\}$ is a basis for $\mathcal{L}^1$. In the following proposition we give the dimensions of $\mathcal{L}^0$ and $\mathcal{L}^1$.

Proposition 2.1. *For $i = 0, 1$, we have*

$$\dim \mathcal{L}^i = \tfrac{1}{2}(n^m + (-1)^i(2r - n)^m).$$

Proof. Let A_k be the set of all $\alpha \in E_n^m$ such that α has exactly $2k$ terms greater than r. Then we have the set partition

$$E_n^m = \bigcup_{k=0}^{[m/2]} A_k.$$

So we are going to count the number of elements in A_k. We can consider every element $\alpha \in A_k$ as an ordered set of m boxes in which $2k$ boxes contain numbers from the set

$$r+1, r+2, \ldots, n.$$

So we must choose $2k$ boxes from m boxes and this is clearly possible in

$$\binom{m}{2k} = \frac{m!}{(2k)!(m-2k)!}$$

ways. Next, we write one of the numbers in the above list in each selected box and this can be done in $(n-r)^{2k}$ ways. Finally, we must write one of the numbers $1, 2, \ldots, r$ in the remaining $m - 2k$ boxes, which is possible in r^{m-2k} ways. So we can obtain $\alpha \in A_k$ in

$$\binom{m}{2k}(n-r)^{2k} r^{m-2k}$$

ways. Thus

$$|A_k| = \binom{m}{2k}(n-r)^{2k} r^{m-2k},$$

and this completes the proof.

3. Symmetry classes of tensors. In this section, we review the notion of a symmetry class of tensors. The interested reader can find a detailed introduction in [5] or [6]. Note that as in Section 1, $V = V^0 \oplus V^1$ is an n-dimensional $\mathbb{Z}_2$-graded complex vector space with $\dim V^0 = r$.

Let S_m denote the full symmetric group of degree m and let $G \leq S_m$ be any subgroup. Suppose χ is an irreducible complex character of G. For any $\sigma \in S_m$, define the *permutation operator* $P_\sigma : V^{\otimes m} \to V^{\otimes m}$ by

$$P_\sigma(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}.$$

It is clear from this definition that $\mathcal{L}^0$ and $\mathcal{L}^1$ are invariant under P_σ and so we can define, for $i = 0, 1$,

$$P^i_\sigma = P_\sigma|_{\mathcal{L}^i},$$

the restriction of P_σ to $\mathcal{L}^i$. The *symmetrizer* corresponding to G and χ is

$$S_\chi = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P_\sigma.$$

Its image is called the *symmetry class of tensors* associated with G and χ and it is denoted by $V_\chi(G)$.

For example, if we let $G = S_m$ and $\chi = \varepsilon$, the alternating character, then we get $\bigwedge^m V$, the mth Grassmann space over V, and if $G = S_m$ and $\chi = 1$, the principal character, then we obtain $V^{(m)}$, the mth symmetric power of V, as symmetry classes of tensors.

Let $v_1, \ldots, v_m$ be arbitrary vectors in V and define the *decomposable symmetrized tensor*

$$v_1 * \cdots * v_m = S_\chi(v_1 \otimes \cdots \otimes v_m).$$

For $\alpha \in \Gamma^m_n$, we use the notation e^*_α for the decomposable symmetrized tensor $e_{\alpha_1} * \cdots * e_{\alpha_m}$. It is clear that $V_\chi(G)$ is generated by all e^*_α, $\alpha \in \Gamma^m_n$. We define an action of G on Γ^m_n by

$$\alpha^\sigma = (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(m)})$$

for any $\sigma \in G$ and $\alpha \in \Gamma^m_n$. Suppose Δ is a set of representatives of all orbits of this action and let G_α denote the stabilizer subgroup of α. Define

$$\Omega = \{\alpha \in \Gamma^m_n : [\chi, 1_{G_\alpha}] \neq 0\},$$

where $[\,,\,]$ denotes the inner product of characters (see [2]). It is well known that $e^*_\alpha \neq 0$ if and only if $\alpha \in \Omega$ (see for example [6]). Suppose $\bar{\Delta} = \Delta \cap \Omega$. For any $\alpha \in \bar{\Delta}$, we have the cyclic subspace

$$V^*_\alpha = \langle e^*_{\alpha^\sigma} : \sigma \in G \rangle.$$

It is known that we have the direct sum decomposition

$$V_\chi(G) = \dot{\sum_{\alpha \in \bar{\Delta}}} V^*_\alpha$$

(see [6] for a proof). It is also known that

$$\dim V_\alpha^* = \chi(1)[\chi, 1_{G_\alpha}],$$

and in particular, if χ is linear then $\dim V^* = 1$ and so the set $\{e_\alpha^* : \alpha \in \bar{\Delta}\}$ is a basis for $V_\chi(G)$. In the general case, let $\alpha \in \bar{\Delta}$ and suppose

$$e^*_{\alpha^{\sigma_1}}, \ldots, e^*_{\alpha^{\sigma_t}}$$

is a basis for V_α^*, with $\sigma_1 = 1$. Let

$$A_\alpha = \{\alpha^{\sigma_1}, \ldots, \alpha^{\sigma_t}\}.$$

Then we define $\hat{\Delta} = \bigcup_{\alpha \in \bar{\Delta}} A_\alpha$. It is clear that $\bar{\Delta} \subseteq \hat{\Delta} \subseteq \Omega$, and the set $\{e_\alpha^* : \alpha \in \hat{\Delta}\}$ is a basis for $V_\chi(G)$. Finally we remind the reader of a formula for the dimension of symmetry classes. We have

$$\dim V_\chi(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) n^{c(\sigma)},$$

where $c(\sigma)$ denotes the number of disjoint cycles (including cycles of length one) in the cycle decomposition of σ.

4. Supersymmetry classes of tensors. Now we return to the $\mathbb{Z}_2$-gradation of $V^{\otimes m}$. As in Section 1, we use the notations $\mathcal{L}^0$ and $\mathcal{L}^1$ for the even and odd parts of this gradation, respectively. Let

$$\mathcal{L}_\chi^i = S_\chi(\mathcal{L}^i)$$

for $i = 0, 1$. Then it is clear that $V_\chi(G) = \mathcal{L}_\chi^0 \oplus \mathcal{L}_\chi^1$ is a $\mathbb{Z}_2$-gradation of $V_\chi(G)$. If we consider $V_\chi(G)$ together with this $\mathbb{Z}_2$-gradation, we call the resulting vector space the *supersymmetry class of tensors associated with* G *and* χ *on* V. Our next aim is to compute the dimensions of the even and odd parts of this supersymmetry class of tensors. To do this, let

$$S_\chi^i = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P_\sigma^i$$

for $i = 0, 1$. It is easy to see that S_χ^i is idempotent and its image is precisely $\mathcal{L}_\chi^i$. So we have

$$\dim \mathcal{L}_\chi^i = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \operatorname{Tr} P_\sigma^i.$$

For any $\sigma \in S_m$, let $c_0(\sigma)$ (similarly $c_1(\sigma)$) denote the number of cycles of even (odd) length in the disjoint cycle decomposition of σ. Note that we have $c(\sigma) = c_0(\sigma) + c_1(\sigma)$.

THEOREM 4.1. *We have*

$$\operatorname{Tr} P_\sigma^0 = \tfrac{1}{2}(n^{c(\sigma)} + n^{c_0(\sigma)}(2r - n)^{c_1(\sigma)}).$$

There is a similar formula for $\operatorname{Tr} P_\sigma^1$.

Proof. Let $f_1, \dots, f_n \in V^*$ be the corresponding dual basis of $e_1, \dots, e_n$. So $f_i(e_j) = \delta_{ij}$ for any i and j. For any $\alpha \in \Gamma_n^m$, define

$$h_\alpha = f_{\alpha_1} \otimes \cdots \otimes f_{\alpha_m}.$$

Note that

$$h_\alpha(v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^m f_{\alpha_i}(v_i).$$

Now let

$$E = \{e_\alpha^\otimes : \alpha \in E_n^m\} \quad \text{and} \quad E^* = \{h_\alpha : \alpha \in E_n^m\}.$$

We have $h_\beta(e_\alpha^\otimes) = \delta_{\alpha\beta}$ for any $\alpha, \beta \in E_n^m$. Hence E^* is the dual basis to E. The (α, α) entry of the matrix representation of P_σ^0 in the basis E is equal to

$$h_\alpha(P_\sigma^0(e_\alpha^\otimes)) = \delta_{\alpha,\alpha^{\sigma^{-1}}}.$$

Hence

$$\operatorname{Tr} P_\sigma^0 = |\{\alpha \in E_n^m : \alpha = \alpha^\sigma\}|.$$

Now let $(t, \sigma(t), \sigma^2(t), \dots, \sigma^{p-1}(t))$ be one of the cycles in the disjoint cycle decomposition of σ. So $\alpha^\sigma = \alpha$ iff

$$\alpha_t = \alpha_{\sigma(t)} = \cdots = \alpha_{\sigma^{p-1}(t)} = k$$

for some fixed k. Hence α is partitioned into $c(\sigma)$ parts and every part has equal terms.

Let $\sigma = \theta_1 \cdots \theta_c$ be the cycle decomposition of σ. To count the number of $\alpha \in E_n^m$ with $\alpha^\sigma = \alpha$, we consider α as a function

$$\{1, \dots, m\} \to \{1, \dots, n\}.$$

Suppose $X = \{1, \dots, m\}$ and let $X(\theta_i)$ be the set of all non-fixed points of θ_i. Then

$$X = \bigcup_{i=1}^c X(\theta_i)$$

is a partition for X, and $\alpha^\sigma = \alpha$ iff for any i,

$$\alpha|_{X(\theta_i)} = \text{const}.$$

Call this constant s_i. If $|X(\theta_i)|$ is an even number, then we can choose $1 \le s_i \le n$ arbitrary. This can be done in $n^{c_0(\sigma)}$ ways. Next, we are going to count the number of cases for those s_i such that $|X(\theta_i)|$ is odd. Let $i_1, \dots, i_l$ be the set of all indices with $|X(\theta_i)|$ odd. Note that $l = c_1(\sigma)$. Define

$$\alpha' = (\alpha_{i_1}, \dots, \alpha_{i_l}).$$

Now $\alpha \in E_n^m$ iff $\alpha' \in E_n^l$. Hence the number of $\alpha \in E_n^m$ with $\alpha^\sigma = \alpha$ is equal to

$$n^{c_0(\sigma)} |E_n^{c_1(\sigma)}|.$$

But using Proposition 1.1, we have

$$|E_n^{c_1(\sigma)}| = \sum_{k=0}^{[c_1(\sigma)/2]} \binom{c_1(\sigma)}{2k} (n-r)^{2k} r^{c_1(\sigma)-2k},$$

and this completes the proof.

NOTE. It is easy to find a basis for $\mathcal{L}_\chi^i$. Let $\bar{\Delta}$ and $\hat{\Delta}$ be as in Section 2. We define

$$\bar{\Delta}_E = \bar{\Delta} \cap E_n^m, \quad \bar{\Delta}_O = \bar{\Delta} \cap O_n^m,$$
$$\hat{\Delta}_E = \hat{\Delta} \cap E_n^m, \quad \hat{\Delta}_O = \hat{\Delta} \cap O_n^m.$$

Then the sets $\{e_\alpha^* : \alpha \in \hat{\Delta}_E\}$ and $\{e_\alpha^* : \alpha \in \hat{\Delta}_O\}$ are bases for $\mathcal{L}_\chi^0$ and $\mathcal{L}_\chi^1$, respectively.

5. On the vanishing of even and odd parts. In this section, we give a necessary and sufficient condition for vanishing of $\mathcal{L}_\chi^i$. Note that the map $\sigma \mapsto P_\sigma^i$ is a representation of S_m on the vector space $\mathcal{L}^i$. Let ξ^i be the corresponding character, i.e.,

$$\xi^i(\sigma) = \operatorname{Tr} P_\sigma^i.$$

We have

$$\dim \mathcal{L}_\chi^i = \chi(1)[\chi, \xi_G^i] = \chi(1)[\chi^{S_m}, \xi^i],$$

by Frobenius reciprocity. So we must compute the irreducible constituents of ξ^i to determine when $\mathcal{L}_\chi^i = 0$. A similar problem (computing the irreducible constituents of $\xi(\sigma) = n^{c(\sigma)}$) was solved in [9]. In what follows, we compute the irreducible constituents of ξ^0 and one can do the same for ξ^1.

First of all, we give a summary of ordinary representations of the symmetric group S_m. The reader is referred to [3] or [7] for a detailed discussion. Ordinary representations of S_m are in one-to-one correspondence with *partitions* of m. Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be any partition of m. We call $h(\lambda) = s$ the *height* of λ. The *Ferrers diagram* of λ is a set of m boxes, arranged in s rows (left aligned) and for any $1 \le i \le s$, the ith row consists of λ_i boxes. A *Young tableau* associated with λ, or a λ-*tableau*, arises from the Ferrers diagram of λ by inserting numbers $1, \ldots, m$ in the boxes. If t is a λ-tableau, then the number in the (i, j) box is denoted by t_{ij}. The symmetric group S_m acts on the set of all λ-tableaux if we set

$$(t^\sigma)_{ij} = (t_{ij})^\sigma.$$

Two tableaux t and t' are said to be *equivalent* iff their corresponding rows contain the same numbers. The equivalence class of t is denoted by $\{t\}$ and it is called a λ-*tabloid*. For any $\sigma \in S_m$, define

$$\{t\}^\sigma = \{t^\sigma\}.$$

So S_m acts on the set of λ-tabloids. Let M^λ be the free vector space generated by the set of all λ-tabloids over the field of complex numbers. Then M^λ is a $\mathbb{C}S_m$-module and we have

$$\dim M^\lambda = \frac{m!}{\lambda_1! \cdots \lambda_s!}.$$

The *Young subgroup* associated with λ is defined as follows:

$$S_\lambda = S_{\{1,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots .$$

Therefore we have $S_\lambda \cong S_{\lambda_1} \times \cdots \times S_{\lambda_s}$. The corresponding character of M^λ is $(1_{S_\lambda})^{S_m}$ (see [7]). We can look at M^λ from another point of view. First choose a left transversal for $H = S_\lambda$ in S_m, say $\theta_1, \ldots, \theta_k$, and form the $\mathbb{C}S_m$-module $\mathbb{C}[\theta_1 H, \ldots, \theta_k H]$. It is known that

$$M^\lambda \cong \mathbb{C}[\theta_1 H, \ldots, \theta_k H].$$

Let t be a λ-tableau and C_t be the column stabilizer subgroup of t. Define

$$\kappa_t = \sum_{\sigma \in C_t} \varepsilon(\sigma)\sigma,$$

where ε is the alternating character of S_m. Now define the λ-polytabloid $p_t = \kappa_t\{t\}$. Let S^λ be the submodule of M^λ generated by the set of all λ-polytabloids. This is called the *Specht module* associated with λ. It is well known that the Specht modules associated with all partitions of m are all of the non-isomorphic irreducible $\mathbb{C}S_m$-modules (see [3] or [7]).

Now, we are ready to talk about the irreducible constituents of M^λ. To do this, we need to define the concept of *majorizing* in the set of all partitions of m. Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\mu = (\mu_1, \ldots, \mu_l)$ be two partitions of m. We say that μ majorizes λ iff for any $1 \le i \le \min\{s, l\}$,

$$\lambda_1 + \cdots + \lambda_i \le \mu_1 + \cdots + \mu_i.$$

In this case we write $\lambda \trianglelefteq \mu$. This is clearly a partial ordering on the set of all partitions of m. A *generalized μ-tableau* of type λ is a function

$$T : \{(i,j) : 1 \le i \le h(\mu),\ 1 \le j \le \mu_i\} \to \{1, \ldots, m\}$$

such that $|T^{-1}(i)| = \lambda_i$ for any $1 \le i \le m$. This generalized tableau is called *semi-standard* if for each i, $j_1 < j_2$ implies $T(i, j_1) \le T(i, j_2)$, and for any j, $i_1 < i_2$ implies $T(i_1, j) < T(i_2, j)$. In other words, T is semi-standard iff every row of T is nondescending and every column of T is ascending. The number of all such semi-standard tableaux is denoted by $K_{\mu\lambda}$ and it is called the *Kostka number*. It is well known that $K_{\mu\lambda} \ne 0$ iff μ majorizes λ (see [6] for example), and that

$$M^\lambda = \sum_{\lambda \trianglelefteq \mu}^{\cdot} K_{\mu\lambda} S^\lambda.$$

We now return to the problem of determining the irreducible constituents of ξ^0 or equivalently of $\mathcal{L}^0$. Let $\mathbb{C}[E_n^m]$ be the free vector space generated by E_n^m over $\mathbb{C}$. We know that S_m acts on E_n^m and so $\mathbb{C}[E_n^m]$ is a $\mathbb{C}S_m$-module. On the other hand, $\mathcal{L}^0$ is a $\mathbb{C}S_m$-module as well. In fact these two modules are isomorphic via the map

$$\alpha \mapsto e_\alpha^\otimes.$$

So we are done if we can obtain all irreducible constituents of $\mathbb{C}[E_n^m]$. Now, let $\mathcal{M}$ be the set of all orbits of E_n^m under the action of S_m. For any $\Omega \in \mathcal{M}$ we have

$$\mathbb{C}[\Omega] \leq_{\mathbb{C}S_m} \mathbb{C}[E_n^m]$$

and

$$\mathbb{C}[E_n^m] = \dot{\sum_{\Omega \in \mathcal{M}}} \mathbb{C}[\Omega].$$

For any α and $1 \leq t \leq n$, we denote the multiplicity of t in α by $m_t(\alpha)$. We define the *multiplicity composition* of α by

$$m(\alpha) = (m_1(\alpha), \ldots, m_n(\alpha)).$$

The non-zero terms of $m(\alpha)$ form a partition of m (the *multiplicity partition* of α), which we denote by λ_α.

PROPOSITION 5.1. *Let $\alpha \in \Omega$. Then*

$$M^{\lambda_\alpha} \cong \mathbb{C}[\Omega].$$

Proof. For simplicity, let $\lambda = \lambda_\alpha$. There is an element $\omega \in \Omega$ such that $S_\lambda = \mathrm{Stab}_{S_m}(\omega)$, where S_λ is the Young subgroup. Suppose $\theta_1, \ldots, \theta_k$ is a left transversal for $H = S_\lambda$ in S_m. We must show that

$$\mathbb{C}[\Omega] \cong \mathbb{C}[\theta_1 H, \ldots, \theta_k H].$$

We know that the action of S_m on Ω is transitive, so $\mathbb{C}[\Omega] = \mathbb{C}S_m[\omega]$. Define

$$\phi : \mathbb{C}[\theta_1 H, \ldots, \theta_k H] \to \mathbb{C}[\Omega] \quad \text{by} \quad \phi(\theta_i H) = \omega^{\theta_i}.$$

It is easy to see that this is the required $\mathbb{C}S_m$-isomorphism.

COROLLARY 5.2. *Let $\Omega \in \mathcal{M}$ and $\alpha \in \Omega$. Then the irreducible constituents of $\mathbb{C}[\Omega]$ are the Specht modules S^μ, in which $\lambda_\alpha \trianglelefteq \mu$.*

COROLLARY 5.3. *The irreducible constituents of ξ^0 are the Specht modules S^μ, where $\lambda_\alpha \trianglelefteq \mu$ for some $\alpha \in E_n^m$.*

Using the equality $\dim \mathcal{L}_\chi^0 = \chi(1)[\chi^{S_m}, \xi^0]$ and Corollary 5.3, we obtain the following theorem.

THEOREM 5.4. *$\mathcal{L}_\chi^0$ does not vanish iff χ^{S_m} has an irreducible constituent of the form λ_α with $\alpha \in E_n^m$.*

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