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POLYNOMIAL ALGEBRA OF CONSTANTS OF THE FOUR VARIABLE LOTKA–VOLTERRA SYSTEM

BY

PIOTR OSSOWSKI and JANUSZ ZIELIŃSKI (Toruń)

Abstract. We describe the ring of constants of a specific four variable Lotka–Volterra derivation. We investigate the existence of polynomial constants in the other cases of Lotka–Volterra derivations, also in n variables.

1. Introduction. Let k be a field of characteristic zero. Let R be a commutative k-algebra. A k-linear mapping $d : R \to R$ is called a *k-derivation* (or simply a *derivation*) of R if $d(ab) = ad(b) + bd(a)$ for all $a, b \in R$. By R^d we denote the kernel of the mapping d. It forms a ring and we call it the *ring of constants* of the derivation d. Then $k \subseteq R^d$ and a *nontrivial constant* of the derivation d is an element of the set $R^d \setminus k$. By $k[X]$ we denote $k[x_1, \ldots, x_n]$, the polynomial ring in n variables. If $f_1, \ldots, f_n \in k[X]$, then there exists exactly one derivation $d : k[X] \to k[X]$ such that $d(x_1) = f_1, \ldots, d(x_n) = f_n$.

There is no general effective procedure for determining the ring of constants. Even for a given specific derivation the problem may be difficult; see for instance various counterexamples to Hilbert's fourteenth problem (for example by Deveney and Finston [1]), the derivations of Jouanolou type (for example Maciejewski et al. [2]), the three variable Lotka–Volterra derivation (Moulin Ollagnier and Nowicki [3]).

The Lotka–Volterra derivations, besides multiple applications in various branches of science, especially in biology, play an important role in the derivation theory itself. A derivation $d : k[X] \to k[X]$ is said to be *factorizable* if $d(x_i) = x_i f_i$, where $f_i \in k[X]$ for $i = 1, \ldots, n$. The most useful case is when all f_i are of degree 1. Examples of such derivations are Lotka–Volterra derivations. How to associate with any given derivation a factorizable derivation having all f_i of degree 1 is shown in [6]. The construction helps to establish new facts on constants of the initial derivation (see, for instance, Nowicki and Zieliński [5]). For details and discussion we refer the reader to [5] and [2].

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The aim of this paper is to present some method of determining the ring of constants. Section 2 contains several properties of specific Lotka–Volterra derivations. In Section 3 we prove Theorem 3.1. It gives a full description of the ring of polynomial constants of the derivation $d : k[x, y, z, t] \to k[x, y, z, t]$ of the form

$$d = x(t - y)\frac{\partial}{\partial x} + y(x - z)\frac{\partial}{\partial y} + z(y - t)\frac{\partial}{\partial z} + t(z - x)\frac{\partial}{\partial t}.$$

It is the main result of the paper. Finally, in Section 4, we make some further considerations on various cases.

Let $\mathbb{N}$ denote the set of nonnegative integers. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by X^α the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n} \in k[X]$ and by $|\alpha|$ the sum $\alpha_1 + \cdots + \alpha_n$. An element of $\mathbb{N}^n$ with ith coordinate equal to 1 and the remaining coordinates equal to 0 is designated by ε_i (moreover, we assume that $\varepsilon_0 = \varepsilon_n$ and $\varepsilon_{n+1} = \varepsilon_1$). A derivation $d : k[X] \to k[X]$ is called *homogeneous of degree s* if the image of a homogeneous form of degree t under d is a homogeneous form of degree $s + t$ for all $t \in \mathbb{N}$.

2. Preliminary lemmas and propositions. Let $R = k[x_1, \ldots, x_n]$. Throughout this section, $n \geq 3$ and $d : R \to R$ is the derivation defined by

$$d(x_i) = x_i(x_{i-1} - x_{i+1}) \tag{2.1}$$

for $i = 1, \ldots, n$, and we adhere to the convention that $x_{n+1} = x_1$ and $x_0 = x_n$. Denote by $R_{(i)}$ the homogeneous component of R of degree i. Let $R^d_{(i)} = R_{(i)} \cap R^d$. Since d is homogeneous, we have $R^d = \bigoplus_{i=0}^{\infty} R^d_{(i)}$.

LEMMA 2.1. *Let $m \geq 1$. Let $\varphi = \sum_{|\alpha|=m} b_\alpha X^\alpha \in R_{(m)}$, where $b_\alpha \in k$. Then $\varphi \in R^d_{(m)}$ if and only if for every $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ such that $|\beta| = m + 1$ we have $\sum_{i=1}^n \beta_i(b_{\beta-\varepsilon_{i-1}} - b_{\beta-\varepsilon_{i+1}}) = 0$.*

Proof. We compute the value of d at $\varphi = \sum_{|\alpha|=m} b_\alpha X^\alpha \in R_{(m)}$, where $m \geq 1$, as follows:

$$\begin{aligned} d(\varphi) &= \sum_{|\alpha|=m} b_\alpha d(X^\alpha) = \sum_{|\alpha|=m} b_\alpha \sum_{i=1}^n \alpha_i X^{\alpha-\varepsilon_i} d(x_i) \\ &= \sum_{|\alpha|=m} b_\alpha \sum_{i=1}^n \alpha_i X^{\alpha-\varepsilon_i} x_i(x_{i-1} - x_{i+1}) \\ &= \sum_{|\alpha|=m} b_\alpha \sum_{i=1}^n \alpha_i (X^{\alpha+\varepsilon_{i-1}} - X^{\alpha+\varepsilon_{i+1}}) \end{aligned}$$

$$= \sum_{|\alpha|=m} \sum_{i=1}^{n} b_\alpha \alpha_i X^{\alpha+\varepsilon_{i-1}} - \sum_{|\alpha|=m} \sum_{i=1}^{n} b_\alpha \alpha_i X^{\alpha+\varepsilon_{i+1}}$$

$$= \sum_{\substack{|\beta|=m+1 \\ \beta_{i-1}>0}} \sum_{i=1}^{n} b_{\beta-\varepsilon_{i-1}} \beta_i X^\beta - \sum_{\substack{|\beta|=m+1 \\ \beta_{i+1}>0}} \sum_{i=1}^{n} b_{\beta-\varepsilon_{i+1}} \beta_i X^\beta.$$

We adopt the convention that $b_\alpha = 0$ when $\alpha_i < 0$ for some $1 \le i \le n$. Therefore

$$d(\varphi) = \sum_{|\beta|=m+1} \sum_{i=1}^{n} b_{\beta-\varepsilon_{i-1}} \beta_i X^\beta - \sum_{|\beta|=m+1} \sum_{i=1}^{n} b_{\beta-\varepsilon_{i+1}} \beta_i X^\beta$$

$$= \sum_{|\beta|=m+1} X^\beta \sum_{i=1}^{n} (b_{\beta-\varepsilon_{i-1}} \beta_i - b_{\beta-\varepsilon_{i+1}} \beta_i).$$

Hence $d(\varphi) = 0$ if and only if $\sum_{i=1}^{n} \beta_i (b_{\beta-\varepsilon_{i-1}} - b_{\beta-\varepsilon_{i+1}}) = 0$ for all $|\beta| = m+1$. ∎

COROLLARY 2.2. *Let* $\varphi = \sum_{|\alpha|=m} b_\alpha X^\alpha \in R^d_{(m)}$, *where* $m \ge 1$. *If* $r, s \in \mathbb{N} \setminus \{0\}$ *and* $r + s = m + 1$, *then* $r b_{r\varepsilon_i+(s-1)\varepsilon_{i+1}} = s b_{(r-1)\varepsilon_i+s\varepsilon_{i+1}}$.

Proof. Let $\beta = r\varepsilon_i + s\varepsilon_{i+1}$. According to Lemma 2.1,

$$\beta_i (b_{\beta-\varepsilon_{i-1}} - b_{\beta-\varepsilon_{i+1}}) + \beta_{i+1} (b_{\beta-\varepsilon_i} - b_{\beta-\varepsilon_{i+2}}) = 0,$$

because $\beta_j = 0$ for $j \notin \{i, i+1\}$. Then $\beta_i = r$, $\beta_{i+1} = s$, $b_{\beta-\varepsilon_{i-1}} = 0$, $b_{\beta-\varepsilon_{i+2}} = 0$, hence

$$-r b_{r\varepsilon_i+(s-1)\varepsilon_{i+1}} + s b_{(r-1)\varepsilon_i+s\varepsilon_{i+1}} = 0. \ ∎$$

Let $\varphi \in R$ and $1 \le q \le n$. Then for every subset $\{i_1, \ldots, i_q\} \subseteq \{1, \ldots, n\}$ we denote by $\varphi^{\{i_1,\ldots,i_q\}}$ the sum of monomials of φ that depend on variables $x_{i_1}, \ldots, x_{i_q}$, that is, $\varphi^{\{i_1,\ldots,i_q\}} = \varphi|_{x_j=0 \text{ for } j \notin \{i_1,\ldots,i_q\}}$.

LEMMA 2.3. *If* $\varphi \in R^d_{(m)}$, *then* $\varphi^{\{i,i+1\}} = c(x_i + x_{i+1})^m$ *for* $c \in k$.

Proof. Let $\varphi^{\{i,i+1\}} = \sum_{r=0}^{m} b_{r\varepsilon_i+(m-r)\varepsilon_{i+1}} x_i^r x_{i+1}^{m-r}$. By Corollary 2.2 we have $r b_{r\varepsilon_i+(m-r)\varepsilon_{i+1}} = (m+1-r) b_{(r-1)\varepsilon_i+(m+1-r)\varepsilon_{i+1}}$ for $r = 1, \ldots, m$.

We show that $b_{r\varepsilon_i+(m-r)\varepsilon_{i+1}} = \binom{m}{r} b_{m\varepsilon_{i+1}}$. We proceed by induction on r. If $r = 1$, then $b_{\varepsilon_i+(m-1)\varepsilon_{i+1}} = m b_{m\varepsilon_{i+1}} = \binom{m}{1} b_{m\varepsilon_{i+1}}$. Let $r > 1$. Then

$$b_{r\varepsilon_i+(m-r)\varepsilon_{i+1}} = \frac{m+1-r}{r} b_{(r-1)\varepsilon_i+(m+1-r)\varepsilon_{i+1}} = \frac{m+1-r}{r} \binom{m}{r-1} b_{m\varepsilon_{i+1}},$$

by the inductive assumption. Therefore $b_{r\varepsilon_i+(m-r)\varepsilon_{i+1}} = \binom{m}{r} b_{m\varepsilon_{i+1}}$. Hence

$$va^{\{i,i+1\}} = \sum_{r=0}^{m} \binom{m}{r} b_{m\varepsilon_{i+1}} x_i^r x_{i+1}^{m-r} = b_{m\varepsilon_{i+1}} (x_i + x_{i+1})^m. \ ∎$$

Proposition 2.4. $R^d_{(1)} = k\sum_{j=1}^n x_j$.

Proof. Let $\varphi = \sum_{j=1}^n b_{\varepsilon_j} x_j \in R^d_{(1)}$. By Lemma 2.3, $b_{\varepsilon_i} x_i + b_{\varepsilon_{i+1}} x_{i+1} = c_i(x_i + x_{i+1})$ for $i = 1, \ldots, n-1$. Thus $b_{\varepsilon_i} = b_{\varepsilon_{i+1}}$ for $i = 1, \ldots, n-1$. Therefore $\varphi = b_{\varepsilon_1}\sum_{j=1}^n x_j$. Obviously then $d(\varphi) = 0$. ∎

Here and throughout, $\operatorname{supp}(\alpha) = \{i : \alpha_i \neq 0\}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

Lemma 2.5. *If $\varphi \in R^d_{(m)}$, then $\varphi = c(\sum_{j=1}^n x_j)^m + \sum b_\alpha X^\alpha$, where the latter sum is taken over all $|\alpha| = m$ such that either $\#\operatorname{supp}(\alpha) \geq 3$, or $\#\operatorname{supp}(\alpha) = 2$ and the two nonzero exponents are not on consecutive variables* (*in the cyclic sense*).

Proof. Let $\varphi \in R^d_{(m)}$. It follows from Lemma 2.3 that for every $1 \leq i \leq n$ there exists $c \in k$ such that $\varphi^{\{i,i+1\}} = c(x_i + x_{i+1})^m$. Then c is the coefficient of x_i^m (and of x_{i+1}^m, and of $x_{i+2}^m, \ldots$) in the polynomial φ. Likewise, $c\binom{m}{l}$ is the coefficient of $x_i^l x_{i+1}^{m-l}$ in φ. Thus $\varphi - c(\sum_{j=1}^n x_j)^m$ does not contain monomials associated to $x_i^l x_{i+1}^{m-l}$ for any $0 \leq l \leq m$, which proves the assertion. ∎

Proposition 2.6. $R^d_{(2)} = \begin{cases} k(\sum x_j)^2 & \text{for } n = 3, \\ k(\sum x_j)^2 + kx_1x_3 + kx_2x_4 & \text{for } n = 4. \end{cases}$

Proof. According to Lemma 2.5, if $\varphi \in R^d_{(2)}$, then

$$\varphi = c\Big(\sum x_j\Big)^2 + \sum_{\substack{|\alpha|=2 \\ \#\operatorname{supp}(\alpha)\geq 3}} b_\alpha X^\alpha + \sum_{0<j-i\notin\{1,n-1\}} b_{ij}x_ix_j$$

for $c \in k$. Since the conditions $\#\operatorname{supp}(\alpha) \geq 3$ and $|\alpha| = 2$ are contradictory, it follows that

$$\sum_{\substack{|\alpha|=2 \\ \#\operatorname{supp}(\alpha)\geq 3}} b_\alpha X^\alpha = 0.$$

For $n = 3$, we also have $\sum_{0<j-i\notin\{1,n-1\}} b_{ij}x_ix_j = 0$, because then there do not exist nonconsecutive variables (in the cyclic sense). For $n = 4$, we easily check that $\varphi = c(\sum x_j)^2 + px_1x_3 + qx_2x_4$ is a constant of d for all $c, p, q \in k$. ∎

As an obvious consequence of the fact that $x_i \mid d(x_i)$ for $i = 1, \ldots, n$ we obtain the following:

Proposition 2.7. *If $A \subseteq \{1, \ldots, n\}$, then for every homogeneous polynomial $\varphi \in R_{(m)}$ we have $d(\varphi^A)^A = d(\varphi)^A$.*

Corollary 2.8. *If $A \subseteq \{1, \ldots, n\}$, then for every $\varphi \in R^d_{(m)}$ we have $d(\varphi^A)^A = 0$.*

LEMMA 2.9. *If* $B \subseteq A \subseteq \{1, \ldots, n\}$ *and* $d(\varphi^A)^A = 0$, *then also* $d(\varphi^B)^B = 0$.

Proof. Let $\varphi^A = \varphi^B + \psi$, where each monomial in ψ has x_j in a positive power for some $j \in A \setminus B$. Then $d(\varphi^A) = d(\varphi^B) + d(\psi)$. If $d(\varphi^A)^A = 0$, then clearly $d(\varphi^A)^B = 0$. Therefore $0 = d(\varphi^A)^B = d(\varphi^B)^B + d(\psi)^B$. Moreover $d(\psi)^B = 0$, because every monomial in $d(\psi)$ has x_j in a positive power for some $j \in A \setminus B$, by the definition of d. Finally, $d(\varphi^B)^B = 0$. ∎

LEMMA 2.10. *If* $\varphi \in R_{(m)}$, $A = \{i, i+1\} \subseteq \{1, \ldots, n\}$ *and* $d(\varphi^A)^A = 0$, *then* $\varphi^A = c(x_i + x_{i+1})^m$ *for some* $c \in k$.

Proof. Let $\varphi^A = \sum_{r=0}^m b_r x_i^{m-r} x_{i+1}^r$. Then

$$\begin{aligned} d(\varphi^A) &= \sum_{r=0}^m b_r (d(x_i^{m-r}) x_{i+1}^r + x_i^{m-r} d(x_{i+1}^r)) \\ &= \sum_{r=0}^m b_r ((m-r) x_i^{m-r} x_{i+1}^r (x_{i-1} - x_{i+1}) + r x_i^{m-r} x_{i+1}^r (x_i - x_{i+2})). \end{aligned}$$

Consequently,

$$\begin{aligned} d(\varphi^A)^A &= \sum_{r=0}^m b_r (r x_i^{m-r+1} x_{i+1}^r - (m-r) x_i^{m-r} x_{i+1}^{r+1}) \\ &= \sum_{r=1}^m r b_r x_i^{m-r+1} x_{i+1}^r - \sum_{r=0}^{m-1} (m-r) b_r x_i^{m-r} x_{i+1}^{r+1} \\ &= \sum_{r=1}^m r b_r x_i^{m-r+1} x_{i+1}^r - \sum_{r=1}^{m} (m-r+1) b_{r-1} x_i^{m-r+1} x_{i+1}^r \\ &= \sum_{r=1}^m (r b_r - (m-r+1) b_{r-1}) x_i^{m-r+1} x_{i+1}^r = 0. \end{aligned}$$

Hence for $r = 1, \ldots, m$ we have $r b_r = (m-r+1) b_{r-1}$, that is, $b_r = \frac{m-r+1}{r} b_{r-1}$. Thus an easy induction on r shows that $b_r = \binom{m}{r} b_0$ for $r = 0, \ldots, m$. Therefore, $\varphi^A = b_0 (x_i + x_{i+1})^m$. ∎

PROPOSITION 2.11. *Let* $n \geq 4$. *If* $\varphi \in R_{(m)}$, $A = \{i, i+1, i+2\} \subseteq \{1, \ldots, n\}$ *and* $d(\varphi^A)^A = 0$, *then* $\varphi^A \in k[x_i + x_{i+1} + x_{i+2}, x_i x_{i+2}]$.

Proof. The proof is by induction on m. Let $m = 1$. By assumption and Lemma 2.9, $d(\varphi^{\{i,i+1\}})^{\{i,i+1\}} = 0$. In view of Lemma 2.10 we have $\varphi^{\{i,i+1\}} = c_1 (x_i + x_{i+1})$. Similarly, we obtain $\varphi^{\{i+1,i+2\}} = c_2 (x_{i+1} + x_{i+2})$. Thus $c_1 = c_2$ and $\varphi^A = c_1 (x_i + x_{i+1} + x_{i+2})$. Now let $m = 2$. Since $d(\varphi^{\{i,i+1\}})^{\{i,i+1\}} = 0$, it follows that $\varphi^{\{i,i+1\}} = c_1 (x_i + x_{i+1})^2$. Analogously $\varphi^{\{i+1,i+2\}} = c_2 (x_{i+1} + x_{i+2})^2$. Therefore, $\varphi^A = c_1 (x_i + x_{i+1} + x_{i+2})^2 + b x_i x_{i+2}$ for some $b \in k$.

Assume $m \geq 3$. Let $\varphi^A = \sum b_\alpha X^\alpha$, where the sum is taken over all α with $|\alpha| = m$ such that $\operatorname{supp}(\alpha) \subseteq \{i, i+1, i+2\}$. We have $\varphi^{\{i,i+1\}} = c_1(x_i + x_{i+1})^m$ and $\varphi^{\{i+1,i+2\}} = c_2(x_{i+1} + x_{i+2})^m$ for $c_1, c_2 \in k$. Thus $c_1 = c_2 =: c$. The terms of the form $x_i^r x_{i+1}^{m-r}$ and $x_{i+1}^r x_{i+2}^{m-r}$ for $r = 0, \ldots, m$ have the same coefficients in φ^A and in $c(x_i + x_{i+1} + x_{i+2})^m$. Therefore

$$\varphi^A = c(x_i + x_{i+1} + x_{i+2})^m + \sum_{\operatorname{supp}(\alpha)=\{i,i+2\}} b_\alpha X^\alpha + \sum_{\operatorname{supp}(\alpha)=\{i,i+1,i+2\}} b_\alpha X^\alpha,$$

that is, $\varphi^A = c(x_i + x_{i+1} + x_{i+2})^m + x_i x_{i+2}\psi$, where $\psi \in R_{(m-2)}$.

Obviously, $\psi^A = \psi$. We show that $d(\psi^A)^A = 0$. First,

$$\begin{aligned} d(\varphi^A) &= cd((x_i + x_{i+1} + x_{i+2})^m) + d(x_i x_{i+2})\psi + x_i x_{i+2} d(\psi) \\ &= cd\Big(\Big(\Big(\sum_{j=1}^n x_j\Big)^m\Big)^A\Big) + d\Big(\Big(\sum_{0<s-r\notin\{1,n-1\}} x_r x_s\Big)^A\Big)\psi + x_i x_{i+2} d(\psi^A). \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &= d(\varphi^A)^A \\ &= cd\Big(\Big(\Big(\sum_{j=1}^n x_j\Big)^m\Big)^A\Big)^A + d\Big(\Big(\sum_{0<s-r\notin\{1,n-1\}} x_r x_s\Big)^A\Big)^A\psi + x_i x_{i+2} d(\psi^A)^A. \end{aligned}$$

Because $(\sum_{j=1}^n x_j)^m$ and $\sum_{0<s-r\notin\{1,n-1\}} x_r x_s$ belong to the ring of constants of the derivation d, it follows from Corollary 2.8 that $d(((\sum_{j=1}^n x_j)^m)^A)^A = 0$ and $d((\sum_{0<s-r\notin\{1,n-1\}} x_r x_s)^A)^A = 0$. Hence indeed $d(\psi^A)^A = 0$.

By the inductive assumption, $\psi = \psi^A \in k[x_i + x_{i+1} + x_{i+2}, x_i x_{i+2}]$. Thus $\varphi^A = c(x_i + x_{i+1} + x_{i+2})^m + x_i x_{i+2}\psi \in k[x_i + x_{i+1} + x_{i+2}, x_i x_{i+2}]$. ∎

3. Main theorem

THEOREM 3.1. *Let $R = k[x_1, \ldots, x_4]$. Let $d : R \to R$ be the derivation of the form*

$$d(x_i) = x_i(x_{i-1} - x_{i+1})$$

for $i = 1, \ldots, 4$. Then

$$R^d = k[x_1 + x_2 + x_3 + x_4, x_1 x_3, x_2 x_4].$$

Proof. It suffices to show that $R^d_{(m)} \subseteq k[x_1 + x_2 + x_3 + x_4, x_1 x_3, x_2 x_4]$ for all $m \geq 1$. We proceed by induction on m. In view of Proposition 2.4, if $\varphi \in R^d_{(1)}$, then $\varphi = c\sum_{j=1}^4 x_j$ for $c \in k$. From Proposition 2.6, if $\varphi \in R^d_{(2)}$, then $\varphi = c_1(\sum_{j=1}^4 x_j)^2 + c_2 x_1 x_3 + c_3 x_2 x_4$ for $c_1, c_2, c_3 \in k$.

Let $\varphi \in R^d_{(3)}$. By Lemma 2.5,

$$\varphi = c\Big(\sum_{j=1}^{4} x_j\Big)^3 + p_1x_1x_3^2 + p_2x_1^2x_3 + q_1x_2x_4^2 + q_2x_2^2x_4 + r_1x_2x_3x_4 + r_2x_1x_3x_4 + r_3x_1x_2x_4 + r_4x_1x_2x_3.$$

Therefore, putting $x_4 = 0$ we get

$$\begin{aligned}\varphi^{\{1,2,3\}} &= c\Big(\sum_{j=1}^{3} x_j\Big)^3 + p_1x_1x_3^2 + p_2x_1^2x_3 + r_4x_1x_2x_3\\ &= c\Big(\sum_{j=1}^{3} x_j\Big)^3 + x_1x_3(p_2x_1 + r_4x_2 + p_1x_3).\end{aligned}$$

According to Corollary 2.8 and Proposition 2.11,

$$\varphi^{\{1,2,3\}} \in k[x_1 + x_2 + x_3, x_1x_3].$$

Hence $p_1 = p_2 = r_4 =: p$. For $\varphi^{\{1,3,4\}}$ we similarly obtain $p_1 = p_2 = r_2 = p$. Analogously, $q_1 = q_2 = r_1 = r_3 =: q$. Thus

$$\varphi = c\Big(\sum_{j=1}^{4} x_j\Big)^3 + px_1x_3\Big(\sum_{j=1}^{4} x_j\Big) + qx_2x_4\Big(\sum_{j=1}^{4} x_j\Big).$$

Assume $m \geq 4$. Let $\varphi \in R^d_{(m)}$. Denote by $\sum_A$ the sum $\sum_{\mathrm{supp}(\alpha)=A} b_\alpha X^\alpha$. Then by Lemma 2.5,

$$\varphi = c\Big(\sum_{j=1}^{4} x_j\Big)^m + \sum_{\{1,3\}} + \sum_{\{2,4\}} + \sum_{\{1,2,3\}} + \sum_{\{1,2,4\}} + \sum_{\{1,3,4\}} + \sum_{\{2,3,4\}} + \sum_{\{1,2,3,4\}}$$

and this decomposition is unique. By Corollary 2.8 and Proposition 2.11,

$$\varphi^{\{1,2,3\}} = c\Big(\sum_{j=1}^{3} x_j\Big)^m + \sum_{\{1,3\}} + \sum_{\{1,2,3\}} \in k[x_1 + x_2 + x_3, x_1x_3].$$

Then we have

$$\sum_{\{1,3\}} + \sum_{\{1,2,3\}} = c_1\Big(\sum_{j=1}^{3} x_j\Big)^{m-2} x_1x_3 + c_2\Big(\sum_{j=1}^{3} x_j\Big)^{m-4} (x_1x_3)^2 + \cdots. \tag{3.1}$$

Let $\Phi_1(u,v) = c_1u^{m-2}v + c_2u^{m-4}v^2 + c_3u^{m-6}v^3 + \cdots \in k[u,v]$. Then $\deg_{(1,2)} \Phi_1$ equals m and Φ_1 is uniquely determined, since $k[x_1 + x_2 + x_3, x_1x_3]$ is a polynomial ring.

Moreover, Φ_1 is uniquely determined by $\sum_{\{1,3\}}$, because c_1 is the coefficient of $x_1^{m-1}x_3$ on the right-hand side of (3.1), whereas $x_1^{m-1}x_3$ appears on the left-hand side in $\sum_{\{1,3\}}$ only, that is, c_1 equals the coefficient of $x_1^{m-1}x_3$

in $\sum_{\{1,3\}}$. Similarly, the coefficient of $x_1^{m-2}x_3^2$ on the right-hand side of (3.1) is equal to $c_2+c_1(m-2)$, while the monomial $x_1^{m-2}x_3^2$ appears only in $\sum_{\{1,3\}}$ on the left-hand side of (3.1). Analogously, by recursion, we conclude that every c_i is determined by $\sum_{\{1,3\}}$.

Let us now consider

$$\varphi^{\{1,3,4\}} = c(x_1+x_3+x_4)^m + \sum_{\{1,3\}} + \sum_{\{1,3,4\}} .$$

Then we have

$$\sum_{\{1,3\}} + \sum_{\{1,3,4\}} = b_1(x_1+x_3+x_4)^{m-2}x_1x_3 + b_2(x_1+x_3+x_4)^{m-4}(x_1x_3)^2 + \cdots .$$

The coefficients $b_1, b_2, \ldots$ are determined by $\sum_{\{1,3\}}$ in the same way as $c_1, c_2, \ldots$, hence $b_i = c_i$ for all i. Consequently, $\sum_{\{1,3\}} + \sum_{\{1,3,4\}} = \Phi_1(x_1 + x_3 + x_4, x_1x_3)$.

Therefore

$$\begin{aligned}\Phi_1\Big(\sum_{j=1}^4 x_j, x_1x_3\Big) &= c_1\Big(\sum_{j=1}^4 x_j\Big)^{m-2} x_1x_3 + c_2\Big(\sum_{j=1}^4 x_j\Big)^{m-4}(x_1x_3)^2 + \cdots \\ &= \sum_{\{1,3\}} + \sum_{\{1,2,3\}} + \sum_{\{1,3,4\}} + x_1x_2x_3x_4\Psi_1\end{aligned}$$

for some $\Psi_1 \in R$.

The reasoning above shows that there exists $\Phi_1 \in k[u,v]$ such that

$$\varphi = c\Big(\sum_{j=1}^4 x_j\Big)^m + \Phi_1\Big(\sum_{j=1}^4 x_j, x_1x_3\Big) + \sum_{\{2,4\}} + \sum_{\{1,2,4\}} + \sum_{\{2,3,4\}} + x_1x_2x_3x_4\bar{\Psi}_1$$

for some $\bar{\Psi}_1 \in R$.

Analogously, there exist $\Phi_2 \in k[u,v]$ and $\Psi_2 \in R$ such that

$$\Phi_2\Big(\sum_{j=1}^4 x_j, x_2x_4\Big) = \sum_{\{2,4\}} + \sum_{\{1,2,4\}} + \sum_{\{2,3,4\}} + x_1x_2x_3x_4\Psi_2.$$

Consequently,

$$\varphi = c\Big(\sum_{j=1}^4 x_j\Big)^m + \Phi_1\Big(\sum_{j=1}^4 x_j, x_1x_3\Big) + \Phi_2\Big(\sum_{j=1}^4 x_j, x_2x_4\Big) + x_1x_2x_3x_4\Psi$$

for some $\Psi \in R$.

All the polynomials $c(\sum_{j=1}^4 x_j)^m$, $\Phi_1(\sum_{j=1}^4 x_j, x_1x_3)$, $\Phi_2(\sum_{j=1}^4 x_j, x_2x_4)$, $x_1x_2x_3x_4$ belong to R^d. Thus $\varphi \in R^d_{(m)}$ implies $\Psi \in R^d_{(m-4)}$. Hence, by the inductive assumption, $\Psi \in k[\sum_{j=1}^4 x_j, x_1x_3, x_2x_4]$. Finally, we deduce that $\varphi \in k[\sum_{j=1}^4 x_j, x_1x_3, x_2x_4]$. ■

4. Further results. It follows from Proposition 2.4 that for every $n \geq 3$ the derivation $d : k[x_1, \dots, x_n] \to k[x_1, \dots, x_n]$ defined by $d(x_i) = x_i(x_{i-1} - x_{i+1})$ for $i = 1, \dots, n$ has a nontrivial polynomial constant. A simple calculation shows that the derivation d always has a nontrivial monomial constant. Namely, we have the following proposition.

PROPOSITION 4.1. *Let d be a derivation of the form* (2.1). *If n is odd, then a monomial f belongs to R^d if and only if $f = c(x_1 \dots x_n)^a$ for $c \in k$ and $a \in \mathbb{N}$. If $n = 2k$, then a monomial f belongs to R^d if and only if $f = c(x_1 x_3 \dots x_{2k-1})^a (x_2 x_4 \dots x_{2k})^b$ for $c \in k$ and $a, b \in \mathbb{N}$.*

By the definition, d has the property:

PROPOSITION 4.2. *The ring of constants R^d is invariant under the action of the subgroup of the group of permutations generated by the cycle* $(2\ 3\ \dots\ n\ 1)$.

Note that this does not mean that each particular constant is invariant. Proposition 4.3 is a simple extension of Proposition 2.6.

PROPOSITION 4.3. $R^d_{(2)} = k(\sum x_j)^2 + k \sum_{0<j-i \notin \{1,n-1\}} x_i x_j$ *for* $n \geq 5$.

Let $d : k[x, y, z, t] \to k[x, y, z, t]$ be a derivation of the form

$$d = x(Dy + t)\frac{\partial}{\partial x} + y(Az + x)\frac{\partial}{\partial y} + z(Bt + y)\frac{\partial}{\partial z} + t(Cx + z)\frac{\partial}{\partial t}, \tag{4.1}$$

where $A, B, C, D \in k$. Then linear algebra calculations give the following proposition.

PROPOSITION 4.4. *Let d be a derivation of the form* (4.1). *Then the ring $k[x, y, z, t]^d$ has a nonzero homogeneous constant of degree 2 if and only if at least one of the following conditions holds:*

(1) $ABCD = 1$,
(2) $A = -1$ *and* $C = -1$,
(3) $B = -1$ *and* $D = -1$,
(4) $ABCD = -1$ *and at least one of the elements A or C equals -1 and at least one of the elements B or D equals -1.*

The next proposition is easily verified.

PROPOSITION 4.5. *Let d be a derivation of the form* (4.1). *Then the ring $k[x, y, z, t]^d$ has a nontrivial monomial constant if and only if at least one of the following two conditions is fulfilled:*

(1) *D and B are negative rational numbers and* $DB = 1$,
(2) *A and C are negative rational numbers and* $AC = 1$.

Let $R = k[x_1, \dots, x_n]$, where $n \geq 3$. From now on, let $d : R \to R$ be the derivation defined by

$$d(x_i) = x_i(x_{i-1} - C_i x_{i+1}),$$

where $C_i \in k$ for $i = 1, \dots, n$. The following propositions are analogs of Lemmas 2.1, 2.3, Proposition 2.4 and Lemmas 2.10, 2.5 respectively. Their proofs are analogous as well.

PROPOSITION 4.6. *Let* $\varphi = \sum_{|\alpha|=m} b_\alpha X^\alpha \in R_{(m)}$, *where* $m \geq 1$. *Then* $\varphi \in R^d_{(m)}$ *if and only if for every* $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ *such that* $|\beta| = m+1$ *we have* $\sum_{i=1}^n \beta_i(b_{\beta-\varepsilon_{i-1}} - C_i b_{\beta-\varepsilon_{i+1}}) = 0$.

PROPOSITION 4.7. *If* $\varphi \in R^d_{(m)}$, *then* $\varphi^{\{i,i+1\}} = c(x_i + C_i x_{i+1})^m$ *for some* $c \in k$.

PROPOSITION 4.8. *If* $C_1 \dots C_n \neq 1$, *then* $R^d_{(1)} = 0$. *If* $C_1 \dots C_n = 1$, *then* $R^d_{(1)} = k(x_1 + C_1 x_2 + C_1 C_2 x_3 + \dots + C_1 \dots C_{n-1} x_n)$.

PROPOSITION 4.9. *If* $\varphi \in R_{(m)}$, $A = \{i, i+1\} \subseteq \{1, \dots, n\}$ *and* $d(\varphi^A)^A = 0$, *then* $\varphi^A = c(x_i + C_i x_{i+1})^m$ *for some* $c \in k$.

PROPOSITION 4.10. *If* $\varphi \in R^d_{(m)}$, *then* $\varphi = a(x_1 + C_1 x_2 + C_1 C_2 x_3 + \dots + C_1 \dots C_{n-1} x_n)^m + \sum b_\alpha X^\alpha$, *where the latter sum is taken over all* α *with* $|\alpha| = m$ *such that either* $\#\operatorname{supp}(\alpha) \geq 3$, *or* $\#\operatorname{supp}(\alpha) = 2$ *and the two nonzero exponents are not on consecutive variables (in the cyclic sense). Moreover, if* $(C_1 \dots C_n)^m \neq 1$, *then* $a = 0$.

We hope that the results presented in the paper will be useful in further investigations of the Lotka–Volterra derivations.

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REFERENCES

[1] J. K. Deveney and D. R. Finston, *A proper* G_a *action on* $\mathbb{C}^5$ *which is not locally trivial*, Proc. Amer. Math. Soc. 123 (1995), 651–655.

[2] A. J. Maciejewski, J. Moulin Ollagnier, A. Nowicki and J.-M. Strelcyn, *Around Jouanolou non-integrability theorem*, Indag. Math. 11 (2000), 239–254.

[3] J. Moulin Ollagnier and A. Nowicki, *Polynomial algebra of constants of the Lotka–Volterra system*, Colloq. Math. 81 (1999), 263–270.

[4] A. Nowicki, *Polynomial Derivations and Their Rings of Constants*, N. Copernicus Univ. Press, Toruń, 1994.

[5] A. Nowicki and J. Zieliński, *Rational constants of monomial derivations*, J. Algebra 302 (2006), 387–418.

[6] J. Zieliński, *Factorizable derivations and ideals of relations*, Comm. Algebra 35 (2007), 983–997.

Piotr Ossowski, Janusz Zieliński
Faculty of Mathematics and Computer Science
N. Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: ossowski@mat.uni.torun.pl
ubukrool@mat.uni.torun.pl

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