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ON A DECOMPOSITION OF POLYNOMIALS IN SEVERAL VARIABLES, II

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Abstract. One considers representation of cubic polynomials in several variables as the sum of values of univariate polynomials taken at linear combinations of the variables.

In the first paper of this series [6] we have defined for a field K the number $M(n,d,K)$ as the least integer M (provided it exists, otherwise ∞) such that for every polynomial $F \in K[x_1,\ldots,x_n]$ of degree d there exist vectors $\boldsymbol{\alpha}_\mu = [\alpha_{\mu 1},\ldots,\alpha_{\mu n}] \in K^n$ and polynomials $f_\mu \in K[z]$ $(1 \le \mu \le M)$ such that

$$F(x_1,\ldots,x_n) = \sum_{\mu=1}^{M} f_\mu\Big(\sum_{\nu=1}^{n} \alpha_{\mu\nu} x_\nu\Big). \tag{1}$$

We have shown that $M(n,d,K) < \infty$ if $\operatorname{char} K = 0$ or $\operatorname{char} K > d$ and studied the cases $d = 2$ and $n = 2$. In this paper we study the next simplest case $d = 3$ and prove two theorems.

THEOREM 1. *For every field K of characteristic different from $2,3$ we have*

$$M(n,3,K) \le \binom{n+1}{2}.$$

THEOREM 2. *For every algebraically closed field K of characteristic different from $2,3$ we have*

$$M(3,3,K) = 5.$$

An analogue of Theorem 1 for forms and $K = \mathbb{C}$ was proved by B. Reznick [3]. Equality (1) for generic cubic forms F over $\mathbb{C}$ has been studied by B. Reichstein [2], but his interesting results have no bearing on our theorems.

The proof of Theorem 1 is based on

LEMMA 1. *For every quadratic form $F \in K[x_1,\ldots,x_n] \setminus K[x_2,\ldots,x_n]$ there exist n linearly independent vectors $\boldsymbol{\alpha}_\mu \in K^n$ with $\alpha_{\mu 1} \ne 0$ and*

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$a_\mu \in K$ $(1 \le \mu \le n)$ *such that*

$$F = \sum_{\mu=1}^{n} a_\mu (\boldsymbol{\alpha}_\mu \boldsymbol{x})^2, \quad \textit{where} \quad \boldsymbol{\alpha x} = \sum_{i=1}^{n} \alpha_i x_i. \tag{2}$$

Proof. We proceed by induction on n. For $n = 1$ the lemma is obviously true. Assume that it is true for all quadratic forms over K in less than n variables.

If $\operatorname{rank} F = r < n$, then there exist linearly independent vectors $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r \in K^n$ and a form $G \in K[y_1, \ldots, y_r]$ such that

$$F = G(\boldsymbol{\beta}_1 \boldsymbol{x}, \boldsymbol{\beta}_2 \boldsymbol{x}, \ldots, \boldsymbol{\beta}_r \boldsymbol{x}).$$

Since $F \notin K[x_2, \ldots, x_n]$ we may assume that $\beta_{11} \neq 0$. Let

$$H = G\left(y_1, y_2 + \frac{\beta_{21}}{\beta_{11}} y_1, \ldots, y_r + \frac{\beta_{r1}}{\beta_{11}} y_1\right).$$

We have

$$F = H\left(\boldsymbol{\beta}_1 \boldsymbol{x}, \boldsymbol{\beta}_2 \boldsymbol{x} - \frac{\beta_{21}}{\beta_{11}} \boldsymbol{\beta}_1 \boldsymbol{x}, \ldots, \boldsymbol{\beta}_r \boldsymbol{x} - \frac{\beta_{r1}}{\beta_{11}} \boldsymbol{\beta}_1 \boldsymbol{x}\right)$$

and $\boldsymbol{\beta}_\varrho \boldsymbol{x} - \frac{\beta_{\varrho 1}}{\beta_{11}} \boldsymbol{\beta}_1 \boldsymbol{x} \in K[x_2, \ldots, x_n]$ $(1 \le \varrho \le r)$, hence $F \notin K[x_2, \ldots, x_n]$ implies $H \notin K[y_2, \ldots, y_r]$. By the inductive assumption

$$H = \sum_{\mu=1}^{r} c_\mu (\boldsymbol{\gamma}_\mu \boldsymbol{y})^2, \qquad \boldsymbol{y} = [y_1, \ldots, y_r],$$

where $c_\mu \in K$, $\boldsymbol{\gamma}_\mu \in K^r$ $(1 \le \mu \le r)$, $\boldsymbol{\gamma}_\mu$ are linearly independent and $\gamma_{\mu 1} \neq 0$. It follows that

$$F = \sum_{\mu=1}^{r} c_\mu \left(\gamma_{\mu 1} \boldsymbol{\beta}_1 \boldsymbol{x} + \sum_{\varrho=2}^{r} \gamma_{\mu\varrho} \left(\boldsymbol{\beta}_\varrho \boldsymbol{x} - \frac{\beta_{\varrho 1}}{\beta_{11}} \boldsymbol{\beta}_1 \boldsymbol{x}\right)\right)^2.$$

Let us put

$$(\alpha_{\mu\nu})_{\substack{\mu \le r \\ \nu \le n}} = (\gamma_{\mu\varrho})_{\substack{\mu \le r \\ \varrho \le r}} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 - \boldsymbol{\beta}_1 \frac{\beta_{21}}{\beta_{11}} \\ \vdots \\ \boldsymbol{\beta}_r - \boldsymbol{\beta}_1 \frac{\beta_{r1}}{\beta_{11}} \end{pmatrix}. \tag{3}$$

Since the matrix $(\gamma_{\mu\varrho})_{\substack{\mu \le r \\ \varrho \le r}}$ is non-singular of order r and the rank of the matrix $\begin{pmatrix} \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\beta}_r \end{pmatrix}$ is r, the rank of the matrix $(\alpha_{\mu\nu})_{\substack{\mu \le r \\ \nu \le n}}$ is r and there exist $n - r$ vectors $\boldsymbol{\alpha}_{r+1}, \ldots, \boldsymbol{\alpha}_n$ in K^n such that $A = (\alpha_{\mu\nu})_{\substack{\mu \le n \\ \nu \le n}}$ is of rank n. We have, by (3), $\alpha_{\mu 1} = \gamma_{\mu 1} \beta_{11} \neq 0$ for $\mu \le r$. Adding, if necessary, the first row of A to rows $r+1,\dots,n$ we achieve that $\alpha_{\mu 1}\neq 0$ for all $\mu\le n$ and (2) is satisfied with $a_\mu = c_\mu$ for $\mu\le r$, $a_\mu=0$ for $\mu>r$.

If $\operatorname{rank} F = n$, let

$$F=\sum_{i,j=1}^{n} a_{ij}x_ix_j, \quad \text{where} \quad a_{ij}=a_{ji}.$$

If $a_{ii}\neq 0$ for at least one $i>1$, then we consider the form

$$G = F - a_{ii}\Big(\sum_{j=1}^{n}\frac{a_{ij}}{a_{ii}}x_j\Big)^2. \tag{4}$$

This form is of rank $n-1$ and it does not depend on x_i, hence $G\notin K[x_2,\dots,x_n]$ and by the inductive assumption

$$G=\sum_{\mu=1}^{n-1} b_\mu(\boldsymbol{\beta}_\mu\boldsymbol{x})^2, \tag{5}$$

where $b_\mu\in K$, $\boldsymbol{\beta}_\mu\in K^n$ $(1\le\mu<n)$, $\boldsymbol{\beta}_\mu$ are linearly independent, $\beta_{\mu 1}\neq 0$ and $b_\mu\neq 0$ (otherwise G would be of rank $<n-1$). If $a_{i1}\neq 0$ it suffices to take in (1)

$$\begin{aligned} a_\mu &= b_\mu, &\quad \boldsymbol{\alpha}_\mu &= \boldsymbol{\beta}_\mu \quad \text{for } \mu<n,\\ a_n &= a_{ii}, &\quad \boldsymbol{\alpha}_n &= \Big[\frac{a_{i1}}{a_{ii}},\dots,\frac{a_{in}}{a_{ii}}\Big]. \end{aligned}$$

If $a_{i1}=0$, then we choose $c\in K$, $c^2\neq 0, \pm a_{ii}/b_{n-1}$, which is possible unless $K=\mathbb{F}_5$, $a_{ii}=\pm b_{n-1}$, and infer from (4), (5) that

$$\begin{aligned} F = \sum_{\mu=1}^{n-2} b_\mu(\boldsymbol{\beta}_\mu\boldsymbol{x})^2 &+ b_{n-1}\Big(\frac{b_{n-1}c^2-a_{ii}}{b_{n-1}c^2+a_{ii}}\boldsymbol{\beta}_{n-1}\boldsymbol{x} + \frac{2a_{ii}c}{b_{n-1}c^2+a_{ii}}\sum_{j=1}^{n}\frac{a_{ij}}{a_{ii}}x_j\Big)^2\\ &+ a_{ii}\Big(\frac{2b_{n-1}c}{b_{n-1}c^2+a_{ii}}\boldsymbol{\beta}_{n-1}\boldsymbol{x} - \frac{b_{n-1}c^2-a_{ii}}{b_{n-1}c^2+a_{ii}}\sum_{j=1}^{n}\frac{a_{ij}}{a_{ii}}x_j\Big)^2. \end{aligned}$$

Hence (2) is satisfied with

$$\begin{aligned} a_\mu &= b_\mu, & \boldsymbol{\alpha}_\mu &= \boldsymbol{\beta}_\mu \quad \text{for } \mu<n-1,\\ a_{n-1} &= b_{n-1}, & \boldsymbol{\alpha}_{n-1} &= \frac{b_{n-1}c^2-a_{ii}}{b_{n-1}c^2+a_{ii}}\boldsymbol{\beta}_{n-1} + \frac{2c}{b_{n-1}c^2+a_{ii}}[a_{i1},\dots,a_{in}],\\ a_n &= a_{ii}, & \boldsymbol{\alpha}_n &= \frac{2b_{n-1}c}{b_{n-1}c^2+a_{ii}}\boldsymbol{\beta}_{n-1} - \frac{b_{n-1}c^2-a_{ii}}{b_{n-1}c^2+a_{ii}}\Big[\frac{a_{i1}}{a_{ii}},\dots,\frac{a_{in}}{a_{ii}}\Big]. \end{aligned}$$

By the choice of c we have $\alpha_{\mu 1}\neq 0$. Also the $\boldsymbol{\alpha}_\mu$ are linearly independent, since otherwise F would be of rank less than n.

In the remaining case $K=\mathbb{F}_5$, $a_{ii}=\varepsilon b_{n-1}$ $(\varepsilon=\pm1)$ it suffices to take

$$\begin{aligned} a_\mu &= b_\mu, & \boldsymbol{\alpha}_\mu &= \boldsymbol{\beta}_\mu \quad \text{for } \mu<n-1,\\ a_{n-1} &= 3b_{n-1}, & \boldsymbol{\alpha}_{n-1} &= \boldsymbol{\beta}_{n-1}+\frac{3-\varepsilon}{2}\left[\frac{a_{i1}}{a_{ii}},\dots,\frac{a_{in}}{a_{ii}}\right],\\ a_n &= 3b_{n-1}, & \boldsymbol{\alpha}_n &= \boldsymbol{\beta}_{n-1}-\frac{3-\varepsilon}{2}\left[\frac{a_{i1}}{a_{ii}},\dots,\frac{a_{in}}{a_{ii}}\right]. \end{aligned}$$

If $a_{ii}=0$ for all $i>1$, but $a_{ij}\neq0$ for some $i>1$, $j>1$ then we make the linear transformation $x_i=x_i'+x_j'$, $x_j=x_i'-x_j'$ and reduce this case to the former.

There remains the case where $a_{ij}=0$ for all $i,j>1$. Then $1<n=\operatorname{rank}F\le2$, so $n=\operatorname{rank}F=2$ and we have

$$\begin{aligned} F &= a_{11}x_1^2+2a_{12}x_1x_2\\ &= \frac14\left(\left(c+\frac{a_{11}}{c}\right)x_1+\frac{2a_{12}}{c}x_2\right)^2-\frac14\left(\left(c-\frac{a_{11}}{c}\right)x_1-\frac{2a_{12}}{c}x_2\right)^2, \end{aligned}$$

where c in K is chosen so that $c^2\neq0,\pm a_{11}$. Such a choice is possible unless $K=\mathbb{F}_5$, $a_{11}=\pm1$. In that case

$$F=3a_{11}(x_1-2a_{11}a_{12}x_2)^2+3a_{11}(x_1-a_{11}a_{12}x_2)^2.$$

Proof of Theorem 1. We proceed by induction on n. For $n=1$ the theorem is obviously true. Assume that $M(n-1,3,K)\le\binom{n}{2}$ and consider a polynomial $F\in K[x_1,\dots,x_n]$ of degree 3. Let F_0 be its leading form.

Since $\operatorname{card}K\ge4$, by Lemma 1 of [6], or by Lemma 4.4.1 of [7], there exists $\boldsymbol{\beta}_1\in K^n$ such that $F_0(\boldsymbol{\beta}_1)\neq0$ and thus $\boldsymbol{\beta}_1\neq\mathbf{0}$. If vectors $\boldsymbol{\beta}_2,\dots,\boldsymbol{\beta}_n$ are chosen in K^n so that $\det(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_n)\neq0$, we can replace in our argument $F(\boldsymbol{x})$ by $F_1(\boldsymbol{y}):=F(\boldsymbol{\beta}_1\boldsymbol{y},\dots,\boldsymbol{\beta}_n\boldsymbol{y})$, where $\boldsymbol{y}=[y_1,\dots,y_n]$ and where the coefficient of y_1^3 in $F_1(\boldsymbol{y})$ is $F_0(\boldsymbol{\beta}_1)\neq0$ (cf. [3]). Hence we may assume without loss of generality that

$$\deg_{x_1}\frac{\partial F_0}{\partial x_1}>0.$$

Then by Lemma 1,

$$\frac{\partial F_0}{\partial x_1}=\sum_{\mu=1}^n a_\mu(\boldsymbol{\alpha}_\mu\boldsymbol{x})^2, \tag{6}$$

where $a_\mu\in K$, $\boldsymbol{\alpha}_\mu\in K^n$, $\alpha_{\mu1}\neq0$ $(1\le\mu\le n)$ and the $\boldsymbol{\alpha}_\mu$ are linearly independent. By the last condition there exist $c_\mu\in K$ $(1\le\mu\le n)$ and $d\in K$ such that

$$\frac{\partial F}{\partial x_1}-\frac{\partial F_0}{\partial x_1}=\sum_{\mu=1}^n c_\mu(\boldsymbol{\alpha}_\mu\boldsymbol{x})+d. \tag{7}$$

Consider now the polynomial

$$G = F - \sum_{\mu=1}^{n} \frac{a_\mu}{3\alpha_{\mu 1}} (\boldsymbol{\alpha}_\mu \boldsymbol{x})^3 - \sum_{\mu=1}^{n} \frac{c_\mu}{2\alpha_{\mu 1}} (\boldsymbol{\alpha}_\mu \boldsymbol{x})^2 - \frac{d}{\alpha_{11}} (\boldsymbol{\alpha}_1 \boldsymbol{x}). \tag{8}$$

By (6)–(8) we have

$$\frac{\partial G}{\partial x_1} = 0,$$

hence $G \in K[x_2, \ldots, x_n]$. By the inductive assumption there exist vectors $\boldsymbol{\alpha}_\mu^* \in K^{n-1}$ and polynomials $g_\mu \in K[z]$ $(n < \mu \le \binom{n+1}{2})$ such that

$$G = \sum_{\mu=n+1}^{\binom{n+1}{2}} g_\mu\Big(\sum_{\nu=1}^{n-1} \alpha_{\mu\nu}^* x_{\nu+1}\Big).$$

The decomposition (1) follows now from (8) with

$$\begin{aligned}
f_1 &= \frac{a_1}{3\alpha_{11}} z^3 + \frac{c_1}{2\alpha_{11}} z^2 + \frac{d}{\alpha_{11}}, \\
f_\mu &= \frac{a_\mu}{3\alpha_{\mu 1}} z^3 + \frac{c_\mu}{2\alpha_{\mu 1}} z^2 \qquad (1 < \mu \le n), \\
f_\mu &= g_\mu, \ \boldsymbol{\alpha}_\mu = [0, \boldsymbol{\alpha}_\mu^*] \qquad \left(n < \mu \le \binom{n+1}{2}\right).
\end{aligned}$$

The proof of Theorem 1 is complete. The idea this proof is taken from the paper of Rosanes [4], §6.

The proof of Theorem 2 is based on four lemmas.

Lemma 2. *For*

$$\begin{aligned}
\phi(x) = {} & a x_1^3 + b x_2^3 + c x_3^3 + 3a_2 x_1^2 x_2 + 3a_3 x_1^2 x_3 + 3b_1 x_2^2 x_1 + 3b_3 x_2^2 x_3 \\
& + 3c_1 x_3^2 x_1 + 3c_2 x_3^2 x_2 + 6m x_1 x_2 x_3 \in K[x_1, x_2, x_3],
\end{aligned}$$

let

$$\begin{aligned}
S(\phi) = {} & abcm - (bca_2a_3 + cab_1b_3 + abc_1c_2) - m(ab_3c_2 + bc_1a_3 + ca_2b_1) \\
& + (ab_1c_2^2 + bc_2a_3^2 + ac_1b_3^2 + ba_2c_1^2 + cb_3a_2^2 + ca_3b_1^2) - m^4 \\
& + 2m^2(b_1c_1 + c_2a_2 + a_3b_3) - 3m(a_2b_3c_1 + a_3b_1c_2) \\
& - (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2) + (c_2a_2a_3b_3 + a_3b_3b_1c_1 + b_1c_1c_2a_2).
\end{aligned}$$

Then for every matrix $A \in K^{3\times 3}$,

$$S(\phi(A\boldsymbol{x})) = S(\phi)(\det A)^4.$$

Moreover, if $\phi = \sum_{i=1}^{3} (\boldsymbol{\alpha}_i \boldsymbol{x})^3$ *for some* $\boldsymbol{\alpha}_i \in K^3$, *then* $S(\phi) = 0$.

Proof. See Salmon [5], Section 221.

LEMMA 3. *Every non-zero ternary cubic form ϕ over an algebraically closed field K of characteristic different from 2, 3 can be transformed by a non-singular linear transformation over K into one of the forms*

$$\begin{aligned} F_1 &= \eta x_2^3 + x_3^3, \quad \eta = 0 \text{ or } 1,\\ F_2 &= x_1^3 + x_2^3 + x_3^3 + 6m x_1x_2x_3, \quad m \in K,\\ F_3 &= x_2^3 + x_3^3 + 6x_1x_2x_3,\\ F_4 &= \varepsilon x_1^3 + 3x_2^2x_3, \quad \varepsilon = 0 \text{ or } 1,\\ F_5 &= 6x_1x_2x_3 + x_3^3,\\ F_6 &= 3x_1^2x_2 + 3x_1x_3^2. \end{aligned}$$

Proof. See Gordan [1]. Gordan gives 10 types of forms, but two of them, $x_1^3 + x_2^3 + x_3^3$ and $6x_1x_2x_3$, are obtained from F_2 for $m = 0$ and $m = -\frac{1}{2}$ respectively. Indeed,

$$x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 = (x_1 + x_2 + x_3)(x_1 + \varrho x_2 + \varrho^2 x_3)(x_1 + \varrho^2 x_2 + \varrho x_3),$$

where $\varrho^3 = 1$, $\varrho \neq 1$.

LEMMA 4. *For $\nu = 2, 3, 4$ we have for some $\boldsymbol{\alpha}_\mu \in K^3$, $a_\mu \in K$ $(1 \le \mu \le 4)$,*

$$F_\nu = \sum_{\mu=1}^{4} a_\mu (\boldsymbol{\alpha}_\mu \boldsymbol{x})^3, \tag{9}$$

where $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$ are linearly independent, $\operatorname{rank} A_\nu < 4$ *and, for any $\alpha_1, \alpha_2, \alpha_3 \in K$, the conditions $\alpha_1 \neq 0$ and* $\operatorname{rank} B_\nu < 5$ *imply that $[\alpha_1, \alpha_2, \alpha_3]$ is a scalar multiple of one of the $\boldsymbol{\alpha}_\mu$.*

Here

$$A_\nu = \begin{pmatrix} \alpha_{11}\alpha_{12} & \dots & \alpha_{41}\alpha_{42}\\ \alpha_{12}^2 & \dots & \alpha_{42}^2\\ \alpha_{11}\alpha_{13} & \dots & \alpha_{41}\alpha_{43}\\ \alpha_{12}\alpha_{13} & \dots & \alpha_{42}\alpha_{43}\\ \alpha_{13}^2 & \dots & \alpha_{43}^2 \end{pmatrix}, \quad B_\nu = \begin{pmatrix} \alpha_{11}^2 & \dots & \alpha_{41}^2 & \alpha_1^2\\ & & & \alpha_1\alpha_2\\ & & & \alpha_2^2\\ & A_\nu & & \alpha_1\alpha_3\\ & & & \alpha_2\alpha_3\\ & & & \alpha_3^2 \end{pmatrix}. \tag{10}$$

Proof. We have, with $\varrho^3 = 1$, $\varrho \neq 1$,

$$\begin{aligned} F_2(\boldsymbol{x}) = (1 - m^3)x_1^3 &+ \frac{1}{3}(mx_1 + x_2 + x_3)^3\\ &+ \frac{1}{3}(mx_1 + \varrho x_2 + \varrho^2 x_3)^3 + \frac{1}{3}(mx_1 + \varrho^2 x_2 + \varrho x_3)^3, \end{aligned}$$

hence

$$A_2 = A_2(m) = \begin{pmatrix} 0 & m & m\varrho & m\varrho^2 \\ 0 & 1 & \varrho^2 & \varrho \\ 0 & m & m\varrho^2 & m\varrho \\ 0 & 1 & 1 & 1 \\ 0 & 1 & \varrho & \varrho^2 \end{pmatrix}.$$

Clearly $\operatorname{rank} A_2 < 4$ and by adding the first three columns of $B_2 = B_2(m)$ to the fourth we obtain

$$\operatorname{rank} B_2(m) = \operatorname{rank} C_2(m) + 2,$$

where

$$C_2(m) = \begin{pmatrix} m & m\varrho & \alpha_1\alpha_2 \\ 1 & \varrho^2 & \alpha_2^2 \\ m & m\varrho^2 & \alpha_1\alpha_3 \\ 1 & \varrho & \alpha_3^2 \end{pmatrix}.$$

If $\operatorname{rank} C_2(m) < 3$, then $\alpha_1\alpha_3 - m\alpha_2^2 = m\alpha_3^2 - \alpha_1\alpha_2 = 0$, hence if $\alpha_1 \neq 0$ we have either $m \neq 0$ and $\frac{m}{\alpha_1}[\alpha_1, \alpha_2, \alpha_3] = [m, 1, 1]$, $[m, \varrho, \varrho^2]$ or $[m, \varrho^2, \varrho]$, or $m = 0$ and $\frac{1}{\alpha_1}[\alpha_1, \alpha_2, \alpha_3] = [1, 0, 0]$.

Further, we have

$$F_3(\boldsymbol{x}) = -x_1^3 + \frac{1}{3}(x_1 + x_2 + x_3)^3 + \frac{1}{3}(x_1 + \varrho x_2 + \varrho^2 x_3)^3 + \frac{1}{3}(x_1 + \varrho^2 x_2 + \varrho x_3)^3,$$

hence

$$A_3 = A_2(1), \quad B_3 = B_2(1)$$

and the assertion follows.

Finally, we have

$$F_4(\boldsymbol{x}) = \varepsilon x_1^3 + \frac{1}{3}(x_3 + 2x_2)^3 + \frac{1}{3}(x_3 - 2x_2)^3 - \frac{1}{4}x_3^3,$$

hence

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 \quad 0 \quad 0 \quad 0 & \alpha_1^2 \\ & \alpha_1\alpha_2 \\ & \alpha_2^2 \\ A_3 & \alpha_1\alpha_3 \\ & \alpha_2\alpha_3 \\ & \alpha_3^2 \end{pmatrix}.$$

Clearly $\operatorname{rank} A_4 < 4$ and $\operatorname{rank} B_4 = \operatorname{rank} C_4 + 2$, where

$$C_4 = \begin{pmatrix} 0 & 0 & \alpha_1\alpha_2 \\ 4 & 4 & \alpha_2^2 \\ 0 & 0 & \alpha_1\alpha_3 \\ 2 & -2 & \alpha_2\alpha_3 \end{pmatrix}.$$

If $\operatorname{rank} C_4 < 3$, then $\alpha_1\alpha_2 = \alpha_1\alpha_3 = 0$, hence if $\alpha_1 \neq 0$, then $\frac{1}{\alpha_1}[\alpha_1, \alpha_2, \alpha_3] = [1, 0, 0]$.

Lemma 5. *For* $\nu = 5$ *we have* (9) *for some* $\boldsymbol{\alpha}_\mu \in K^3$, $a_\mu \in K$; $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$ *are linearly independent and the condition* $\operatorname{rank} B_5 < 5$, *where* B_5 *is given by the formula* (10), *implies that* $[\alpha_1, \alpha_2, \alpha_3]$ *is a scalar multiple of one of the* $\boldsymbol{\alpha}_\mu$.

Proof. We have

$$F_5(\boldsymbol{x}) = \frac{1}{48}(3x_1 - 2x_2 - 2x_3)^3 - \frac{1}{48}(3x_1 - 2x_2 - 2x_3)^3 - \frac{1}{96}(3x_1 - 2x_2 - 4x_3)^3 + \frac{1}{96}(3x_1 - 2x_2 - 4x_3)^3,$$

hence

$$B_5 = \begin{pmatrix} 9 & 9 & 9 & 9 & \alpha_1^2 \\ -6 & -6 & 6 & 6 & \alpha_1\alpha_2 \\ 4 & 4 & 4 & 4 & \alpha_2^2 \\ -6 & 6 & -12 & 12 & \alpha_1\alpha_3 \\ 4 & -4 & -8 & 8 & \alpha_2\alpha_3 \\ 4 & 4 & 16 & 16 & \alpha_3^2 \end{pmatrix}.$$

It is easily seen that

$$\operatorname{rank} B_5 = \operatorname{rank} C_5 + 2,$$

where

$$C_5 = \begin{pmatrix} 9 & 9 & \alpha_1^2 \\ -6 & 6 & \alpha_1\alpha_2 \\ 4 & 4 & \alpha_2^2 \\ 4 & 16 & \alpha_3^2 \end{pmatrix}.$$

If $\operatorname{rank} C_5 < 3$ then $-48\alpha_1^2 + 108\alpha_2^2 = 0$, $48\alpha_1\alpha_2 + 120\alpha_2^2 - 48\alpha_3^2 = 0$, hence $3[\alpha_1, \alpha_2, \alpha_3] = \alpha_1[3, -2, -2]$, $\alpha_1[3, -2, 2]$, $\alpha_1[3, 2, -4]$ or $\alpha_1[3, 2, 4]$.

Proof of Theorem 2. We shall show first that

$$M(3, 3, K) \geq 5.$$

Indeed, suppose that

$$3x_1^2x_2 + 3x_1x_2^2 = \sum_{\mu=1}^{4} f_\mu(\boldsymbol{\alpha}_\mu \boldsymbol{x}), \qquad \text{where } \boldsymbol{\alpha}_\mu \in K^3,\ f_\mu \in K[z]. \tag{11}$$

Since the left hand side is homogeneous we may assume that the f_μ are monomials, and since K is algebraically closed, that $f_\mu = z^3$ $(1 \leq \mu \leq 4)$. Then for each $\mu \leq 4$,

$$G_\mu := 3x_1^2x_2 + 3x_1x_3^2 - (\boldsymbol{\alpha}_\mu \boldsymbol{x})^3$$

is the sum of three cubes of linear forms and by Lemma 2 we have $S(G_\mu) = 0$. However by the same lemma and a tedious computation

$$S(G_\mu) = -\alpha_{\mu 2}^3,$$

hence $\alpha_{\mu 2} = 0$ for all $\mu \le 4$. This contradicts (11), since the left hand side depends on x_2, while the right does not.

We shall now show that

$$M(3,3,K) \le 5.$$

Let $F \in K[x]$ be a polynomial of degree 3 with the highest homogeneous part F_0. Since the statement of the theorem is invariant with respect to non-singular linear transformations we may assume by virtue of Lemma 3 that $F_0(\boldsymbol{x}) = F_\nu(\boldsymbol{x})$, where $1 \le \nu \le 6$. Also we may assume that $F(\mathbf{0}) = 0$, since a constant can be added to any polynomial f_μ. If $\nu = 1$ we have

$$F_0(\boldsymbol{x}) = F_1(\boldsymbol{x}) = \eta x_2^3 + x_3^3.$$

On the other hand, by Theorem 3 of [6],

$$F(\boldsymbol{x}) - F_1(\boldsymbol{x}) = \sum_{\mu=1}^{3} f_\mu(\boldsymbol{\alpha}_\mu \boldsymbol{x}), \qquad \boldsymbol{\alpha}_\mu \in K^3,\ f_\mu \in K[z],$$

hence (1) holds with $M = 5$ and

$$\begin{aligned} \boldsymbol{\alpha}_4 &= [0,1,0], \quad & f_4 &= \eta z^3, \\ \boldsymbol{\alpha}_5 &= [0,0,1], \quad & f_5 &= z^3. \end{aligned}$$

If $1 < \nu \le 5$ we have, by Lemmas 4 and 5,

$$F_0(x) = F_\nu(x) = \sum_{\mu=1}^{4} a_\mu (\boldsymbol{\alpha}_\mu \boldsymbol{x})^3. \tag{12}$$

Now, let

$$F(x) - F_0(x) = \sum_{i,j=1}^{3} b_{ij} x_i x_j + \sum_{i=1}^{3} c_i x_i, \quad b_{ij} = b_{ji}, \tag{13}$$

and let

$$D = \det \begin{pmatrix} & b_{11} \\ & b_{12} \\ B_\nu & b_{22} \\ & b_{13} \\ & b_{23} \\ & b_{33} \end{pmatrix} = \sum_{1 \le i \le j \le 3} c_{ij} \alpha_i \alpha_j .$$

If $\nu \le 4$, in view of the condition $\operatorname{rank} A_\nu = 0$ we have $c_{11} = 0$, hence the equation

$$\sum_{1 \le i \le j \le 3} c_{ij} \alpha_i \alpha_j = 0 \tag{14}$$

has infinitely many pairwise linearly independent solutions with $\alpha_1 \ne 0$. Thus there exists a vector $[\alpha_{51}, \alpha_{52}, \alpha_{53}]$ with $\alpha_{51} \ne 0$ that is not a scalar multiple of any $\boldsymbol{\alpha}_\mu$ $(1 \le \mu \le 4)$ and satisfies $D = 0$. According to Lemma 4, for $\alpha_i = \alpha_{5i}$ $(1 \le i \le 3)$ we have $\operatorname{rank} B_\nu = 5$. It follows by Kronecker–Capelli's theorem that the system of linear equations

$$\sum_{\mu=1}^{5} b_\mu \alpha_{\mu i}\alpha_{\mu j} = b_{ij} \qquad (1 \le i \le j \le 3)$$

is solvable for b_μ in K $(1 \le \mu \le 5)$. Also the system

$$\sum_{\mu=1}^{3} d_\mu \alpha_{\mu i} = c_i \qquad (1 \le i \le 3)$$

is solvable for d_μ in K $(1 \le \mu \le 3)$, because of the linear independence of $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$. Therefore, with $d_4 = d_5 = 0$,

$$F(x) - F_0(x) = \sum_{\mu=1}^{5} b_\mu(\boldsymbol{\alpha}_\mu \boldsymbol{x})^2 + \sum_{\mu=1}^{5} d_\mu(\boldsymbol{\alpha}_\mu \boldsymbol{x}),$$

which together with (12) gives (1) with

$$f_\mu = \alpha_\mu z^3 + b_\mu z^2 + d_\mu z, \quad a_5 = 0.$$

If $\nu = 5$, then the equation (14) has infinitely many pairwise linearly independent solutions. Thus there exists a vector $[\alpha_{51}, \alpha_{52}, \alpha_{53}]$ that is not a scalar multiple of any $\boldsymbol{\alpha}_\mu$ $(1 \le \mu \le 4)$ and satisfies $D = 0$. According to Lemma 5, for $\alpha_i = \alpha_{5i}$ $(1 \le i \le 3)$ we have $\operatorname{rank} B_5 = 5$. The remainder of the proof is identical with the one given above.

There remains the most difficult case $\nu = 6$. Here, as the proof of $M(3,3,K) \ge 5$ shows, the above approach is impossible and we argue as follows.

Let again (13) hold and consider first the case where $b_{22} \ne 0$, or $b_{22} = 0$, $b_{23} = 0$. We choose $c \in K$ as a solution to the equation

$$cb_{22} = b_{23} \tag{15}$$

and then choose α_{43}, α_{53} in K satisfying the conditions

$$c\alpha_{43}\alpha_{53} + \alpha_{43} + \alpha_{53} = 0, \quad \alpha_{43}\alpha_{53}(\alpha_{43} - \alpha_{53}) \ne 0.$$

This gives

$$\begin{vmatrix} \alpha_{43} & \alpha_{53} & -c \\ \alpha_{43}^2 & \alpha_{53}^2 & 1 \\ \alpha_{43}^3 & \alpha_{53}^3 & 0 \end{vmatrix} = 0, \qquad \begin{vmatrix} \alpha_{43} & \alpha_{53} \\ \alpha_{43}^2 & \alpha_{53}^2 \end{vmatrix} \ne 0$$

and the systems of linear equations

$$\begin{aligned} &a_4\alpha_{43} + a_5\alpha_{53} = -c, \quad a_4\alpha_{43}^2 + a_5\alpha_{53}^2 = 1, \quad a_4\alpha_{43}^3 + a_5\alpha_{53}^3 = 0; \\ &b_4\alpha_{43} + b_5\alpha_{53} = b_{13} - cb_{12}, \quad b_4\alpha_{43}^2 + b_5\alpha_{53}^2 = b_{33} - c^2 b_{22} \end{aligned} \tag{16}$$

are solvable for a_4, a_5, b_4, b_5 in K.

Then we choose $\alpha_{12}, \alpha_{22}, \alpha_{32}$ in K such that

$$(a_4 + a_5)\alpha_{12}\alpha_{22}\alpha_{32} + \alpha_{12}\alpha_{22} + \alpha_{12}\alpha_{32} + \alpha_{22}\alpha_{32} = 0,$$
$$(\alpha_{22} - \alpha_{12})(\alpha_{32} - \alpha_{12})(\alpha_{32} - \alpha_{22}) \neq 0.$$

This gives

$$\begin{vmatrix} 1 & 1 & 1 & -a_4 - a_5 \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & 1 \\ \alpha_{12}^2 & \alpha_{22}^2 & \alpha_{32}^2 & 1 \\ \alpha_{12}^3 & \alpha_{22}^3 & \alpha_{32}^3 & 0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{12}^2 & \alpha_{22}^2 & \alpha_{32}^2 \end{vmatrix} \neq 0, \qquad \begin{vmatrix} 1 & 1 & 1 \\ \alpha_{12} & \alpha_{22} & 0 \\ c\alpha_{12} & c\alpha_{22} & \alpha_{43} \end{vmatrix} \neq 0$$

and the systems of linear equations

$$\sum_{\mu=1}^{3} a_\mu = -a_4 - a_5, \ \sum_{\mu=1}^{3} a_\mu \alpha_{\mu 2} = 1, \ \sum_{\mu=1}^{3} a_\mu \alpha_{\mu 2}^2 = 0, \ \sum_{\mu=1}^{3} a_\mu \alpha_{\mu 2}^3 = 0;$$

$$\sum_{\mu=1}^{3} b_\mu = b_{11} - b_4 - b_5, \ \sum_{\mu=1}^{3} b_\mu \alpha_{\mu 2} = b_{12}, \ \sum_{\mu=1}^{3} b_\mu \alpha_{\mu 2}^2 = b_{22}; \tag{17}$$

$$\sum_{\mu=1}^{2} d_\mu + d_4 = c_1, \ \sum_{\mu=1}^{2} d_\mu \alpha_{\mu 2} = c_2, \ \sum_{\mu=1}^{2} d_\mu c\alpha_{\mu 2} + d_4 \alpha_{43} = c_3$$

are solvable for $a_1, a_2, a_3, b_1, b_2, b_3, d_1, d_2, d_4$ in K. Then we put $d_3 = d_5 = 0$,

$$\boldsymbol{\alpha}_\mu = \begin{cases} [1, \alpha_{\mu 2}, c\alpha_{\mu 2}] & \text{for } \mu \leq 3, \\ [1, 0, \alpha_{\mu 3}] & \text{for } \mu = 4, 5, \end{cases}$$

$$f_\mu = a_\mu z^3 + b_\mu z^2 + d_\mu z \qquad (1 \leq \mu \leq 5),$$

and verify that (12)–(13) and (15)–(17) imply (1).

Consider now the case where $b_{22} = 0$, $b_{23} \neq 0$. We take c as the solution to the equation

$$cb_{23} + b_{12} = b_{33}. \tag{18}$$

Then we choose $\alpha_{\mu 3}$ as 5 distinct roots of the equation

$$\alpha_3^5 = d(c\alpha_3 + 1)^3, \tag{19}$$

where d is chosen so that the equation has distinct roots. It follows that $c\alpha_{\mu 3} + 1 \neq 0$ and we take

$$\alpha_{\mu 2} = \frac{\alpha_{\mu 3}^2}{c\alpha_{\mu 3} + 1} \qquad (1 \leq \mu \leq 5). \tag{20}$$

Let us consider the matrix

$$A = \begin{pmatrix} 1 & \dots & 1 \\ \alpha_{12} & \dots & \alpha_{52} \\ \alpha_{12}^2 & \dots & \alpha_{52}^2 \\ \alpha_{13} & \dots & \alpha_{53} \\ \alpha_{12}\alpha_{13} & \dots & \alpha_{52}\alpha_{53} \end{pmatrix}$$

and suppose that a linear combination of its successive rows with coefficients $e_1, \dots, e_5$, respectively, equals $\mathbf{0}$. From (20) we obtain

$$e_1 + e_2 \frac{\alpha_{\mu 3}^2}{c\alpha_{\mu 3}+1} + e_3 \frac{\alpha_{\mu 3}^4}{(c\alpha_{\mu 3}+1)^2} + e_4\alpha_{\mu 3} + e_5 \frac{\alpha_{\mu 3}^3}{c\alpha_{\mu 3}+1} = 0,$$

hence

$$\begin{aligned}(e_3 + e_5 c)\alpha_{\mu 3}^4 + (e_2 c + e_4 c^2 + e_5)\alpha_{\mu 3}^3 + (e_1 c^2 + e_2 + 2e_4 c)\alpha_{\mu 3}^2 \\ + (2e_1 c + e_4)\alpha_{\mu 3} + e_1 = 0\end{aligned}$$

and since the left hand side is a polynomial of degree at most 4 in $\alpha_{\mu 3}$, and $\alpha_{13}, \dots, \alpha_{53}$ are distinct, we have

$$e_3 + e_5 c = 0, \quad e_2 c + e_4 c^2 + e_5 = 0, \quad e_1 c^2 + e_2 + 2e_4 c = 0,$$
$$2e_1 c + e_4 = 0, \quad e_1 = 0,$$

which implies $e_\mu = 0$ $(1 \le \mu \le 5)$. Thus the rows of A are linearly independent and the systems of linear equations

$$\begin{gathered}\sum_{\mu=1}^{5} a_\mu = 0, \quad \sum_{\mu=1}^{5} a_\mu \alpha_{\mu 2} = 1, \quad \sum_{\mu=1}^{5} a_\mu \alpha_{\mu 2}^2 = 0, \\ \sum_{\mu=1}^{5} a_\mu \alpha_{\mu 3} = 0, \quad \sum_{\mu=1}^{5} a_\mu \alpha_{\mu 2}\alpha_{\mu 3} = 0;\end{gathered} \tag{21}$$

$$\begin{gathered}\sum_{\mu=1}^{5} b_\mu = b_{11}, \quad \sum_{\mu=1}^{5} b_\mu \alpha_{\mu 2} = b_{12}, \quad \sum_{\mu=1}^{5} b_\mu \alpha_{\mu 2}^2 = 0, \\ \sum_{\mu=1}^{5} b_\mu \alpha_{\mu 3} = b_{13}, \quad \sum_{\mu=1}^{5} a_\mu \alpha_{\mu 2}\alpha_{\mu 3} = b_{23};\end{gathered} \tag{22}$$

$$\sum_{\mu=1}^{5} d_\mu = c_1, \quad \sum_{\mu=1}^{5} d_\mu \alpha_{\mu 2} = c_2, \quad \sum_{\mu=1}^{5} d_\mu \alpha_{\mu 3} = c_3 \tag{23}$$

are solvable for a_μ, b_μ, d_μ in K.

In view of (18) and of the identities

$$\begin{gathered}\alpha_{\mu 2}^3 = d\alpha_{\mu 3}, \quad \alpha_{\mu 3}^2 = c\alpha_{\mu 2}\alpha_{\mu 3} + \alpha_{\mu 2}, \quad \alpha_{\mu 2}^2\alpha_{\mu 3} = d(c\alpha_{\mu 3}+1), \\ \alpha_{\mu 2}\alpha_{\mu 3}^2 = c\alpha_{\mu 2}^2\alpha_{\mu 3} + \alpha_{\mu 2}^2, \quad \alpha_{\mu 3}^3 = c\alpha_{\mu 2}\alpha_{\mu 3}^2 + \alpha_{\mu 2}\alpha_{\mu 3},\end{gathered}$$

which follow from (19)–(20), the resulting a_μ, b_μ, d_μ satisfy also the equations

$$\sum_{\mu=1}^{5} a_\mu \alpha_{\mu2}^3 = 0, \quad \sum_{\mu=1}^{5} a_\mu \alpha_{\mu3}^2 = 1, \quad \sum_{\mu=1}^{5} a_\mu \alpha_{\mu2}^2 \alpha_{\mu3} = 0;$$

$$\sum_{\mu=1}^{5} a_\mu \alpha_{\mu2} \alpha_{\mu3}^2 = 0, \quad \sum_{\mu=1}^{3} a_\mu \alpha_{\mu3}^3 = 0;$$

$$\sum_{\mu=1}^{3} b_\mu \alpha_{\mu3}^2 = cb_{23} + b_{12} = b_{33},$$

hence, by (12) and (13), (1) holds with

$$\boldsymbol{\alpha}_\mu = [1, \alpha_{\mu2}, \alpha_{\mu3}], \quad f_\mu = a_\mu z^3 + b_\mu z^2 + d_\mu z \quad (1 \le \mu \le 5).$$

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