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SHARP SPECTRAL ASYMPTOTICS AND WEYL FORMULA FOR ELLIPTIC OPERATORS WITH NON-SMOOTH COEFFICIENTS—PART 2

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Abstract. We describe the asymptotic distribution of eigenvalues of self-adjoint elliptic differential operators, assuming that the first-order derivatives of the coefficients are Lipschitz continuous. We consider the asymptotic formula of Hörmander's type for the spectral function of pseudodifferential operators obtained via a regularization procedure of non-smooth coefficients.

1. Introduction. This paper presents a refinement of [22], where we consider the asymptotic behaviour of eigenvalues for an elliptic differential operator A with non-smooth coefficients acting on a compact (boundaryless) smooth manifold M with a density dx. More precisely A is defined as a self-adjoint operator in $L^2(M,dx)$ asssociated with a quadratic form which can be expressed in local coordinates as

$$\sum_{|\alpha|,|\beta|\le m}(a_{\alpha,\beta}D^\alpha\varphi, D^\beta\psi)\quad \text{ for } \varphi,\psi\in C_0^\infty(\mathbb{R}^d)$$

where $(\cdot,\cdot)$ is the scalar product of $L^2(\mathbb{R}^d)$, $a_{\alpha,\beta}=\overline{a}_{\beta,\alpha}\in L^\infty(\mathbb{R}^d)$ and the ellipticity hypothesis means that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha,\beta}(x)\xi^{\alpha+\beta}\ge c|\xi|^{2m}$$

with $c>0$. We assume that the first order derivatives of top order coefficients (i.e. of $a_{\alpha,\beta}$ with $|\alpha|=|\beta|=m$ in local coordinates) are Lipschitz continuous. Then we have

THEOREM 1.1. *The spectrum of A under the above hypotheses is discrete, bounded from below and the counting function $N(A,\lambda)$ (i.e. the number of eigenvalues less than λ, counted with multiplicities) satisfies the Weyl formula*

$$N(A,\lambda)=\omega\lambda^{d/(2m)}(1+O(\lambda^{-1/(2m)}))\qquad (\lambda\to\infty) \tag{1.1}$$

where $\omega>0$ is a constant and d is the dimension of the manifold M.

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The above result is a consequence of the estimate stated in [22, Theorem 1.2] for the pseudodifferential operator A' obtained as a suitable regularization of A. Following Hörmander's well-known approach of [3], instead of A' we study the pseudodifferential operator $P = A'^{1/(2m)}$ of order 1 and the corresponding spectral asymptotics is described in Theorem 1.2 below.

This result is used in [20] to derive the Weyl formula for boundary value problems (cf. [20, Theorem 2.1b] stated without proof). One more result is stated in [20] without proof: the property of finite propagation speed for e^{-itP} formulated in [20, Proposition 2.5b]; we give its proof in Section 3 of this paper. Concerning earlier results for boundary value problems we refer to [2], [5–6], [10] in the case of smooth coefficients, and to [9], [11–14] in the case of irregular coefficients; concerning the related spectral asymptotics for differential or pseudodifferential operators we refer to [1], [4], [8] and [15]. We note (cf. [21]) that Theorem 1.2 can also be used to obtain the Weyl formula for the integrated density of states for transitive, ergodic, elliptic differential operators in $\mathbb{R}^d$ (e.g. operators with almost periodic coefficients).

The precise formulation of Theorem 1.2 uses the following notation of [7]. If $r \geq 0$, $m \in \mathbb{R}$, $0 \leq \delta < \varrho \leq 1$ and X is open in $\mathbb{R}^{d'}$, then $S^m_{\varrho,\delta}(r)(X \times \mathbb{R}^d)$ is the class of functions $a \in C^\infty(X \times \mathbb{R}^d)$ satisfying

$$|\partial_\xi^\alpha \partial_x^{\alpha'} a(x,\xi)| \leq C_{\alpha,\alpha'} \langle \xi \rangle^{m-\varrho|\alpha|+\delta(|\alpha'|-r)_+} \tag{1.2}$$

for $(x,\xi) \in X \times \mathbb{R}^d$, $\alpha \in \mathbb{N}^d$, $\alpha' \in \mathbb{N}^{d'}$, where $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$ and s_+ denotes the positive part of the real number s.

If $r = 0$ then $S^m_{\varrho,\delta}(0)(X \times \mathbb{R}^d) = S^m_{\varrho,\delta}(X \times \mathbb{R}^d)$ is the usual Hörmander class of symbols of type ϱ, δ. We abbreviate $S^m_{\varrho,\delta}(r)(\mathbb{R}^d \times \mathbb{R}^d) = S^m_{\varrho,\delta}(r)$.

We denote by H^s ($s \in \mathbb{R}$) the Sobolev space on $\mathbb{R}^d$ and write $R \in \Psi^{-\infty}$ if R is a linear operator on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ having continuous extensions $H^{-s} \to H^s$ for every $s \in \mathbb{R}$. Then $\Psi^{-\infty}$ is a Fréchet space with seminorms $\|R\|_{B(H^{-n},H^n)}$, $n \in \mathbb{N}$, where $B(\mathcal{X},\mathcal{X}')$ denotes the Banach space of bounded linear operators $\mathcal{X} \to \mathcal{X}'$. A subset of a Fréchet space is called bounded when it is bounded with respect to each seminorm.

If A' is the operator described in [22, Theorem 1.2] and $P = A'^{1/(2m)}$, then due to [22, Lemma 2.1] we have $P = p(x,D) + R$ with $R \in \Psi^{-\infty}$ and

$$p \in S^1_{1,\delta}(2), \tag{1.3}$$

$$|p(x,\xi)| \geq c|\xi| \quad \text{if } |\xi| \geq C, \tag{1.4}$$

$$|\nabla_\xi p(x,\xi)| \geq c \quad \text{if } |\xi| \geq C, \tag{1.5}$$

for certain constants $C, c > 0$. We shall prove

THEOREM 1.2 *Let $0 \leq \delta < 1$ and assume that $P = p(x,D) + R$ is self-adjoint in $L^2(\mathbb{R}^d)$ with $R \in \Psi^{-\infty}$ and p satisfying* (1.3)–(1.5). *Then the spectral projectors $E(P,\lambda) \in \Psi^{-\infty}$ have smooth integral kernels $e(P,\cdot,\cdot,\lambda)$*

and

$$e(P,y,y,\lambda) = \omega(p,y,\lambda)(1+O(\lambda^{-1})) \tag{1.6}$$

with

$$\omega(p,y,\lambda) = (2\pi)^{-d} \int\limits_{\operatorname{Re} p(y,\xi)<\lambda} d\xi,$$

uniformly with respect to $y \in \mathbb{R}^d$.

Since the above result implies [22, Theorem 1.2] with no restrictions on $0 \le \delta < 1$, we obtain Theorem 1.1 as explained in [22]. The plan of the proof of Theorem 1.2 is the following. The starting point is the decomposition of the operator P given in the following lemma (proved in the Appendix):

LEMMA 1.3. *Let* P *be as in Theorem* 1.2 *and* $\delta < \varrho < 1$. *Then there exist*

$$\hat{P} = \hat{p}(x,D) + \hat{R}, \qquad \check{P} = \check{p}(x,D) + \check{R}, \tag{1.7}$$

which are self-adjoint operators in $L^2(\mathbb{R}^d)$ *such that* $P = \hat{P} + \check{P}$, $\hat{R}, \check{R} \in \Psi^{-\infty}$ *and*

$$\hat{p} \in S^1_{1,1-\varrho}(2), \qquad \check{p} \in S^{1-2(1-\varrho)}_{1,\delta} \cap S^{\varrho}_{1,\delta}(1). \tag{1.8}$$

Moreover the conditions (1.4), (1.5) *hold with* $\hat{p}$ *in place of* p.

Now we can note that the theory of Fourier integral operators described in [22, Section 4] still holds for $\hat{P}$ in place of P. In Section 2 we recall the consequences of the Egorov theorem proved in [22] and give some refinements in terms of new classes of pseudodifferential operators. These properties are used in Section 3 to obtain the finite propagation speed by using the Kato–Trotter formula (cf. [16]),

$$\sup_{\tau \in [-\theta;\,\theta]} \| e^{-i\tau P}\varphi - (e^{-i\tau \hat{P}/n} e^{-i\tau \check{P}/n})^n \varphi \| \to 0 \quad \text{as } n \to \infty, \tag{1.9}$$

where $\varphi \in L^2(\mathbb{R}^d)$ and $\theta > 0$. To obtain the asymptotic formula (1.6) we use the parabolic approximation of [22, Proposition 3.1] and clearly it suffices to check that [22, Corollary 3.3] still holds without the restriction $\delta < 1/2$. In fact the condition $\delta < 1/2$ was needed for "integrations by parts" described in [22, Proposition 3.4] and the task is to formulate a similar statement valid in the general case $0 \le \delta < 1$. Our approach still uses the Kato–Trotter formula and the calculus of commutators leads to new classes of operators considered in Section 4. The introduction of these classes allows us to give a new statement of "integrations by parts" in Section 5 and to end the proof by reasoning in a similar way to [22, Section 5].

2. Preliminary notations. We introduce some classes of pseudodifferential operators depending on parameters τ and v. We fix $\theta > 0$ small enough and we consider some regularity conditions with respect to $\tau \in [-\theta;\theta]$. The parameter v is an element of a given set V and we assume that the constants are independent of $v \in V$ in all estimates. We write $(v,\tau) \in V_\theta = V \times [-\theta;\theta]$. Set $S^m_\varrho = S^m_{\varrho,1-\varrho}$; moreover, $A \in \Psi^m_{\varrho,\delta}$ means that $A - a(x,D) \in \Psi^{-\infty}$ with $a \in S^m_{\varrho,\delta}$. For $m \in \mathbb{R}$ and $0 \le \delta < \varrho \le 1$ we write

$$A = \{A_v(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}^m_{\varrho,\delta}[V] \tag{2.1}$$

if $A_v(\tau) = a_v(\tau,x,D) + R_v(\tau)$, where $\{R_v\}_{v\in V}$ is a bounded subset of the Fréchet space $C^\infty([-\theta;\theta], \Psi^{-\infty})$ and for every $(l,\alpha',\alpha) \in \mathbb{N}\times\mathbb{N}^d\times\mathbb{N}^d$,

$$|\partial^l_\tau \partial^{\alpha'}_x \partial^\alpha_\xi a_v(\tau,x,\xi)| \le C_{l,\alpha',\alpha}\langle\xi\rangle^{m-\varrho|\alpha|+\delta|\alpha'|+l(1-\varrho)}, \tag{2.2(a)}$$

$$|\partial^l_\tau \partial^{\alpha'}_x \partial^\alpha_\xi a_v(0,x,\xi)| \le C_{l,\alpha',\alpha}\langle\xi\rangle^{m-|\alpha|+\delta|\alpha'|+l(1-\varrho)}, \tag{2.2(b)}$$

with some constants $C_{l,\alpha',\alpha}$.

LEMMA 2.1. *Let $m, \widetilde{m} \in \mathbb{R}$, let A be given by* (2.1) *and consider*

$$\widetilde{A} = \{\widetilde{A}_v(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}^{\widetilde{m}}_{\varrho,\delta}[V]. \tag{2.3}$$

Then

$$A\widetilde{A} = \{A_v(\tau)\widetilde{A}_v(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}^{m+\widetilde{m}}_{\varrho,\delta}[V],$$
$$[A,\, \widetilde{A}] = \{[A_v(\tau),\, \widetilde{A}_v(\tau)]\}_{(v,\tau)\in V_\theta} \in \mathcal{A}^{m+\tilde{m}-\varrho+\delta}_{\varrho,\delta}[V],$$
$$[A_v(\tau),\, x_j] = A^+_v(\tau) + \tau A^-_v(\tau)$$

with $A^\pm = \{A^\pm_v(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}^{m^\pm}_{\varrho,\delta}[V]$, where $m^+ = m-1$, $m^- = m+1-2\varrho$ and x_j stands for the operator of multiplication by the jth coordinate.

Proof. The assertions concerning $A\tilde{A}$, $[A,\, \widetilde{A}]$ are obvious and the last assertion follows as in the proof of [22, Lemma 5.2]. ■

Let $m \in \mathbb{R}$, $r \ge 0$ and $1/2 < \varrho \le 1$. We write

$$A = \{A_v(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}^m_\varrho((r))[V] \tag{2.4(r)}$$

whenever $A_v(\tau) = a_v(\tau,x,D) + R_v(\tau)$ with $\{R_v\}_{v\in V}$ forming a bounded subset of $C^\infty([-\theta;\theta], \Psi^{-\infty})$ and the symbols a_v are such that

$$|\partial^l_\tau \partial^{\alpha'}_x \partial^\alpha_\xi a_v(\tau,x,\xi)| \le C_{l,\alpha',\alpha}\langle\xi\rangle^{m-|\alpha|+(1-\varrho)(|\alpha|+|\alpha'|+l-r)_+}, \tag{2.5(a)}$$

$$|\partial^l_\tau \partial^{\alpha'}_x \partial^\alpha_\xi a_v(0,x,\xi)| \le C_{l,\alpha',\alpha}\langle\xi\rangle^{m-|\alpha|+(1-\varrho)(|\alpha'|+l-r)_+}, \tag{2.5(b)}$$

with some constants $C_{l,\alpha',\alpha}$ (for every $(l,\alpha',\alpha) \in \mathbb{N}\times\mathbb{N}^d\times\mathbb{N}^d$), where as before s_+ denotes the positive part of the real number s.

LEMMA 2.2. *Let $\varrho \ge 2/3$ and let A satisfy* (2.4(r)) *with either $r=0$ or $r=1$. If $\hat{P}$ is as in Lemma* 1.3, *then*

$$\{e^{i\tau\hat{P}} A_v(\tau) e^{-i\tau\hat{P}}\}_{(v,\tau)\in V_\theta} \in \mathcal{A}^m_\varrho((r))[V].$$

Proof. For simplicity we skip the index v. If (2.4(0)) holds and $\mathcal{C}_\theta = (]-\theta;\theta[\times\mathbb{R}^d)\times\mathbb{R}^d$, then due to [22, Corollary 4.2] we have

$$e^{i\tau\hat{P}}A(\tau)e^{-i\tau\hat{P}} = \widetilde{A}(\tau) = \widetilde{a}(\tau,x,D) + \widetilde{R}(\tau)$$

with $\widetilde{a} \in S^m_\varrho(\mathcal{C}_\theta)$ and $\widetilde{R} \in C^\infty([-\theta;\theta];\varPsi^{-\infty})$. Therefore the estimates (2.5(a)) hold with $r = 0$ and $\widetilde{a}$ in place of a. Since $\widetilde{A}(0) = A(0)$, to complete the proof in the case $r = 0$ it remains to show (2.5(b)) for $l \geq 1$, $r = 0$. We write $\partial_{\hat{P}}A(\tau) = \partial^1_{\hat{P}}A(\tau) = \partial_\tau A(\tau) + [i\hat{P}, A(\tau)]$ and $\partial^{l+1}_{\hat{P}}A = \partial_{\hat{P}}(\partial^l_{\hat{P}}A)$ for $l \in \mathbb{N}\setminus\{0\}$, hence $\partial^l_\tau\widetilde{A}|_{\tau=0} = \partial^l_{\hat{P}}A|_{\tau=0}$ and

$$(2.6)\quad (\partial^k_\tau A|_{\tau=0} \in \varPsi^{m+k(1-\varrho)}_{1,1-\varrho} \text{ for } k = 0 \text{ and } 1) \;\Rightarrow\; \partial_{\hat{P}}A|_{\tau=0} \in \varPsi^{m+1-\varrho}_{1,1-\varrho}.$$

From (2.6) and $\partial^k_\tau\partial_{\hat{P}}A = \partial_{\hat{P}}\partial^k_\tau A$ we have

$$(2.7)\quad (\partial^k_\tau A|_{\tau=0} \in \varPsi^{m+k(1-\varrho)}_{1,1-\varrho} \text{ for all } k \in \mathbb{N}) \\ \Rightarrow\; (\partial^k_\tau\partial_{\hat{P}}A|_{\tau=0} \in \varPsi^{m+(k+1)(1-\varrho)}_{1,1-\varrho} \text{ for all } k \in \mathbb{N})$$

and by induction with respect to $l \in \mathbb{N}\setminus\{0\}$ we obtain

$$(2.8)\quad (\partial^k_\tau A|_{\tau=0} \in \varPsi^{m+k(1-\varrho)}_{1,1-\varrho} \text{ for all } k \in \mathbb{N}) \\ \Rightarrow\; (\partial^k_\tau\partial^l_{\hat{P}}A|_{\tau=0} \in \varPsi^{m+(k+l)(1-\varrho)}_{1,1-\varrho} \text{ for all } k \in \mathbb{N},\; l \in \mathbb{N}\setminus\{0\}),$$

hence the estimates (2.5(b)) hold with $r = 0$ and $\widetilde{a}$ in place of a.

Consider now the case $r = 1$. Then [22, Corollary 4.2] ensures

$$\widetilde{a} - \widetilde{a}_0 \in S^{m+1-2\varrho}_\varrho(\mathcal{C}_\theta) \quad \text{with} \quad \widetilde{a}_0(\tau,x,\xi) = a(\tau,\vartheta(\tau,x,\xi)),$$

where $\vartheta : \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d\times\mathbb{R}^d$ is the Hamitonian flow of $\hat{p}_0 = \operatorname{Re}\hat{p}$,

$$\vartheta(t,y,\eta) = \exp(tH_{\hat{p}_0})(y,\eta) = (x(t,y,\eta),\xi(t,y,\eta)),$$

i.e. $\partial_t x(t,y,\eta) = \partial_\xi\hat{p}_0(\vartheta(t,y,\eta))$, $\partial_t\xi(t,y,\eta) = -\partial_x\hat{p}_0(\vartheta(t,y,\eta))$ and $\vartheta(0,y,\eta) = (y,\eta)$. Since $b \in S^{m+1-2\varrho}_\varrho(\mathcal{C}_\theta)$ with $\varrho \geq 2/3$ implies $\partial_x b \in S^{m+2-3\varrho}_\varrho(\mathcal{C}_\theta) \subset S^m_\varrho(\mathcal{C}_\theta)$ and $\partial_\xi b \in S^{m+1-3\varrho}_\varrho(\mathcal{C}_\theta) \subset S^{m-1}_\varrho(\mathcal{C}_\theta)$, it is clear that (2.5(a)) holds with $r = 1$ and $b = \widetilde{a} - \widetilde{a}_0$ in place of a. Moreover the properties of the Hamiltonian flow ϑ described in [22, Lemma 4.1] give the estimates (2.5(a)) with $r = 1$ for $\widetilde{a}_0$ by using [22, Lemma 2.3b].

To complete the proof it remains to show estimates (2.5(b)) with $r = 1$ and $\widetilde{a}$ in place of a. Since $\widetilde{A}(0) = A(0) \in \varPsi^m_{1,1-\varrho}$ and $\partial_\tau\widetilde{A}(0) = \partial_{\hat{P}}A(0) \in \varPsi^m_{1,1-\varrho}$, it suffices to consider (2.5(b)) with $l \geq 2$. We complete the proof using (2.8) with $\partial_{\hat{P}}A$ in place of A, which gives the implication

$$(\partial^k_\tau\partial_{\hat{P}}A|_{\tau=0} \in \varPsi^{m+k(1-\varrho)}_{1,1-\varrho} \text{ for all } k \in \mathbb{N}) \\ \Rightarrow (\partial^l_{\hat{P}}(\partial_{\hat{P}}A)|_{\tau=0} = \partial^{l+1}_{\hat{P}}A|_{\tau=0} \in \varPsi^{m+l(1-\varrho)}_{1,1-\varrho} \text{ for all } l \in \mathbb{N}\setminus\{0\}).\ \blacksquare$$

3. Finite propagation speed. We set $\mathbb{C}_- = \{t \in \mathbb{C} : \operatorname{Im} t < 0\}$, $\overline{\mathbb{C}}_- = \mathbb{C}_- \cup \mathbb{R}$ and we prove

PROPOSITION 3.1. *Let P be as in Theorem 1.2 and let $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi_1 = 1$ on a neighbourhood of* $\operatorname{supp}\chi_2$. *If $\theta = \theta(\chi_1, \chi_2) > 0$ is small enough, then $\{(1-\chi_1)e^{-itP}\chi_2\}_{\{t\in\overline{\mathbb{C}}_-:\,|t|<\theta\}}$ is a bounded subset of $\Psi^{-\infty}$, where $1-\chi_1$ and χ_2 are considered as operators of multiplication by the corresponding functions.*

Replacing P by $P + CI$ with a constant $C > 0$ sufficiently large we can assume that $P \geq I$ and since P is elliptic of degree 1, for every $s \in \mathbb{R}$ we can find constants $C_s, C_s' > 0$ such that

$$\|e^{-itP}\varphi\|_{H^s} \leq C_s\|P^s e^{-itP}\varphi\| \leq C_s\|P^s\varphi\| \leq C_s'\|\varphi\|_{H^s} \quad \text{for } t \in \overline{\mathbb{C}}_-. \tag{3.1}$$

Let $\widetilde{\chi}_2 \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi_1 = 1$ on a neighbourhood of $\operatorname{supp}\widetilde{\chi}_2$ and $\widetilde{\chi}_2 = 1$ on a neighbourhood of $\operatorname{supp}\chi_2$. If $t = \tau - i\varepsilon$ with $\tau = \operatorname{Re} t$ and $\varepsilon > 0$, then $(1-\chi_1)e^{-itP}\chi_2$ can be written in the form

$$(1-\chi_1)e^{-i\tau P}\widetilde{\chi}_2 e^{-\varepsilon P}\chi_2 + (1-\chi_1)e^{-i\tau P}(1-\widetilde{\chi}_2)e^{-\varepsilon P}\chi_2. \tag{3.2}$$

Thus it suffices to prove Proposition 3.1 for $\tau = \operatorname{Re} t$ and $\widetilde{\chi}_2$ in place of t and χ_2. Indeed, it is well known that $\{e^{-\varepsilon P}\}_{0<\varepsilon<\theta}$ is a bounded subset of $\Psi^0_{1,\delta}$ (for every $\theta > 0$), hence the last term of (3.2) belongs to a bounded subset of $\Psi^{-\infty}$.

In the next step of the proof we consider the operator $\check{P}$ described in Lemma 1.3 assuming that $\varrho < 1$. We note that $\check{P}$ is not elliptic and it is not possible to obtain $e^{-it\check{P}} \in B(H^s) := B(H^s, H^s)$ reasoning as in (3.1).

LEMMA 3.2 *Let $\theta > 0$ and $\chi_1, \widetilde{\chi}_2$ be as above. Then*

$$\{(1-\chi_1)e^{-i\tau\check{P}}\widetilde{\chi}_2\}_{\tau\in[-\theta;\theta]}$$

is a bounded subset of $\Psi^{-\infty}$.

Proof. Let $s \in \mathbb{R}$. We show that there is a constant $C_s > 0$ such that

$$\|e^{-i\tau\check{P}}\varphi\|_{H^s} \leq C_s\|\varphi\|_{H^s} \quad \text{for } \tau \in [-\theta;\theta]. \tag{3.3(s)}$$

Using the duality it suffices to show (3.3(s)) for $s \geq 0$ and it is clear that (3.3(0)) holds. Then writing

$$Ae^{-i\tau\check{P}}\varphi = e^{-i\tau\check{P}}A\varphi + \tau\int_0^1 dz\, e^{-i\tau(1-z)\check{P}}[i\check{P}, A]e^{-i\tau z\check{P}}\varphi \tag{3.4}$$

with $A = \langle D\rangle^\kappa$ and assuming (3.3(s)) for a given $s \geq 0$ we can estimate the norm $\|e^{-i\tau\check{P}}\varphi\|_{H^{s+\kappa}}$ by

$$\|Ae^{-i\tau\check{P}}\varphi\|_{H^s} \leq \|e^{-i\tau\check{P}}A\varphi\|_{H^s} + \sup_{0\leq z\leq 1}\theta C_s\|[\check{P}, A]e^{-i\tau z\check{P}}\varphi\|_{H^s}. \tag{3.5(s)}$$

If $0 < \kappa \le 1 - \varrho$, then using (1.8) we obtain

$$[\check{P}, A] \in \Psi_{1,\delta}^{\kappa+\varrho-1} \subset B(H^s) \quad \text{for every } s \in \mathbb{R} \tag{3.6}$$

and the right hand side of (3.5(s)) can be estimated by $C_{s,\kappa}\|\varphi\|_{H^{s+\kappa}}$, implying (3.3($s+\kappa$)).

Due to (3.3(s)) it remains to show that for every $s \ge 0$ there is a constant $C_s > 0$ such that

$$\|(1-\chi_1)e^{-i\tau\check{P}}\varphi\|_{H^s} \le C_s\|\varphi\| \quad \text{if } \operatorname{supp}\varphi \subset \operatorname{supp}\widetilde{\chi}_2,\ \tau \in [-\theta;\theta]. \tag{3.7(s)}$$

It is clear that (3.7(0)) holds. Setting $A = \langle D\rangle^\kappa(1-\chi_1)$ we still have (3.6) if $0 < \kappa \le 1 - \varrho$, and introducing $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi_1 = 1$ on a neighbourhood of $\operatorname{supp}\chi$, $\chi = 1$ on a neighbourhood of $\operatorname{supp}\widetilde{\chi}_2$, we have $[\check{P}, A]\chi \in \Psi^{-\infty}$. Therefore assuming that (3.7(s)) holds for a given $s \ge 0$ and χ in place of χ_1 we obtain

$$\|[\check{P}, A](1-\chi)e^{-i\tau z\check{P}}\varphi\|_{H^s} \le C_s\|\varphi\| \quad \text{if } \operatorname{supp}\varphi \subset \operatorname{supp}\widetilde{\chi}_2,\ z\tau \in [-\theta;\theta],$$

and (3.5(s)) implies (3.7($s+\kappa$)). ∎

Now, for $\tau \in \mathbb{R}$ we define

$$\hat{U}_\tau = e^{-i\tau\hat{P}}, \quad \hat{U}_\tau^* = e^{i\tau\hat{P}}, \quad \check{U}_\tau = e^{-i\tau\check{P}}, \quad \check{U}_\tau^* = e^{i\tau\check{P}}, \tag{3.8}$$

where $\check{P}$ and $\hat{P}$ are as in Lemma 1.3 with $\max\{2/3, \delta, 1-\delta\} < \varrho < 1$.

We consider a map $\widetilde{\sigma} : V \to [-1;1]$ and

$$A^0 = \{A_v^0(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}_\varrho^{m_0}((1))[V]. \tag{3.9}$$

Then setting

$$A_v^{\widetilde{\sigma}}(\tau) = \hat{U}^*_{\tau\widetilde{\sigma}(v)} A_v^0(\tau) \hat{U}_{\tau\widetilde{\sigma}(v)} \tag{3.10}$$

and reasoning as in the proof of Lemma 2.2 we obtain

$$A^{\widetilde{\sigma}} = \{A_v^{\widetilde{\sigma}}(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}_\varrho^{m_0}((1))[V]. \tag{3.11}$$

Next we consider another map $\sigma : V \to [-1;1]$ and applying the formula (3.4) we can write

$$\check{U}^*_{\tau\sigma(v)} A_v^{\widetilde{\sigma}}(\tau) \check{U}_{\tau\sigma(v)} = A_v^{\widetilde{\sigma}}(\tau) + \tau\sigma(v) Y_v^{\sigma,\widetilde{\sigma}}(\tau) \tag{3.12(a)}$$

with

$$Y_v^{\sigma,\widetilde{\sigma}}(\tau) = \int_0^1 dz\, \check{U}^*_{\tau z\sigma(v)}[i\check{P},\, A_v^{\widetilde{\sigma}}(\tau)]\check{U}_{\tau z\sigma(v)}. \tag{3.12(b)}$$

However,

$$\{[\check{P},\, A_v^{\widetilde{\sigma}}(\tau)]\}_{(v,\tau)\in V_\theta} \in \mathcal{A}_{\varrho,\delta}^{m_0+\varrho-1}[V] \tag{3.13}$$

and due to (3.3(s)) for every $s \in \mathbb{R}$ there is a constant $C_s > 0$ such that

$$\|Y_v^{\sigma,\widetilde{\sigma}}(\tau)\|_{B(H^s, H^{s-m_0-\varrho+1})} \le C_s \quad \text{for } (v,\tau) \in V_\theta \tag{3.14}$$

for every $\sigma, \widetilde{\sigma} : V \to [-1;1]$.

Our aim is to use the properties of $\check{U}_\tau$ and $\hat{U}_\tau$ to complete the proof of Proposition 3.1 via the Kato–Trotter formula (1.9). For this reason we are going to study the powers of products $\check{U}_{\tau\sigma(v)}\hat{U}_{\tau\sigma(v)}$ where $0 \le \sigma(v) \le 1$. To begin we set

$$U_\tau = \hat{U}_\tau \check{U}_\tau, \quad U_\tau^k = (\hat{U}_\tau \check{U}_\tau)^k, \tag{3.8$'$}$$

where $\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then $U^{-1}_{\tau\sigma(v)} A_v^{\widetilde{\sigma}}(\tau) U_{\tau\sigma(v)}$ equals

$$\check{U}^*_{\tau\sigma(v)} A_v^{\widetilde{\sigma}+\sigma}(\tau) \check{U}_{\tau\sigma(v)} = A_v^{\widetilde{\sigma}+\sigma}(\tau) + \tau\sigma(v) Y_v^{\sigma,\sigma+\widetilde{\sigma}}(\tau) \tag{3.15}$$

and assuming that $\hat{n} : V \to \mathbb{N}$ is such that

$$\hat{n}\sigma : V \to [-1;1] \quad \text{where} \quad \hat{n}\sigma(v) = \hat{n}(v)\sigma(v), \tag{3.16}$$

by induction we find that $U^{-\hat{n}(v)}_{\tau\sigma(v)} A_v^0(\tau) U^{\hat{n}(v)}_{\tau\sigma(v)}$ can be expressed as

$$A_v^{\hat{n}\sigma}(\tau) + \tau\sigma(v) \sum_{1 \le n(v) \le \hat{n}(v)} U^{n(v)-\hat{n}(v)}_{\tau\sigma(v)} Y_v^{\sigma,n\sigma}(\tau) U^{\hat{n}(v)-n(v)}_{\tau\sigma(v)}, \tag{3.17}$$

where $n\sigma : V \to [-1;1]$ denotes the map $v \mapsto n(v)\sigma(v)$.

LEMMA 3.3. *Let* $\chi_1, \widetilde{\chi}_2 \in C_0^\infty(\mathbb{R}^d)$ *be such that* $\chi_1 = 1$ *on a neighbourhood of* $\operatorname{supp} \widetilde{\chi}_2$ *and let* $\theta = \theta(\chi_1, \widetilde{\chi}_2) > 0$ *be small enough. Then for every* $s \in \mathbb{R}$ *there is a constant* $C_s > 0$ *such that*

$$\|(1-\chi_1) U^{\hat{n}(v)}_{\tau\sigma(v)} \varphi\|_{H^s} \le C_s \|\varphi\| \quad \textit{if} \ \operatorname{supp}\varphi \subset \operatorname{supp}\widetilde{\chi}_2, \ (v,\tau) \in V_\theta, \tag{3.18(s)}$$

for any maps $\sigma : V \to [-1;1]$ *and* $\hat{n} : V \to \mathbb{N}$ *satisfying* (3.16).

Proof. In the first step we check that for every $s \in \mathbb{R}$ there is a constant $C_s > 0$ such that

$$\|U^{\hat{n}(v)}_{\tau\sigma(v)} \varphi\|_{H^s} \le C_s \|\varphi\|_{H^s} \quad \text{for } (v,\tau) \in V_\theta, \tag{3.19(s)}$$

for all $\sigma : V \to [-1;1]$ and $\hat{n} : V \to \mathbb{N}$ satisfying (3.16).

Clearly the above asssertion holds for $s = 0$. Assume that it holds for a given $s \ge 0$. Since (3.16) still holds with $-\hat{n}(v)$ in place of $\hat{n}(v)$, we can use (3.19(s)) with $-\hat{n}(v)$ in place of $\hat{n}(v)$ to obtain (3.19($-s$)). Moreover the condition $1 \le n(v) \le \hat{n}(v)$ ensures $(n-\hat{n})\sigma : V \to [-1;1]$ and (3.19(s)) holds with $(\hat{n}-n)(v)$ in place of $n(v)$. Therefore writing the composition of (3.17) with $U^{\hat{n}(v)}_{\tau\sigma(v)}$ we can estimate $\|A_v^0(\tau) U^{\hat{n}(v)}_{\tau\sigma(v)} \varphi\|_{H^s}$ by

$$C_s \|A_v^{\hat{n}\sigma}(\tau)\varphi\|_{H^s} + \sup_{1 \le n(v) \le \hat{n}(v)} \theta C_s \|Y_v^{\sigma,n\sigma}(\tau) U^{(\hat{n}-n)(v)}_{\tau\sigma(v)} \varphi\|_{H^s}. \tag{3.20(s)}$$

Taking $A_v^0(\tau) = \langle D \rangle^\kappa$ with $0 < \kappa \le 1 - \varrho$ we have (3.11) with $m_0 = \kappa$ and (3.14) allows us to estimate the right hand side of (3.20(s)) by $C_{s,\kappa}\|\varphi\|_{H^{s+\kappa}}$, implying (3.19($s+\kappa$)).

Next we note that the assertion of Lemma 3.3 holds if $s = 0$. Assume that it holds for a given $s \geq 0$ and set $A_v^0(\tau) = \langle D\rangle^\kappa(1-\chi_1)$ with $0 < \kappa \leq 1-\varrho$ and χ defined as below (3.7(s)). Then choosing $\theta = \theta(\chi_1, \chi) > 0$ small enough we find that $\{A_v^{\hat{n}\sigma}(\tau)\chi\}_{(v,\tau)\in V_\theta}$ is a bounded subset of $\Psi^{-\infty}$. Indeed, applying the theory of Fourier integral operators of [22, Section 4] to $\hat{P}$ we obtain the finite propagation speed for $e^{-it\hat{P}}$ in a standard way (cf. Egorov theorem stated as [22, Corollary 4.2]).

Therefore introducing $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^d)$ such that $\chi = 1$ on a neighbourhood of $\operatorname{supp}\widetilde{\chi}$ and $\chi_1 = 1$ on a neighbourhood of $\operatorname{supp}\widetilde{\chi}$, we see that $\{Y_v^{\sigma,n\sigma}(\tau)\widetilde{\chi}\}_{(v,\tau)\in V_\theta}$ is a bounded subset of $\Psi^{-\infty}$ due to Lemma 3.2. Thus (3.20(s)) allows estimating the norm $\|(1-\chi_1)U_{\tau\sigma(v)}^{\hat{n}(v)}\varphi\|_{H^{s+\kappa}}$ by

$$C_s'\|\varphi\| + \sup_{1\leq n(v)\leq \hat{n}(v)} C_s'\|Y_v^{\sigma,n\sigma}(\tau)(1-\widetilde{\chi})U_{\tau\sigma(v)}^{(n-\hat{n})(v)}\varphi\|_{H^s}, \tag{3.21}$$

and $\operatorname{supp}\varphi \subset \operatorname{supp}\widetilde{\chi}_2$ allows estimating the last term of (3.21) by $C_s''\|\varphi\|$ due to the assertion (3.18(s)) with $\widetilde{\chi}$ in place of χ_1. ■

End of proof of Proposition 3.1. We take $V = \mathbb{N}\setminus\{0\}$, $s \geq 0$ and applying Lemma 3.3 with $\hat{n}(v) = v$ and $\sigma(v) = 1/v$ we find $\theta = \theta(\chi_1, \widetilde{\chi}_2) > 0$ such that $\{\widetilde{\chi}_2(U_{\tau/v}^v)^*(1-\chi_1)\}_{(v,\tau)\in V_\theta}$ is a bounded subset of $B(H^{-s}, L^2)$. Therefore we find constants C_s such that

$$\|\widetilde{\chi}_2(U_{\tau/v}^v)^*(1-\chi_1)\langle D\rangle^s\varphi\| \leq C_s\|\varphi\| \quad \text{for } (v,\tau)\in V_\theta,\ \varphi\in H^s.$$

Thus the Kato–Trotter formula (1.9) implies

$$\|\widetilde{\chi}_2 e^{i\tau P}(1-\chi_1)\langle D\rangle^s\varphi\| \leq C_s\|\varphi\| \quad \text{for } \varphi\in H^s,\ \tau\in[-\theta;\theta],$$

i.e. $\{\widetilde{\chi}_2 e^{i\tau P}(1-\chi_1)\}_{\tau\in[-\theta;\theta]}$ is a bounded subset of $B(H^{-s}, L^2)$ for every $s \geq 0$, completing the proof by (3.1). ■

4. Commutator estimates. We keep the notations of Section 3 assuming that $0 \leq \delta < 1$ and $\hat{P}$, $\check{P}$ are as in Lemma 1.3 with $\max\{2/3, \delta, 1-\delta\} < \varrho < 1$. Moreover $\kappa = \min\{1-\varrho, \varrho-\delta\}$ and as before σ is a map $V \to [-1;1]$. We begin by defining classes $\mathcal{Y}_\sigma^m[V]$ preserved by conjugations with $\check{U}_s$ similarly to [19]. We write

$$Y = \{Y_v(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{Y}_\sigma^m[V] \tag{4.1}$$

if there exist $N \in \mathbb{N}$, polynomials $w_1, \ldots, w_N : \mathbb{R}^{N+1} \to \mathbb{C}$, real-valued polynomials $w_{k,k'} : \mathbb{R}^N \to \mathbb{R}$ for $k, k' = 1, \ldots, N$ and operators

$$A_{k,k'} = \{A_{k,k',v}(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}_{\varrho,\delta}^{m(k,k')}[V]$$

with $\sum_{1\leq k'\leq N} m(k,k') \leq m$ $(k = 1, \ldots, N)$ such that

$$Y_v(\tau) = \sum_{1\leq k\leq N} \int_{[0;1]^N} dz\, w_k(\tau,z)Y_{k,1,v}(\tau,z)Y_{k,2,v}(\tau,z)\ldots Y_{k,N,v}(\tau,z)$$

with

$$Y_{k,k',v}(\tau,z)=\check{U}^*_{\tau w_{k,k'}(z)\sigma(v)}A_{k,k',v}(\tau)\check{U}_{\tau w_{k,k'}(z)\sigma(v)}.$$

LEMMA 4.1. *Let* $m,\widetilde{m}\in\mathbb{R}$, *let* Y *be given by* (4.1) *and*

$$\widetilde{Y}=\{\widetilde{Y}_v(\tau)\}_{(v,\tau)\in V_\theta}\in\mathcal{Y}^{\widetilde{m}}_\sigma[V]. \tag{4.2}$$

Then

$$Y\widetilde{Y}=\{Y_v(\tau)\widetilde{Y}_v(\tau)\}_{(v,\tau)\in V_\theta}\in\mathcal{Y}^{m+\widetilde{m}}_\sigma[V], \tag{4.3}$$

$$[A,\widetilde{Y}]=\{[A_v(\tau),\widetilde{Y}_v(\tau)]\}_{(v,\tau)\in V_\theta}\in\mathcal{Y}^{m+\widetilde{m}-\kappa}_\sigma[V], \tag{4.4}$$

where A *is given by* (2.4(1)).

Proof. The assertion (4.3) is obvious and the proof of (4.4) follows the reasoning described in [19]. More precisely, we introduce

$$\overline{Y}_{k,k',v}(\tau,z)=\int_0^1 dz'\,\sigma(v)w_{k,k'}(z)\check{U}^*_{\tau z'w_{k,k'}(z)\sigma(v)}[A_v(\tau),i\check{P}]\check{U}_{\tau z'w_{k,k'}(z)\sigma(v)}$$

and express $[A_v(\tau),Y_{k,k',v}(\tau)]$ in the form

$$\check{U}^*_{\tau w_{k,k'}(z)\sigma(v)}[A_v(\tau),A_{k,k',v}(\tau)]\check{U}_{\tau w_{k,k'}(z)\sigma(v)}+\tau[\overline{Y}_{k,k',v}(\tau),Y_{k,k',v}(\tau)].$$

We complete the proof by observing that $\{[A_v(\tau),i\check{P}]\}_{(v,\tau)\in V_\theta}\in\mathcal{A}^{m-1+\varrho}_{\varrho,\delta}[V]$ and $\{[A_v(\tau),A_{k,k',v}(\tau)]\}_{(v,\tau)\in V_\theta}\in\mathcal{A}^{m+m(k,k')-\varrho+\delta}_{\varrho,\delta}[V]$. ■

Let $m\in\mathbb{R}$, $\sigma:V\to[-1;1]$ and $\overline{n}:V\to\mathbb{Z}$. Then we write

$$Z=\{Z_v(\tau)\}_{(v,\tau)\in V_\theta}\in\widetilde{\mathcal{Z}}^m_{\overline{n},\sigma}[V] \tag{4.5}$$

if there exist $C_0>0$ and $N\in\mathbb{N}$, $\hat{n}_1,\dots,\hat{n}_N:V\to\mathbb{Z}$ such that

$$\hat{n}_1(v)+\ldots+\hat{n}_N(v)=\bar{n}(v),\qquad (|\hat{n}_1(v)|+\ldots+|\hat{n}_N(v)|)|\sigma(v)|\le C_0$$

and

$$Z_v(\tau)=Y_{1,v}(\tau)U^{\hat{n}_1(v)}_{\tau\sigma(v)}Y_{2,v}(\tau)U^{\hat{n}_2(v)}_{\tau\sigma(v)}\ldots Y_{N,v}(\tau)U^{\hat{n}_N(v)}_{\tau\sigma(v)}$$

where $\{Y_{k,v}(\tau)\}_{(v,\tau)\in V_\theta}\in\mathcal{Y}^{m(k)}_\sigma[V]$ with $m(1)+\ldots+m(N)\le m$.

LEMMA 4.2. *Let* $m\in\mathbb{R}$, $\sigma:V\to[-1;1]$, $\overline{n}:V\to\mathbb{Z}$ *and* $\hat{n}:V\to\mathbb{Z}$ *be such that* $\hat{n}\sigma:V\to[-1;1]$. *Let* Z *be given by* (4.5) *and* $\sigma_1:\mathbb{N}\times V\to[-1;1]$ *be such that* $\sigma_1(n,v)=\sigma(v)$. *If* A^0 *and* $A^{\widetilde{\sigma}}$ *are given by* (3.9)–(3.10), *then there is* $C>0$ *such that*

$$\begin{aligned}A^{\hat{n}\sigma}_v(\tau)Z_v(\tau)&=Z_v(\tau)A^{(\hat{n}+\overline{n})\sigma}_v(\tau)\\&\quad+\sum_{0\le n'<N}Z_{n',v}(\tau)+\tau\sigma(v)\sum_{N\le n\le C|\sigma(v)|^{-1}}Z_{n,v}(\tau)\end{aligned} \tag{4.6}$$

with some $\{Z_{n,v}(\tau)\}_{(n,v,\tau)\in\mathbb{N}\times V\times[-\theta;\theta]}\in\widetilde{\mathcal{Z}}^{m+m_0-\kappa}_{\overline{n},\sigma_1}[\mathbb{N}\times V]$, *where* $\theta>0$ *is small enough.*

Proof. Assume first that $N = 1$. Then the assertion follows immediately from Lemma 4.1 and (3.17). Indeed, if $N = 1$ then (4.6) holds with

$$Z_{0,v}(\tau) = [A_v^{\hat{n}\sigma}(\tau), Y_{1,v}(\tau)] U_{\tau\sigma(v)}^{\hat{n}_1(v)},$$
$$Z_{n,v}(\tau) = Y_{1,v}(\tau) U_{\tau\sigma(v)}^{n} Y_{n,v}^{0}(\tau) U_{\tau\sigma(v)}^{\hat{n}_1(v)-n} \quad (n \geq 1),$$

where for $1 \leq n \leq \hat{n}(v)$ we have $Y_{n,v}^{0}(\tau) = Y_v^{\sigma,(\hat{n}+n)\sigma}$, introduced in (3.12(b)), and for $\hat{n}(v) < n \leq C|\sigma(v)|^{-1}$ we set $Y_{n,v}^{0}(\tau) = 0$. Therefore (3.13) ensures

$$\{Y_{n,v}^{0}(\tau)\}_{(n,v,\tau)\in\mathbb{N}\times V\times[-\theta;\theta]} \in \mathcal{Y}_{\sigma_1}^{m_0-\kappa}[\mathbb{N}\times V]. \tag{4.7}$$

It is clear that the assertion for $N = 1$ follows from (4.4) and (4.7). For general $N \in \mathbb{N}$, it suffices to repeat the analogous reasoning N times. ∎

Let $m \in \mathbb{R}$, $\sigma : V \to [-1; 1]$ and $\overline{n} : V \to \mathbb{Z}$. Then we write

$$Z = \{Z_v(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{Z}_{\overline{n},\sigma}^{m}[V] \tag{4.8}$$

if there exist $C > 0$, $N \in \mathbb{N}$ and

$$\{Z_{k,n,v}(\tau)\}_{(n,v,\tau)\in\mathbb{N}\times V\times[-\theta;\theta]} \in \widetilde{\mathcal{Z}}_{\overline{n},\sigma_1}^{m}[\mathbb{N}\times V] \quad (k = 0, \ldots, N)$$

such that

$$Z_v(\tau) = \sum_{0\leq k\leq N} \sigma(v)^k \sum_{0\leq n\leq C|\sigma(v)|^{-k}} Z_{k,n,v}(\tau). \tag{4.8$'$}$$

It is easy to see that this notation allows reformulating Lemma 4.2 as

Corollary 4.3. *Let $m \in \mathbb{R}$, $\sigma : V \to [-1; 1]$, $\overline{n} : V \to \mathbb{Z}$ and let Z be given by* (4.8) *as above. If A^0 is given by* (3.9), *$A_v^{\widetilde{\sigma}}$ is defined by* (3.10) *and*

$$\widetilde{Z}_v(\tau) = A_v^0(\tau) Z_v(\tau) - Z_v(\tau) A_v^{\overline{n}\sigma}(\tau), \tag{4.9}$$

then $\{\widetilde{Z}_v(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{Z}_{\overline{n},\sigma}^{m+m_0-\kappa}[V]$.

Lemma 4.4. *Let $\widetilde{\sigma} : V \to [-1; 1]$ and*

$$B_v^{\widetilde{\sigma}}(\tau) = \hat{U}_{\tau\widetilde{\sigma}(v)}^{*} x_j \hat{U}_{\tau\widetilde{\sigma}(v)} \tag{4.10}$$

If $\widetilde{Y}$ is given by (4.2) *and $m^+ = \widetilde{m} - 1$, $m^- = \widetilde{m} - \kappa$, then*

$$[B_v^{\widetilde{\sigma}}(\tau), \widetilde{Y}_v(\tau)] = Y_v^{+}(\tau) + \tau Y_v^{-}(\tau) \text{ with } Y^{\pm} = \{Y_v^{\pm}(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{Y}_{\sigma}^{m^{\pm}}[V].$$

Proof. First of all we check that $\{\widetilde{A}_v(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}_{\varrho,\delta}^{\widetilde{m}}[V]$ ensures

$$[B_v^{\widetilde{\sigma}}(\tau), \widetilde{A}_v(\tau)] = \widetilde{A}_v^{+}(\tau) + \tau\widetilde{A}_v^{-}(\tau) \quad \text{with} \quad \{\widetilde{A}_v^{\pm}(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}_{\varrho,\delta}^{m^{\pm}}[V]$$

and $m^+ = \widetilde{m} - 1$, $m^- = \widetilde{m} - \kappa$.

Indeed, this follows from Lemma 2.1 and the fact that

$$B_v^{\widetilde{\sigma}}(\tau) = x_j + \tau A_v(\tau) \quad \text{with} \quad A_v(\tau) = \int_0^1 \partial_\tau B_v^{\widetilde{\sigma}}(z\tau)\,dz,$$

$$\{\partial_\tau B_v^{\widetilde{\sigma}}(\tau)\}_{(v,\tau)\in V_\theta} = \{\widetilde{\sigma}(v)\hat{U}^*_{\tau\widetilde{\sigma}(v)}\ [i\hat{P}, x_j]\,\hat{U}_{\tau\widetilde{\sigma}(v)}\}_{(v,\tau)\in V_\theta} \in \mathcal{A}^0_\varrho((1))[V],$$

i.e. $A_v = \{A_v(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}^0_\varrho((1))[V]$. Next, similarly to (3.13), we find

$$\{[B_v^{\widetilde{\sigma}}(\tau), i\check{P}]\}_{(v,\tau)\in V_\theta} \in \mathcal{A}^{-1+\varrho}_{\varrho,\delta}[V]$$

and to complete the proof it remains to replace $A_v(\tau)$ by $B_v^{\widetilde{\sigma}}(\tau)$ in the proof of Lemma 4.1. ■

Using Lemmas 2.1 and 4.4 it is easy to follow the reasoning of the proof of Lemma 4.2 with $B^{\widetilde{\sigma}}$ in place of $A^{\widetilde{\sigma}}$. In particular we obtain

COROLLARY 4.5. *Let Z and $B_v^{\widetilde{\sigma}}(\tau)$ be given by* (4.8) *and* (4.10). *Set*

$$\widetilde{Z}_v(\tau) = x_j Z_v(\tau) - Z_v(\tau) B_v^{\overline{n}\sigma(v)}(\tau). \tag{4.11}$$

If $m^+ = m-1$ and $m^- = m-\kappa$, then

$$\widetilde{Z}_v(\tau) = \widetilde{Z}_v^+(\tau) + \tau\widetilde{Z}_v^-(\tau) \quad \textit{with} \quad \{Z_v^\pm(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{Z}^{m^\pm}_{\overline{n},\sigma}[V].$$

5. End of proof. We recall that following the parabolic construction of [22, Proposition 3.1] we obtain Theorem 1.2 from the estimates of [22, Theorem 2.4] and reasoning as at the end of [22, Section 3] we can see that Theorem 1.2 follows from

PROPOSITION 5.1. *Let $l_0 \in \mathbb{N}$, $m_0 \in \mathbb{R}$ and $q \in S^{m_0}_{1,\delta}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$. Let $\theta, \theta' > 0$ be small enough and set*

$$\Xi_0(\theta,\theta') = \{t \in \mathbb{C} : 0 < -\operatorname{Im} t < \theta'|\operatorname{Re} t| \ \textit{and}\ |t| < \theta\}, \tag{5.1}$$

$$\varXi(\theta,\theta') = \{(t,\tau') : t \in \varXi_0(\theta,\theta') \ \textit{and}\ \tau' \in [0;\operatorname{Re} t]\}. \tag{5.2}$$

If $l \ge m_0 + l_0 + d + 1$ and δ_y is the Dirac mass at y, then

$$\sup_{\substack{(t,\tau)\in\varXi(\theta,\theta')\\ y\in\mathbb{R}^d}} |t^l \langle P^{l_0} e^{-i\tau P}\delta_y, \operatorname{Op}(q e^{i(\tau-t)p_0})^*\delta_y\rangle| < \infty. \tag{5.3}$$

Now, we assume that $y \in B(y_0, r)$ where $y_0 \in \mathbb{R}^d$, $r > 0$ is small enough (independent of y_0) and all estimates are uniform with respect to y_0. We consider a set V' of indices and introduce

$$\begin{aligned} V = \{v = (y,t,n,\tau',v') :& \\ & y \in B(y_0,r),\ n \in \mathbb{N}\setminus\{0\},\ (t,\tau') \in \varXi(\theta,\theta'),\ v' \in V'\}. \end{aligned} \tag{5.4}$$

We introduce functions $\sigma : V \to [-1;1]$ and $\hat{n} : V \to \mathbb{N}$, setting

$$\sigma(v) = \sigma(y,t,n,\tau',v') = 1/n, \qquad \hat{n}(v) = \hat{n}(y,t,n,\tau',v') = n, \tag{5.5}$$

and write $Z \in \mathcal{Z}^m_\sigma[V]$ if $Z \in \mathcal{Z}^m_{\overline{n},\sigma}[V]$ with $\overline{n}(v) = 0$ for all $v \in V$.

Let $k \in \mathbb{Z}$ and let $m(k)$, $m'(k)$, $m''(k)$ be some real numbers. We consider the following conditions:

$$(5.6(k)) \qquad \{q_{k,v}\}_{v\in V} \text{ is a bounded subset of } S^{m(k)}_{1,\delta}$$

(i.e. (1.2) holds with $a = q_{k,v}$, $m = m(k)$, $\varrho = 1$, $r = 0$, $x \in X = \mathbb{R}^d$ and constants $C_{\alpha,\alpha'}$ independent of $v \in V$),

$$(5.7(k)) \qquad \{A_{k,v}(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{A}^{m'(k)}[V],$$

$$(5.8(k)) \qquad \{Z_{k,v}(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{Z}^{m''(k)}_\sigma[V],$$

and introduce the notation

$$(5.9) \quad J(q_{k,v}, A_{k,v}, Z_{k,v})(\tau) = \langle U^{\hat{n}(v)}_{\tau\sigma(v)} A_{k,v}(\tau)\delta_y, Z_{k,v}(\tau)\mathrm{Op}(q^\sharp_{k,v} e^{i(\tau-t)p_0})^*\delta_y\rangle,$$

where $v = (y, t, n, \tau', v') \in V$ and $q^\sharp_{k,v}(x, \xi, x') = q_{k,v}(x', \xi)$.

Proposition 5.2. *Let $\kappa = \min\{\varrho - \delta, 1 - \varrho\}$. Assume that $q_{0,v}$, $A_{0,v}$, $Z_{0,v}$ satisfy respectively* (5.6(k)), (5.7(k)), (5.8(k)) *with $k = 0$. Then there exist $k_1 \in \mathbb{N}$ and $q_{k,v}$, $A_{k,v}$, $Z_{k,v}$ satisfying respectively* (5.6(k)), (5.7(k)), (5.8(k)) *for $k = \pm 1, \ldots, \pm k_1$ with*

$$(5.10(a)) \quad m(k) + m'(k) + m''(k) \le m(0) + m'(0) + m''(0) - 1 \textit{ for } k > 0,$$

$$(5.10(b)) \quad m(k) + m'(k) + m''(k) \le m(0) + m'(0) + m''(0) - \kappa \textit{ for } k < 0,$$

such that

$$(5.11) \quad tJ(q_{0,v}, A_{0,v}, Z_{0,v})(\tau) = \sum_{1\le k\le k_1} (J(q_{k,v}, A_{k,v}, Z_{k,v}) + tJ(q_{-k,v}, A_{-k,v}, Z_{-k,v}))(\tau) + O(1)$$

uniformly with respect to $\{(v,\tau) : v = (y, t, n, \tau', v') \in V$ and $\tau' = \tau\}$.

Proof that Proposition 5.1 follows from Proposition 5.2. First of all we note that due to the H^s-estimates of Section 3, the conditions (5.6(k)), (5.7(k)), (5.8(k)) imply that the families of operators

$$\{\mathrm{Op}(q^\sharp_{k,v} e^{i(\tau-t)p_0})\}_{(v,\tau)\in V_\theta}, \quad \{A_{k,v}(\tau)\}_{(v,\tau)\in V_\theta}, \quad \{Z_{k,v}(\tau)\}_{(v,\tau)\in V_\theta}$$

are bounded in $B(H^s, H^{s-m(k)})$, $B(H^s, H^{s-m'(k)})$, $B(H^s, H^{s-m''(k)})$ respectively, hence there is a constant $C > 0$ (independent of $(v,\tau) \in V \times [-\theta;\theta]$) such that

$$m(k) + m'(k) + m''(k) \le -d - 1 \;\Rightarrow\; |J(q_{k,v}, A_{k,v}, Z_{k,v})(\tau)| \le C.$$

Thus reasoning as in [22, after Proposition 5.1] we can take k_1 large enough and forget all the terms $J(q_{-k,v}, A_{-k,v}, Z_{-k,v})$, i.e. in place of (5.11) we have

$$tJ(q_{0,v}, A_{0,v}, Z_{0,v})(\tau) = \sum_{1\le k\le k_1} J(q_{k,v}, A_{k,v}, Z_{k,v})(\tau) + O(1). \tag{5.12}$$

Then reasoning as at the beginning of [22, Section 5] we note that iterating this assertion l times we can write (5.12) with t^l in place of t and $q_{k,v}$, $A_{k,v}$, $Z_{k,v}$ satisfying (5.6(k)), (5.7(k)), (5.8(k)) with

$$m(k) + m'(k) + m''(k) \le m(0) + m'(0) + m''(0) - l \quad \text{for } k = 1, \dots, k_l.$$

This general statement is analogous to [22, Proposition 3.4], and to obtain (5.3) it suffices to take

$$q_{0,v}(x,\xi) = q(y,\xi,x), \quad A_{0,v}(\tau) = P^{-d}, \quad Z_{0,v}(\tau) = P^{d+l_0}.$$

Indeed, since $P^{-d}\delta_y \in L^2$, the Kato–Trotter formula (1.9) allows us to write (5.3) in the form

$$\sup_{\substack{(t,\tau)\in\Xi(\theta,\theta')\\ y\in\mathbb{R}^d}} |\lim_{n\to\infty} t^l\, J(q_{0,v}, A_{0,v}, Z_{0,v})(\tau)|_{\tau'=\tau}| < \infty. \ \blacksquare$$

Proof of Proposition 5.2. Step 1. First of all we note that as at the beginning of the proof of [22, Proposition 5.1], using a suitable partition of unity we may assume that

$$\operatorname{supp} q_{0,v} \subset B(y_0, 2r) \times \varGamma_j(c), \tag{5.13}$$

where $c > 0$ is small enough and

$$\varGamma_{\pm j}(c) = \{\xi \in \mathbb{R}^d : \pm\partial_{\xi_j} p_0(y_0,\xi) > 2c\} \quad \text{for } j = 1, \dots, d. \tag{5.14}$$

As in [22], we fix $j \in \{1, \dots, d\}$. If $\chi^0_{j,c,r} \in S^0_{1,0}$ is such that $\operatorname{supp}\chi^0_{j,c,r} \subset B(y_0, 3r) \times \varGamma_j(c/2)$ and $\chi^0_{j,c,r} = 1$ on $B(y_0, 2r) \times \varGamma_j(c)$ (cf. [22, Lemma 3.2]), then

$$\{(1-\chi^0_{j,c,r})(x,D)\operatorname{Op}(q^\sharp_{0,v} e^{i(t-\tau)p_0})^*\}_{(v,\tau)\in V_\theta} \quad \text{is bounded in } \Psi^{-\infty}. \tag{5.15}$$

Then Corollary 4.5 alllows us to write

$$U^n_{\tau/n} x_j U^{-n}_{\tau/n} = \hat U_\tau x_j \hat U^*_\tau + \widetilde Z_{1,v}(\tau) + \tau \widetilde Z_{-1,v}(\tau) \tag{5.16}$$

with

$$\{\widetilde Z_{1,v}(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{Z}^{-1}_\sigma[V], \quad \{\widetilde Z_{-1,v}(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{Z}^{-\kappa}_\sigma[V].$$

Moreover we have

$$\hat U_\tau x_j \hat U^*_\tau = x_j - \tau\widetilde P_1(\tau) \quad \text{with} \quad \widetilde P_1(\tau) = \int_0^1 dz\, \hat U_{z\tau}[i\hat P, x_j]\hat U^*_{z\tau}, \tag{5.17}$$

hence we can consider $\widetilde P_1$ as an element of $\mathcal{A}^0_\varrho((1))$. For $y \in \mathbb{R}^d$ let

$$P_y = p_y(x,D) \quad \text{with} \quad p_y(x,\xi) = \partial_{\xi_j} p_0(y,\xi)$$

(i.e. the symbol $p_y \in S^0_1$ is independent of the x-variable) and for $v = (y,t,n,\tau',v') \in V$ let

$$\widetilde{P}_v(\tau) = \frac{\tau'}{\overline{t}}\widetilde{P}_1(\tau) + \left(1 - \frac{\tau'}{\overline{t}}\right)P_y = \widetilde{p}_v(\tau,x,D) + R_{v,\tau}, \tag{5.18}$$

where $\{R_{v,\tau}\}_{(v,\tau)\in V_\theta}$ is bounded in $\Psi^{-\infty}$.

As in [22, Section 5], assuming $\theta,\theta' > 0$ small enough we obtain

$$|\widetilde{p}_v(\tau,x,\xi)| \geq \frac{c}{4} \quad \text{for } v \in V,\ (\tau,x,\xi) \in [-\theta;\theta]\times B(y_0,3r)\times \Gamma_j(c/2). \tag{5.19}$$

Step 2. We note that there exist $\widetilde{Z}_{-2,v}$ satisfying (5.8(−2)) with $m''(-2) = m''(0) - \kappa$ and

$$\{\widetilde{Z}_{0,v}(\tau)\}_{(v,\tau)\in V_\theta} \in \mathcal{Z}^{m''(0)}_\sigma[V],$$

such that for $v = (y,t,n,\tau',v') \in V$ and $\tau' \in [0;\operatorname{Re} t]$ we have

$$\begin{aligned} &Z_{0,v}(\tau)\mathrm{Op}(q^\sharp_{0,v}e^{i(t-\tau)p_0})^* \\ &\qquad = (\widetilde{Z}_{0,v}(\tau)\widetilde{P}_v(\tau) + Z_{-2,v}(\tau))\mathrm{Op}(q^\sharp_{0,v}e^{i(t-\tau)p_0})^*. \end{aligned} \tag{5.20}$$

Indeed, since $\widetilde{p}_v$ is uniformly elliptic in $B(y_0,3r)\times\Gamma_j(c/2) \supset \operatorname{supp}\chi^0_{j,c,r}$ (due to (5.19)), it remains to use (5.15) as in [22, Section 5].

Step 3. We write $y = (y_1,\dots,y_d) \in \mathbb{R}^d$ and note that there exist $Z_{\pm3,v}$ satisfying (5.8(±3)) with $m''(3) = m''(0) - 1$, $m''(-3) = m''(0) - \kappa$ and

$$\begin{aligned} \overline{t}\widetilde{Z}_{0,v}(\tau)\widetilde{P}_v(\tau) &= (\tau'\widetilde{P}_1(\tau) - (x_j - y_j))\widetilde{Z}_{0,v}(\tau) \\ &\quad + \widetilde{Z}_{0,v}(\tau)((\overline{t} - \tau')P_y + (x_j - y_j)) \\ &\quad + Z_{3,v}(\tau) + \overline{t}Z_{-3,v}(\tau). \end{aligned} \tag{5.21}$$

Indeed, it suffices to estimate the commutator of $\widetilde{Z}_{0,v}$ with x_j applying Corollary 4.5, and with $\widetilde{P}_1(\tau)$ applying Corollary 4.3.

Step 4. There exist $q_{\pm4,v}$ satisfying (5.6(±4)) with $m(4) = m(0) - 1$, $m(-4) = m(0) - \kappa$ and

$$\begin{aligned} &((\overline{t} - \tau)P_y + (x_j - y_j))\mathrm{Op}(q^\sharp_{1,v}e^{i(t-\tau)p_0})^*\delta_y \\ &\qquad = \mathrm{Op}((q^\sharp_{4,v} + tq^\sharp_{-4,v})e^{i(t-\tau)p_0})^*\delta_y. \end{aligned} \tag{5.22}$$

Indeed, we integrate by parts as in the proof of [22, Proposition 5.1].

Step 5. There exist $Z_{\pm1,v}$ satisfying (5.8(±1)) with $m''(1) = m''(0) - 1$, $m''(-1) = m''(0) - \kappa$ and $A_{\pm5,v}$ satisfying (5.7(±5)) with $m'(5) = m'(0) - 1$, $m'(-5) = m'(0) - \kappa$ such that

$$\begin{aligned} &\widetilde{Z}_{0,v}(\tau)^*(\tau\widetilde{P}_1(\tau) + y_j - x_j)U^n_{\tau/n}A_{0,v}(\tau)\delta_y \\ &= (Z_{1,v} + tZ_{-1,v})(\tau)^*U^n_{\tau/n}A_{0,v}(\tau)\delta_y + \widetilde{Z}_{0,v}(\tau)^*U^n_{\tau/n}(A_{5,v} + tA_{-5,v})(\tau)\delta_y. \end{aligned} \tag{5.23}$$

Indeed, Lemma 2.1 ensures the existence of $A_{\pm 5,v}$ satisfying

$$(y_j - x_j)A_{0,v}(\tau)\delta_y = [A_{0,v}(\tau),\, x_j - y_j]\delta_y = (A_{5,v} + tA_{-5,v})(\tau)\delta_y$$

and (5.16) can be written as

$$U^n_{\tau/n}(y_j - x_j) = (\tau\widetilde{P}_1(\tau) + y_j - x_j - \widetilde{Z}_{1,v}(\tau) - \tau\widetilde{Z}_{-1,v}(\tau))U^n_{\tau/n},$$

hence (5.23) holds if

$$Z_{1,v}(\tau)^* = \widetilde{Z}_{0,v}(\tau)^*\widetilde{Z}_{1,v}(\tau), \qquad Z_{-1,v}(\tau)^* = \frac{\tau'}{t}\widetilde{Z}_{0,v}(\tau)^*\widetilde{Z}_{-1,v}(\tau).$$

Step 6. Let $Z_{\pm 1,v}$, $Z_{-2,v}$, $Z_{\pm 3,v}$, $q_{\pm 4,v}$, $A_{\pm 5,v}$ be as above, $Z_{2,v}(\tau) = 0$ and

$$\begin{aligned} q_{k,v} &= q_{0,v}, & m(k) &= m(0) & &\text{for } k = \pm 1, \pm 2, \pm 3, \pm 5,\\ A_{k,v} &= A_{0,v}, & m'(k) &= m'(0) & &\text{for } k = \pm 1, \pm 2, \pm 3, \pm 4. \end{aligned}$$

Then (5.20)–(5.23) with $\tau = \tau'$ give the equality (5.11). ■

6. Appendix: Proof of Lemma 1.3. Let $\gamma \in \mathcal{S}(\mathbb{R}^d)$ and set $\gamma_\alpha(x) = x^\alpha\gamma(x)$ for $\alpha \in \mathbb{N}^d$. We assume that $\int\gamma = 1$ and $\int\gamma_\alpha = 0$ if $|\alpha| = 1$. We introduce $\hat{h}$ and $\check{h} = p - \hat{h}$ by

$$\hat{h}(x,\xi) = \int p(y,\xi)\gamma((x-y)\langle\xi\rangle^{1-\varrho})\langle\xi\rangle^{(1-\varrho)d}\,dy, \tag{A.1}$$

$$\check{h}(x,\xi) = \int (p(x,\xi) - p(y,\xi))\gamma((x-y)\langle\xi\rangle^{1-\varrho})\langle\xi\rangle^{(1-\varrho)d}\,dy, \tag{A.2}$$

$$\partial_x^\alpha\hat{h}(x,\xi) = \int \partial_x^\alpha p(x-y,\xi)\gamma(y\langle\xi\rangle^{1-\varrho})\langle\xi\rangle^{(1-\varrho)d}\,dy. \tag{A.3}$$

A simple analysis of (A.3) (cf. [7] or [18, Proposition 6.3]) allows us to conclude that $\partial_x^\alpha p \in S^1_{1,\delta} \Rightarrow \partial_x^\alpha\hat{h} \in S^1_{1,1-\varrho}$ if $|\alpha| \le 2$. Using the Taylor formula

$$p(x,\xi) - p(y,\xi) + \sum_{1\le j\le d}\partial_{x_j}p(x,\xi)(x_j - y_j) = \sum_{|\alpha|=2}p_\alpha(x,y,\xi)(x-y)^\alpha$$

and $\int(x_j - y_j)\gamma((x-y)\langle\xi\rangle^{1-\varrho})\,dy = 0$, we can write (A.2) in the form

$$\check{h}(x,\xi) = \sum_{|\alpha|=2}\langle\xi\rangle^{-2(1-\varrho)}\int p_\alpha(x,y,\xi)\gamma_\alpha((x-y)\langle\xi\rangle^{1-\varrho})\langle\xi\rangle^{(1-\varrho)d}\,dy. \tag{A.4}$$

Now it is easy to check that $p_\alpha \in S^1_{1,\delta}(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ for $|\alpha| = 2$ implies $\check{h} \in S^{1-2(1-\varrho)}_{1,\delta}$. Similarly introducing $\widetilde{p}_\alpha \in S^1_{1,\delta}(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ ($|\alpha| = 1$) by

$$\partial_x p(x,\xi) - \partial_x p(x-y,\xi) = \sum_{|\alpha|=1}\widetilde{p}_\alpha(x,y,\xi)y^\alpha,$$

we get $\partial_x \check{h} = \partial_x p - \partial_x \hat{h} \in S^{1-(1-\varrho)}_{1,\delta}$, using (A.3) to express $(\partial_x p - \partial_x \hat{h})(x,\xi)$ as

$$\int (\partial_x p(x,\xi) - \partial_x p(x-y,\xi))\gamma(y\langle\xi\rangle^{1-\varrho})\langle\xi\rangle^{(1-\varrho)d}\,dy$$
$$= \sum_{|\alpha|=1} \langle\xi\rangle^{-(1-\varrho)} \int \widetilde{p}_\alpha(x,y,\xi)\gamma_\alpha(y\langle\xi\rangle^{1-\varrho})\langle\xi\rangle^{(1-\varrho)d}\,dy.$$

Then $\check{P} = \frac{1}{2}(\check{h}(x,D) + \check{h}(x,D)^*)$ and $\hat{P} = \frac{1}{2}(\hat{h}(x,D) + \hat{h}(x,D)^*)$ are essentially self-adjoint on $\mathcal{S}(\mathbb{R}^d)$ by Nelson's commutator lemma (cf. [16] with $N = I - \Delta$) and conditions (1.4)–(1.5) for p imply analogous conditions for $\hat{h}$ and $\hat{p}$ due to (1.8) (note that $|\nabla_\xi \check{h}(x,\xi)| \le C\langle\xi\rangle^{-2(1-\varrho)}$). ∎

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