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THE SET OF POINTS AT WHICH A MORPHISM OF AFFINE SCHEMES IS NOT FINITE

BY

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Abstract. Assume that X, Y are integral noetherian affine schemes. Let $f : X \to Y$ be a dominant, generically finite morphism of finite type. We show that the set of points at which the morphism f is not finite is either empty or a hypersurface. An example is given to show that this is no longer true in the non-noetherian case.

1. Introduction. Let $f : X \to Y$ be a morphism of affine varieties over an algebraically closed field k. Let $y \in Y$. We say that f *is not finite* at y if there exists no open affine neighborhood U of y such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. If $k = \mathbb{C}$, then f is not finite at y iff there exists a sequence $x_n \to \infty$ such that $f(x_n) \to y$. The set of all points at which f is not finite will be denoted by S_f.

In [2], [3] the first author proved that for a polynomial, generically finite dominant mapping $f : X \to Y$ of affine varieties, the set S_f is either empty or a hypersurface.

The aim of this paper is to generalize this result to the case of a dominant, generically finite morphism of finite type of affine integral noetherian schemes. The main result is that even under such general assumptions the set S_f is either empty or a hypersurface.

2. Preliminaries. We use the terminology and notation as in [4]. Let $A \subset B$ be arbitrary rings. We say that B is a *finite ring extension* of A if B is a finitely generated A-module. A morphism $f : X \to Y$ of schemes is called *finite* if there exists a covering of Y by open affine subsets $V_i = \mathrm{Spec}(A_i)$ such that for each i, $f^{-1}(V_i)$ is affine, equal to $\mathrm{Spec}(B_i)$, where B_i is a finite ring extension of A_i. The morphism $f : X \to Y$ is finite if and only if for every covering of Y by open affine subsets $V_i = \mathrm{Spec}(A_i)$, the sets $f^{-1}(V_i)$ are affine, equal to $\mathrm{Spec}(B_i)$, where B_i is a finite ring extension of A_i (see e.g. [4]). In particular if f is finite, then for every open subset $V = \mathrm{Spec}(A) \subset Y$

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the set $f^{-1}(V)$ is affine, equal to $\mathrm{Spec}(B)$, where B is a finite ring extension of A.

DEFINITION 2.1. Let $f : X \to Y$ be a morphism of schemes and let $y \in Y$. We say that f is *finite at* y if there exists an open affine neighborhood U of y such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite.

If $Y = \mathrm{Spec}(A)$ is affine, then f is finite at $y \in Y$ if and only if there exists $h \in A$ such that $y \in D(h) = \mathrm{Spec}(A_h)$ and $f|_{f^{-1}(D(h))} : f^{-1}(D(h)) \to D(h)$ is finite. For a morphism $f : X \to Y$, we will denote by S_f the set of points at which f is not finite. Of course f is finite iff S_f is an empty set.

Let $X = \mathrm{Spec}(A)$ be an integral affine scheme. By a *rational function* on the scheme X we mean an element of the field A_0 (field of fractions of the ring A). We say that a rational function ξ *has a pole at a point* x iff $\xi \notin A_x = \mathcal{O}_{X,x}$. For a rational function ξ on the scheme X we will denote by P_ξ the set of points at which ξ has a pole. This set is a closed proper subset of X. Indeed, we have the following:

PROPOSITION 2.2. *Let $\xi = b/a$ be a rational function on an integral scheme $X = \mathrm{Spec}(A)$. Then $P_\xi = V(((a) : (b)))$.*

Proof. Let $p \in P_\xi$. Then $\xi \notin A_p$, i.e., $\xi \neq x/r$ for all $x \in A$ and $r \in A \backslash p$. Suppose $((a) : (b)) \not\subset p$ and take $r \in ((a) : (b)) \setminus p$. Thus $rb = xa$ for some $x \in A$, but this is impossible if $\xi \notin A_p$. Therefore $((a) : (b)) \subset p$, i.e., $p \in V(((a) : (b)))$.

Conversely, assume that $((a) : (b)) \subset p$. If $p \notin P_\xi$, then $b/a \in A_p$. This means that there are $x \in A$ and $r \in A \backslash p$ such that $rb = xa$. Consequently, $r \in ((a) : (b)) \subset p$. This contradiction finishes the proof. ∎

In what follows we need the following results:

PROPOSITION 2.3. *Let A and $B = A[x_1, \dots, x_n]$ be integral domains such that the field B_0 is a finite extension of A_0. Let $f_i \in A_0[T]$, $i = 1, \dots, n$, be minimal* (*monic*) *polynomials of x_i over A_0. Assume that A is a normal ring. Then B is finite over A if and only if $f_i \in A[t]$ for all $i = 1, \dots, n$.*

Proof. Assume that B is finite over A. Fix $i \in \{1, \dots, n\}$. Since x_i is integral over A, there exists a monic polynomial $g_i \in A[X]$ such that $g_i(x_i) = 0$.

Note that $g_i = f_i h$ in $A_0[X]$. Indeed, $g_i = f_i h + r$ in $A_0[X]$ where $\deg r < \deg f_i$. Moreover, $r(x_i) = g_i(x_i) - f_i(x_i)h(x_i) = 0$ and by the minimality of f_i, we have $r = 0$. Since $g_i \in A[X]$, we conclude by [1, Theorem 3.2.2, p. 114] that $f_i \in A[X]$.

The converse implication is obvious. ∎

PROPOSITION 2.4. *Let $f : X \to Y$ be a morphism of finite type, and let $V = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(B)$ be open affine subschemes of X and Y, respectively. If $U \subset f^{-1}(V)$, then B is a finitely generated A-algebra.*

Proof. We start with the proof of the following:

LEMMA 2.5. *Let X be a scheme and let $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(B)$ be open affine subschemes of X. For every $x \in U \cap V$, there exists an open affine subscheme $W_x \subset U \cap V$ of X such that $W_x = \mathrm{Spec}(A_h) = \mathrm{Spec}(B_g)$, where $h \in A$ and $g \in B$.*

Proof. Let $x \in U \cap V$. Since $U \cap V$ is open in U, there exists $\widetilde{h} \in A$ such that $x \in D_U(\widetilde{h}) := \{x \in U : \widetilde{h} \notin x\} \subset U \cap V$. Consequently, we can assume that $U \subset V$. Let $g \in B = \mathcal{O}_X(V)$ be such that $x \in D_V(g) \subset U$. Consider the restriction mapping $\varrho_U^V : \mathcal{O}_X(V) = B \to A = \mathcal{O}_X(U)$ and take $h = \varrho_U^V(g)$. It is easy to see that $\mathrm{Spec}(A_h) = D_V(h) = D_U(g) = \mathrm{Spec}(B_g)$. ∎

Now we can continue the proof of Proposition 2.4. Since f is of finite type, there exist open affine subschemes $U_1, \ldots, U_s$ of X such that $U_1 \cup \ldots \cup U_s = f^{-1}(V)$. Moreover, $U_i = \mathrm{Spec}(B_i)$, where B_i is a finitely generated A-algebra. For every $x \in U$, there is an index i_x such that $x \in U \cap U_{i_x}$. By the lemma above, there exists an open affine subscheme $W_x \subset X$ such that $x \in W_x \subset U \cap U_{i_x}$ and W_x is of the form $W_x = \mathrm{Spec}((B_i)_{h_x}) = \mathrm{Spec}(B_{g_x})$. Since $(B_i)_{h_i}$ is a finitely generated A-algebra, so is B_{g_x}. The family $\mathcal{A} = \{\mathrm{Spec}(B_{g_x})\}_{x \in U}$ is an open covering of $U = \mathrm{Spec}(B)$. Since every affine scheme is quasi-compact, we can choose a finite subcovering $\{\mathrm{Spec}(B_{g_1}), \ldots, \mathrm{Spec}(B_{g_r})\} \subset \mathcal{A}$. Note that $(g_1, \ldots, g_r) = 1$. Let $h_1, \ldots, h_r \in B$ be such that $h_1 g_1 + \ldots + h_r g_r = 1$. Since B_{g_i} is a finitely generated A-algebra, we can write $B_{g_i} = A[s_{i,1}/g_i^{k_1}, \ldots, s_{i,n_i}/g_i^{k_{n_i}}]$.

Take an $s \in B$. Observe that $s/1 \in B_{g_i} = A[s_{i,1}/g_i^{k_1}, \ldots, s_{i,n_i}/g_i^{k_{n_i}}]$ for $i = 1, \ldots, r$. This implies that there is a natural number k_i such that $g_i^{k_i} s \in A[g_i, s_{i,1}, \ldots, s_{i,n_i}]$. Set $k = \max_{i=1,\ldots,r} k_i$. Now we can write

$$s = (h_1 g_1 + \ldots + h_r g_r)^{rk} s = \sum_{\alpha \in \mathbb{N}^r,\, |\alpha| = rk} \frac{|\alpha|!}{\alpha!} (h_1 g_1, \ldots, h_r g_r)^\alpha s.$$

Thus $s \in A[g_1, \ldots, g_r, h_1, \ldots, h_r, s_{1,1}, \ldots, s_{r,n_r}]$. ∎

3. Basic definitions. Let us recall some basic definitions.

DEFINITION 3.1. Let A be a ring and let p be a prime ideal in A. The *height* $\mathrm{ht}(p)$ is the upper bound of the lengths of chains of distinct prime ideals

$$p_0 \subset p_1 \subset \ldots \subset p_d = p$$

of A. The *height* $\mathrm{ht}(\mathbf{a})$ of any ideal $\mathbf{a}$ is the number

$$\mathrm{ht}(\mathbf{a}) = \inf_{p \supset \mathbf{a}} \mathrm{ht}(p),$$

where p ranges over all the prime ideals containing $\mathbf{a}$.

Now let us recall the notion of a Krull ring.

Definition 3.2. A ring A is called a *Krull ring* if it is integral and

1) for every prime ideal p of height 1 the ring A_p is a discrete valuation ring,
2) $A = \bigcap_{\mathrm{ht}(p)=1} A_p$,
3) for any non-zero $r \in A$, there exist only finitely many prime ideals p of height 1 such that $(r) \subset p$.

We have the following fundamental theorem of Nagata (see [5]):

Theorem 3.3. *The normalization of an integral noetherian ring is a Krull ring.*

Now we pass to the definition of a hypersurface.

Definition 3.4. Let X be a topological space and let $Z \subset X$ be an irreducible closed subset. The *codimension* of Z in X, denoted by $\mathrm{codim}_X Z$, is the upper bound of the lengths of chains of distinct closed irreducible subsets of X:

$$Z = Z_0 \subset Z_1 \subset \ldots \subset Z_n.$$

If $Z \subset X$ and all irreducible components of Z have codimension one, we say that Z is a *hypersurface.*

For an ideal $\mathbf{a}$ of a ring A we denote by $\mathrm{Ass}(\mathbf{a})$ the set of all associated prime ideals of $\mathbf{a}$. It is easy to see that the following proposition holds:

Proposition 3.5. *Let A be a ring and let $\mathbf{a}$ be an ideal in A. The subset $V(\mathbf{a}) \subset \mathrm{Spec}(A)$ is a hypersurface if and only if for every ideal $p \in \mathrm{Ass}(\mathbf{a})$ we have $\mathrm{ht}(p) = 1$.*

Example 3.6. If A is a Krull ring and $r \in A$ is a non-zero element, then the subset $V((r)) \subset \mathrm{Spec}(A)$ is a hypersurface (see [1]).

4. Krull schemes. In this section we prove our main theorem in the Krull case.

Proposition 4.1. *Let X, Y be affine integral schemes, and let $f : X \to Y$ be a dominant, generically finite morphism of finite type. If Y is normal, then S_f is the union of the sets of poles of finitely many rational functions on Y.*

Proof. Let $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. The rings A and B are integral domains, and A is also a normal ring. The morphism $f : X \to Y$ is given by a morphism of rings $\varphi : A \to B$. By Proposition 2.4 the morphism φ makes B a finitely generated A-algebra. Since φ is a monomorphism, we can identify A with $\varphi(A)$ and consequently we can write $B = A[x_1, \ldots, x_n]$.

Moreover, $f : X \to Y$ is a dominant, generically finite morphism of finite type, so the field B_0 is a finite extension of A_0. Let

$$P_i = T^{n_i} + a_1^i T^{n_i - 1} + \ldots + a_{n_i}^i \in A_0[T],$$

for $i = 1, \ldots, n$, be the minimal polynomial of x_i over A_0.

We will show that the set

$$S = \bigcup_{i,j} \{y \in Y : a_i^j \notin A_y = \mathcal{O}_{Y,y}\}$$

is equal to S_f. By Proposition 2.2 and irreducibility of Y the set S is a closed proper subset of Y. Let $y \in Y \setminus S$. Then $a_i^j \in \mathcal{O}_{Y,y}$. Thus there exists a neighborhood $D(h)$, where $h \in A$, such that $a_i^j|_{D(h)} \in A_h = \mathcal{O}_Y(D(h))$. It is easy to see that $P_i \in A_0[T] \subset (A_h)_0[T]$, where $i = 1, \ldots, n$, is a minimal polynomial for $x_i|_{D(h)}$ over $(A_h)_0$. Since $a_i^j|_{D(h)} \in \mathcal{O}_Y(D(h))$, we see that $P_i \in A_h[T]$. By Lemma 2.3, the mapping $f|_{f^{-1}(D(h))} : f^{-1}(D(h)) \to D(h)$ is finite. This implies that f is finite at y.

Conversely, let f be finite at $y \in Y$. This means that there exists $h \in A$ such that $y \in D(h)$ and the mapping $f|_{f^{-1}(D(h))} : f^{-1}(D(h)) \to D(h)$ is finite. Now Lemma 2.3 yields that $P_i \in A_h[T]$. This implies that $a_i^j|_{D(h)} \in A_h = \mathcal{O}_Y(D(h))$ and consequently $D(h) \subset Y \setminus S$. ■

The converse statement is also true:

PROPOSITION 4.2. *Let Y be a normal integral affine scheme, and let $\xi_1, \ldots, \xi_r$ be rational functions on Y. There exist an affine integral scheme X and a dominant, generically finite morphism of finite type $f : X \to Y$ such that $S_f = \bigcup_{i=1}^r P(\xi_i)$.*

Proof. It is sufficient to take $X = \mathrm{Spec}(A[\xi_1, \ldots, \xi_r])$ and $f : X \to Y$ given by the inclusion $A \hookrightarrow A[\xi_1, \ldots, \xi_r]$. ■

Let us recall that if A is a Krull ring, we say that a scheme $\mathrm{Spec}(A)$ is a *Krull scheme*. Let X be a Krull scheme and let f be a rational function on X. Then the set P_f of poles of f is either empty or it has pure codimension one. Indeed, we have the following:

PROPOSITION 4.3. *Let $X = \mathrm{Spec}(A)$ be a Krull scheme and $f \in A_0$. Then the set of poles of f, i.e. the set $P_f = \{p \in X : f \notin A_p\}$, is either empty or of the form*

$$P_f = \bigcup_{i=1}^{r} V(p_i),$$

where the p_i are prime ideals with $\mathrm{ht}(p_i) = 1$. In particular, P_f is either empty or a hypersurface.

Proof. Let $f = b/a$. In virtue of Proposition 2.2, $P_f = V\left(((a):(b))\right)$. Since A is a Krull ring, the principal ideal (a) has a primary decomposition, say $(a) = \bigcap_{i=1}^{r} q_i$. Moreover, every associated prime ideal $\sqrt{q_i}$ is minimal and has height one.

We have $((a):(b)) = \bigcap (q_i : (b))$. We also know that

$$\sqrt{(q_i:(b))} = \begin{cases} A, & b \notin p_i = \sqrt{q_i}, \\ \sqrt{q_i}, & b \in p_i = \sqrt{q_i}. \end{cases}$$

Thus if $((a):(b)) \subset p$, then there exists i such that $\sqrt{q_i} \subset p$. ∎

Since a Krull ring is a normal ring (see [1], 4.2.5), by Proposition 4.3 we have our first main result:

THEOREM 4.4. *Let X, Y be affine integral schemes and let $f : X \to Y$ be a dominant, generically finite morphism of finite type. If Y is a Krull scheme, then S_f is either empty or a hypersurface.*

5. Noetherian schemes. In this section we prove our theorem in the noetherian case.

THEOREM 5.1. *Let X, Y be affine integral schemes and let $f : X \to Y$ be a dominant, generically finite morphism of finite type. If Y is noetherian, then S_f is either empty or a hypersurface.*

Proof. Let $\widetilde{Y}$ be a normalization of the scheme Y, say $\widetilde{Y} = \operatorname{Spec}(\widetilde{A})$, where $\widetilde{A}$ is the integral closure of A in the field A_0. Let $\pi_Y : \widetilde{Y} \to Y$ be the morphism given by the inclusion $A \hookrightarrow \widetilde{A}$. Set $\widetilde{X} = \operatorname{Spec}(\widetilde{A}[x_1, \ldots, x_n])$ and let $\pi_X : \widetilde{X} \to X$ be given by the inclusion $A[x_1, \ldots, x_n] \hookrightarrow \widetilde{A}[x_1, \ldots, x_n]$. We have the natural morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ such that the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. The morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a dominant, generically finite morphism of finite type. Since $\widetilde{A}$ is a Krull ring (see Theorem 3.3), by Proposition 4.3, the set $S_{\widetilde{f}}$ is either empty or a hypersurface.

Now we prove that $\pi_Y(S_{\widetilde{f}}) \subset S_f$. For $h \in A$, we have $\pi_Y^{-1}(D_Y(h)) = D_{\widetilde{Y}}(h)$ and $D_Y(h) = \operatorname{Spec}(A_h)$, $D_{\widetilde{Y}}(h) = \operatorname{Spec}(\widetilde{A}_h)$. Of course $\widetilde{A}_h$ is an integral extension of A_h. Moreover, $f^{-1}(D_Y(h)) = D_X(h)$ and $\pi_X^{-1}(D_X(h)) = D_{\widetilde{X}}(h)$. The ring $(\widetilde{A}[x_1, \ldots, x_n])_h = \widetilde{A}_h[x_1, \ldots, x_n]$ is an integral extension of $(A[x_1, \ldots, x_n])_h = A_h[x_1, \ldots, x_n]$. Thus if $f|_{D_X(h)} : D_X(h) \to D_Y(h)$ is a finite morphism, then $\widetilde{A}_h[x_1, \ldots, x_n]$ is an integral extension of A. It follows that $\widetilde{A}_h[x_1, \ldots, x_n]$ is an integral extension of $\widetilde{A}_h$. Since $\widetilde{A}_h[x_1, \ldots, x_n]$ is a finitely generated $\widetilde{A}_h$-algebra, $\widetilde{A}_h[x_1, \ldots, x_n]$ is a finite ring extension of $\widetilde{A}_h$. Thus $\widetilde{f}|_{D_{\widetilde{X}}(h)} : D_{\widetilde{X}}(h) \to D_{\widetilde{Y}}(h)$ is a finite morphism, which proves the inclusion $\pi_Y(S_{\widetilde{f}}) \subset S_f$.

Conversely, we have $S_f \subset \pi_Y(S_{\widetilde{f}})$. It is enough to show that if $y \notin \pi_Y(S_{\widetilde{f}})$, then $y \notin S_f$. Let $y \notin \pi_Y(S_{\widetilde{f}})$. Then there is an open neighborhood U of y disjoint from $\pi_Y(S_{\widetilde{f}})$. The morphism $g := \pi_Y \circ \widetilde{f} : g^{-1}(U) \to U$ is integral, as a composition of integral morphisms. Consequently, so is $f \circ \pi_X : g^{-1}(U) \to U$ and hence also $f : f^{-1}(U) \to U$. Since f is of finite type it is finite, and consequently $y \notin S_f$.

It remains to prove that $\pi_Y(S_{\widetilde{f}})$ is either empty or a hypersurface. Note that $S_{\widetilde{f}}$ is the union of irreducible hypersurfaces $V(P_i)$, where $P_i \in \mathrm{Ass}(a_i)$ for some non-zero and non-invertible element $a_i \in \widetilde{A}$ (see Propositions 2.2 and 3.5).

In fact, we can assume that $a_i \in A$. Indeed, the irreducible equations of x_i over $\widetilde{A}$ are the same as over A, in particular, the coefficients of these equations are of the type a/b, where $a, b \in A$. Since the morphism $\pi_Y : \widetilde{Y} \to Y$ is integral, we have $\pi_Y(V(P_i)) = V(P_i \cap A)$.

Hence (by the Krull theorem) it is enough to prove that if a is a non-zero and non-invertible element of A, and $P_i \in \mathrm{Ass}_{\widetilde{A}}((a))$, then $P_i \cap A$ is a minimal ideal in the set $\mathrm{Ass}_A((a))$. But this can be done exactly as in the proof of Theorem 4.7.2 of [1], pp. 199–200. ∎

To end this paper, we show that our results can be generalized neither to the non-noetherian nor to the normal non-Krull case.

EXAMPLE 5.2. For $k \in \mathbb{N}$ we construct a ring R_k such that:

1) R_k is a normal domain,
2) there exists $p \in R_k$ such that the ideal (p) is prime and $\mathrm{ht}(p) \geq k$.

We proceed by induction. Let $R_1 = \mathbb{Z}$. Having defined a ring R_k, put $R_{k+1} = R_k + (R_k)_0 X + (R_k)_0 X^2 + \ldots \subset (R_k)_0[X]$. The ring R_{k+1} is a normal domain.

Indeed, if $\xi \in (R_{k+1})_0 = (R_k)_0(X)$ is an integral element over R_{k+1}, then it is also integral over $(R_k)_0[X]$. Thus $\xi \in (R_k)_0[X]$, say $\xi = \xi_0 + \xi_1 X + \ldots + \xi_d X^d$. Consequently, if $\xi^n + a_{n-1}\xi^{n-1} + \ldots + a_0 = 0$, where $a_i = a_{i,0} + a_{i,1}X + \ldots + a_{i,d_i}X^{d_i} \in R_k$, then $\xi_0^n + a_{n-1,0}\xi_0^{n-1} + \ldots + a_{0,0} = 0$. Hence $\xi_0 \in R_k$ and $\xi \in R_{k+1}$.

Let $(p) = p_k \supset p_{k-1} \supset \ldots \supset p_0 = 0$ be a sequence of distinct prime ideals in R_k. Now let $\widetilde{p}_i = \{f = f_0 + f_1 X_{k+1} + \ldots + f_d X_{k+1}^d \in R_{k+1} : f_0 \in p_i\}$. It is easy to see that $(p) = \widetilde{p}_k \supset \widetilde{p}_{k-1} \supset \ldots \supset \widetilde{p}_0 \supset 0$ is a sequence of distinct prime ideals in R_{k+1}. Consequently, $\mathrm{ht}(p) \geq k+1$.

Let $f : \mathrm{Spec}(R_k[1/p]) \to \mathrm{Spec}(R_k)$ be the morphism given by the inclusion $R_k \hookrightarrow R_k[1/p]$. By the proof of Proposition 4.1 and Proposition 2.2, applied to f, we see that $S_f = V(p)$ and consequently $\operatorname{codim} S_f \geq k$.

REFERENCES

[1] S. Balcerzyk and T. Józefiak, *Commutative Noetherian and Krull Rings*, PWN, Warszawa, and Horwood, Chichester, 1989.

[2] Z. Jelonek, *Topological characterization of finite mappings*, Bull. Polish Acad. Sci. Math. 49 (2001), 279–283.

[3] —, *Testing sets for properness of polynomial mappings*, Math. Ann. 315 (1999), 1–35.

[4] R. Hartshorne, *Algebraic Geometry*, Springer, New York, 1987.

[5] M. Nagata, *Local Rings*, Interscience Publ., New York, 1962.

[6] I. R. Shafarevich, *Basic Algebraic Geometry*, Springer, Heidelberg, 1974.

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