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ON THE GRAM–SCHMIDT ORTHONORMALIZATONS OF SUBSYSTEMS OF SCHAUDER SYSTEMS

BY

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Abstract. In one of the earliest monographs that involve the notion of a Schauder basis, Franklin showed that the Gram–Schmidt orthonormalization of a certain Schauder basis for the Banach space of functions continuous on $[0,1]$ is again a Schauder basis for that space. Subsequently, Ciesielski observed that the Gram–Schmidt orthonormalization of any Schauder system is a Schauder basis not only for $C[0,1]$, but also for each of the spaces $L^p[0,1]$, $1 \leq p < \infty$. Although perhaps not probable, the latter result would seem to be a plausible one, since a Schauder system is closed, in the classical sense, in each of the L^p-spaces. This closure condition is not a sufficient one, however, since a great variety of subsystems can be removed from a Schauder system without losing the closure property, but it is not always the case that the orthonormalizations of the residual systems thus obtained are Schauder bases for each of the L^p-spaces. In the present work, this situation is examined in some detail; a characterization of those subsystems whose orthonormalizations are Schauder bases for each of the spaces $L^p[0,1]$, $1 \leq p < \infty$, is given, and a class of examples is developed in order to demonstrate the sorts of difficulties that may be encountered.

1. In 1927 Schauder [9] presented a denumerable set of functions, defined and continuous on a closed, bounded interval, in terms of which every continuous function thereon defined has a unique series expansion that converges uniformly to the function. This idea, in a somewhat more restrictive form, had appeared in Faber's [4] earlier study of the Haar orthogonal system; nevertheless, the members of the systems from which the series expansions are formed are commonly termed Schauder functions.

A Schauder system is by no means an orthogonal family, but Franklin [5] showed that the Gram–Schmidt orthonormalization of a Schauder system is, again, a system of representation, or, in the current terminology, a Schauder basis for the continuous functions defined on the fundamental interval. Moreover, in his seminal studies of the Franklin systems, Ciesielski [2, 3] showed that these orthonormal systems are also Schauder bases for each of the L^p-spaces, $1 \leq p < \infty$, associated with the interval of definition, I.

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Now a Schauder system is incredibly rich, in the sense that any finite subset, and many infinite subsets, of the system can be discarded, and the residual subsystem will continue to be closed in each of the spaces $L^p(I)$.

Of course, the Gram–Schmidt orthonormalization of a subsystem obtained in this manner will not be a Schauder basis for $C(I)$, but it may be a basis for some subspace of $C(I)$ and/or for some nontrivial collection of the L^p-spaces. Indeed, in an earlier work [13], some results of this kind were obtained, and, in the present note, a somewhat more complete theory is given.

2. The Schauder systems herein considered are collections of piecewise-linear functions associated with sequences

$$\{\pi_n = \{t_{nk} : 0 \leq k \leq 2^n\}\}_{n=0}^{\infty}$$

of subdivisions of [0,1] for which the following conditions are satisfied:

$$t_{00} = 0, \quad t_{01} = 1,$$
$$t_{nk} < t_{n+1,2k+1} < t_{n,k+1} \quad \forall k = 0, \ldots, 2^n - 1, \ \forall n = 0, 1, \ldots,$$
$$t_{n+1,2k} = t_{n,k} \quad \forall k = 0, \ldots, 2^n, \ \forall n = 0, 1, \ldots,$$

and

$$\lim_n \|\pi_n\| = 0.$$

If, in addition, there is a positive constant, λ, such that

$$\frac{t_{n,j+1} - t_{nj}}{t_{n,k+1} - t_{n,k}} \leq \lambda \quad \forall j, k = 0, \ldots, 2^n, \ \forall n = 1, 2, \ldots,$$

the associated Schauder system will be termed *regular*. (In the case of the standard Faber–Schauder system, based upon the partitions of $[0, 1]$ determined by the dyadic rationals $t_{nk} = k/2^n$, one has, for example, $\lambda = 2$.)

The first two elements of the system are the constant function $\varphi_{00} = 1$ and the identity function φ_{01}. The remaining functions are defined in blocks, or ranks, of sizes 2^{n-1} $(n = 1, 2, \ldots)$ as follows: $\varphi_{n,k-1}$, the kth element of the nth block, takes the value 1 at $t_{n,2k-1}$, has for support the interval $(t_{n,2k-2}, t_{n,2k})$, and is linear on each of the intervals $(t_{n,2k-2}, t_{n,2k-1})$ and $(t_{n,2k-1}, t_{n,2k})$. Finally, the functions are arranged in lexicographical order by taking $\varphi_0 = \varphi_{00}, \varphi_1 = \varphi_{01}$, and, for $k = 1, \ldots, 2^n$ and $n = 0, 1, \ldots,$ $\phi_{2^n+k} = \varphi_{n+1,k-1}$.

If Φ is any Schauder system, then both Φ and $\mathrm{GS}\,\Phi$, its Gram–Schmidt orthonormalization, are Schauder bases for $C[0, 1]$. (See, for example, [7].) Moreover, Ciesielski [2] has shown that $\mathrm{GS}\,\Phi$ is a Schauder basis for each space $L^p[0, 1]$, $1 \leq p < \infty$.

Of course, if one deletes from Φ one or more of its members, then neither the residual system, Φ_ϱ, nor its Gram–Schmidt orthonormalization will be a

Schauder basis for $C[0,1]$, but it was shown in [13] that, for certain classes of deletions, $\mathrm{GS}\,\Phi_\varrho$ will be a basis for the L^p-spaces.

THEOREM A. *Let* $\Phi_\varrho = \{\varphi_1, \varphi_2, \ldots\}$ *be a subsystem of a Schauder system*, Φ, *and let* $\{E_n\}_{n=1}^\infty$ *be the sequence of supports of the elements of* Φ_ϱ. *If* Φ_ϱ *satisfies both*

(i) $\mu(\limsup_n E_n) = 1$, *and*

(ii) *if for all* $\varphi \in \Phi_\varrho$ *and all* $\psi \in \Phi$, $\operatorname{supp}\psi \subset \operatorname{supp}\varphi$ *implies* $\psi \in \Phi_\varrho$, *then for all* $p \in [1,\infty)$, $\mathrm{GS}\,\Phi_\varrho$ *is a Schauder basis for* $L^p[0,1]$.

THEOREM B. *If* Φ *is the Faber–Schauder system*, *and if* Φ_ϱ *is the residual system obtained from* Φ *by deleting any one of its elements*, *then* $\mathrm{GS}\,\Phi_\varrho$ *is a Schauder basis for each space* $L^p[0,1]$, $p \in [1,\infty)$.

Following the trail marked by Ciesielski, the proofs of these theorems are completed by demonstrating the boundedness of the sequence $\{S_n\}_{n=1}^\infty$ of the partial-sum operators associated with $\mathrm{GS}\,\Phi_\varrho$, since, in each case, the system is total in each of the L^p-spaces [14]. For Theorem B, this proved to be a surprisingly complex process, and the technique developed, in [13], did not appear to be easily extendable to systems obtained from Φ by deleting an arbitrary finite subset of its members. Subsequently, examples of cofinite, residual systems have been given for which an application of that technique cannot lead to the desired result. Thus, a different approach to the problem is essential. Such an approach is suggested by the method employed by Kaczmarz and Steinhaus [7] in their treatment of the Franklin system. One begins by showing that, for a cofinite system, Φ_ϱ, associated with a regular Schauder system, Φ, $\mathrm{GS}\,\Phi_\varrho$ is a Schauder basis for a certain Banach subspace of $C[0,1]$.

3. If $\Phi_\delta = \{\varphi_{i_1}, \ldots, \varphi_{i_m}\}$, and if $\Phi_\varrho = \Phi \setminus \Phi_\delta$, then every finite linear combination of elements of Φ_ϱ will be obliged to satisfy a finite number of convex, linear constraints. The latter may be described in the following manner: Let S be the subset of $[0,1]$ that contains each of the endpoints of the support of each element of Φ_δ, and for each $j = 1, \ldots, m$ with $i_j \neq 0$, let τ_j be the point at which φ_{i_j} attains its maximum value, and if $i_1 = 0$ (i.e., if $\varphi_0 \in \Phi_\delta$), let $\tau_1 = 0$. Finally, let $T = \{\tau_1, \ldots, \tau_m\}$, and let $U = S \cup T$.

With one possible exception, the aforementioned constraints relate the values that a Φ_ϱ-polynomial must assume on T to its values on S; viz., to each $\tau_j \neq 0$, there will correspond a set, $\{\alpha_{ij} : i = 1, \ldots, s\}$, such that

$$p(\tau_j) = \sum_{i=1}^{s} \alpha_{ij} p(u_i), \quad \sum_{i=1}^{s} \alpha_{ij} = 1 \quad \forall j.$$

The exceptional constraint will arise if $\varphi_0 \in \Phi_\delta$. In this case, p will be obliged to satisfy, in addition, either $p(0) = 0$ or $p(0) = p(1) = 0$, according as $\varphi_1 \notin \Phi_\delta$ or $\varphi_1 \in \Phi_\delta$. For example, if (only) $\varphi_{3,1} = \varphi_6$, $\varphi_{5,5} = \varphi_{22}$ and $\varphi_{6,10} = \varphi_{43}$ were omitted from the Faber–Schauder system, p would be obliged to satisfy three conditions:

$$p\big(\tfrac{3}{8}\big) = \tfrac{1}{2}p\big(\tfrac{1}{4}\big) + \tfrac{1}{2}p\big(\tfrac{1}{2}\big),$$
$$p\big(\tfrac{11}{32}\big) = \tfrac{1}{4}p\big(\tfrac{1}{4}\big) + \tfrac{1}{2}p\big(\tfrac{5}{16}\big) + \tfrac{1}{4}p\big(\tfrac{1}{2}\big),$$
$$p\big(\tfrac{21}{64}\big) = \tfrac{1}{8}p\big(\tfrac{1}{4}\big) + \tfrac{3}{4}p\big(\tfrac{5}{16}\big) + \tfrac{1}{8}p\big(\tfrac{1}{2}\big).$$

Let $C_\varrho[0,1]$ be the subspace of $C[0,1]$ whose elements satisfy the same constraints as do the Φ_ϱ-polynomials.

THEOREM 1. *If Φ is a regular Schauder system, and if Φ_ϱ is a cofinite subsystem of Φ, then $\Psi = \mathrm{GS}\,\Phi_\varrho$ is a Schauder basis for $C_\varrho[0,1]$.*

Proof. The following demonstration is simply an elaboration of the elegant argument devised by Kaczmarz and Steinhaus in order to show that the Franklin system is a Schauder basis for $C[0,1]$.

Let $\Phi_\varrho = \{\varphi_{k(j)} : j = 1, 2, \ldots\}$, and for each $n \in \mathbb{N}$, let $W_n = \{w_1, \ldots, w_{r(n)}\}$ be the subset of $[0,1]$ that contains each vertex of each Schauder function φ_i, $i = 0, \ldots, k(n)$. The partition, π_n^*, of $[0,1]$ determined by W_n will be either one of the π_m defined above, or a refinement of one of those partitions.

Let f be an arbitrary element of $C_\varrho[0,1]$, and let $s_n = \sum_{i=1}^n c_i\psi_i$ be the nth partial sum of the Ψ-Fourier series of f. Then s_n is a polygonal function whose vertices have abscissae that are elements of W_n.

Let the positive number ε be specified. Then there exists a $\delta = \delta(\varepsilon) > 0$ such that, for all t_1, t_2 in $[0,1]$,

$$|f(t_1) - f(t_2)| < \varepsilon \quad \text{whenever } |t_1 - t_2| < \delta.$$

For all sufficiently large n, one has $\|\pi_n^*\| < \delta$. For such an n, and for each i in $\{1, \ldots, r(n)\}$, let

$$h_i = |f(w_i) - s_n(w_i)|.$$

If w_{i_1} and w_{i_2} are adjacent elements of W_n, then [7, 123–124]

$$\int_{w_{i_2}}^{w_{i_1}} (f(t) - s_n(t))^2\,dt \ge \frac{d}{15}\{\max(h_{i_1}, h_{i_2}) - \varepsilon\}^2 - \varepsilon^2 d,$$

where d is the length of the interval of (w_{i_1}, w_{i_2}).

Let $M = \max\{h_1, \ldots, h_{r(n)}\}$, let $M = h_j$, and suppose that $M > 7\varepsilon$.

Consider now those elements w_i, not in U, lying to the left of w_j for which $h_i \le 7\varepsilon$. Designate the largest of these by w_p. Should there be no such w_i, let $w_p = 0$. Similarly, denote by w_q the smallest of those w_i, greater than w_j, not in U for which $h_i \le 7\varepsilon$. Should there be no such w_i, let $w_q = 1$.

If there is any element $t \in U$ that does not lie within the interval (w_p, w_q), let $w_{p(t)}$ (resp. $w_{q(t)}$) be the largest (resp. smallest) element $w_i \in W_n \setminus U$ left (resp. right) of w_j such that $h_i \le 7\varepsilon$. As before, it may happen that $w_{p(t)} = 0$ or $w_{q(t)} = 1$. If there remain other elements of U that do not lie within $(w_p, w_q) \cup (w_{p(t)}, w_{q(t)})$, one continues this process so as to obtain, at last, a finite collection of nonoverlapping intervals $I_1, \dots, I_l$ such that each element of U lies in the interior of some I_k, and, for each element w_k of $(W_n \setminus U) \cap \bigcup_{k=1}^{l} \operatorname{Int} I_k$, one has $h_k > 7\varepsilon$.

Let these intervals be so ordered that, for $s < t$, I_s lies to the left of I_t, and let the elements of $\operatorname{cl} I_k \cap W_n$ be $v_{k1} < v_{k2} < \ldots < v_{ki_k}$.

One defines the polygonal function σ_n as follows:

$$\begin{aligned}
\sigma_n(v_{11}) &= \begin{cases} f(0), & \text{if } v_{11} = 0, \\ s_n(v_{11}), & \text{if } v_{11} \ne 0; \end{cases} \\
\sigma_n(v_{k1}) &= s_n(v_{k1}), \qquad 1 < k \le l; \\
\sigma_n(v_{kj}) &= f(v_{kj}), \qquad 1 \le k \le l,\ 1 < j < i_k; \\
\sigma_n(v_{ki_k}) &= s_n(v_{ki_k}), \qquad 1 \le k < l; \\
\sigma_n(v_{li_l}) &= \begin{cases} s_n(v_{li_l}), & \text{if } v_{li_l} \ne 1, \\ f(1), & \text{if } v_{li_l} = 1; \end{cases}
\end{aligned}$$

$$\sigma_n(t) = s_n(t), \qquad \forall t \in [0,1] \setminus \bigcup_{k=1}^{l} \operatorname{cl} I_k.$$

Let $E = \bigcup_{k=1}^{l} \operatorname{cl} I_k$, and let

$$J(s_n) = \int_E (f(t) - s_n(t))^2 \, dt = \sum_{k=1}^{l} \int_{I_k} (f(t) - s_n(t))^2 \, dt.$$

Then

$$\begin{aligned}
J(s_n) \ge{}& \frac{d_1 + d_2}{15}(M - \varepsilon)^2 - (d_1 + d_2)\varepsilon^2 \\
&+ \frac{|E| - (d_1 + d_2) - \sum_{k=1}^{q}(d_{k1} + d_{k2})}{15}(6\varepsilon)^2 \\
&- \Big(|E| - (d_1 + d_2) - \sum_{k=1}^{q}(d_{k1} + d_{k2})\Big)\varepsilon^2,
\end{aligned}$$

where d_1 and d_2 are the lengths of the fundamental intervals that abut at w_j, d_{k1} and d_{k2} are the lengths of the corresponding intervals that abut at the kth element of U, and q is the number of distinct elements of U.

On the other hand, one has

$$J(\sigma_n) := \int_E (f(t) - \sigma_n(t))^2 \, dt \le |E|\varepsilon^2 + \sum_{k=1}^{l} (d_{k3} + d_{k4})(8\varepsilon)^2,$$

where

$$d_{k3} = |(v_{k1}, v_{k2})| \quad \text{and} \quad d_{k4} = |(v_{ki_{k-1}}, v_{ki_k})|, \qquad k = 1, \dots, l.$$

Because σ_n agrees with f on all of S, it follows that σ_n is an element of the span of $\{\psi_1, \dots, \psi_n\}$; thus,

$$\int_0^1 (f(t) - s_n(t))^2\,dt \le \int_0^1 (f(t) - \sigma_n(t))^2\,dt,$$

and since σ_n coincides with s_n on all of $[0,1] \setminus E$,

$$\int_E (f(t) - s_n(t))^2\,dt \le \int_E (f(t) - \sigma_n(t))^2\,dt.$$

It follows that

$$\frac{d_1 + d_2}{15}(M - \varepsilon)^2 \le \Big(|E| - \sum_{k=1}^{q}(d_{k1} + d_{k2})\Big)\big(2 - \tfrac{36}{15}\big)\varepsilon^2 + \frac{d_1 + d_2}{15}(36\varepsilon^2) + \Big(\sum_{k=1}^{q}(d_{k1} + d_{k2})\Big)\varepsilon^2 + \sum_{k=1}^{l}(d_{k3} + d_{k4})(64\varepsilon^2).$$

Taking account of the possibility that either d_1 or d_2 might be zero, one finds that

$$M \le (1 + \sqrt{36 + 130\lambda s})\varepsilon,$$

where λ is the regularity constant for Φ, and thus, s_n converges uniformly to f.

Since Ψ is a Schauder basis for $C_\varrho[0,1]$, it follows that the corresponding sequence of partial-sum operators,

$$\{S_n^\varrho : C_\varrho[0,1] \to C_\varrho[0,1]\}_{n=1}^\infty,$$

is bounded.

For each f in $C_\varrho[0,1]$,

$$S_n^\varrho f = \int_0^1 K_n(\cdot, t) f(t)\,dt,$$

where K_n is the nth Dirichlet kernel associated with Ψ; thus, there is a positive constant, A, such that, for all n,

$$\sup_{\substack{f \in C_\varrho[0,1], \\ \|f\|_\infty \le 1}} \sup_{x \in [0,1]} \left| \int_0^1 K_n(x,t) f(t)\,dt \right| = \sup_{x \in [0,1]} \sup_{\substack{f \in C_\varrho[0,1], \\ \|f\|_\infty \le 1}} \left| \int_0^1 K_n(x,t) f(t)\,dt \right| \le A.$$

Let x be a fixed element of $[0,1]$. Then $\operatorname{sgn} K_n(x, \cdot)$ is a step function, and there is a sequence $\{f_m\}_{m=1}^\infty$ of continuous functions such that $\|f_m\|_\infty \le 1$

for all m, and

$$\lim_m f_m(t) = \operatorname{sgn} K_n(x,t).$$

From the Lebesgue theorem of dominated convergence it follows that

$$\lim_m \int_0^1 K_n(x,t) f_m(t)\,dt = \int_0^1 |K_n(x,t)|\,dt.$$

Finally, for each m, there exists a g_m in $C_\varrho[0,1]$ such that

$$\|g_m\|_\infty \le 1 \quad \text{and} \quad \|f_m - g_m\|_1 < 2^{-m}.$$

Hence,

$$\Big|\int_0^1 K_n(x,t) g_m(t)\,dt - \int_0^1 K_n(x,t) f_m(t)\,dt\Big| \\ \le \|K_n(x,\cdot)\|_\infty \|g_m - f_m\|_1 \to 0 \qquad \text{as } m \to \infty,$$

from which follows

$$\int_0^1 |K_n(x,t)|\,dt = \lim_m \int_0^1 K_n(x,t) g_m(t)\,dt \le A,$$

and, because x may be chosen arbitrarily,

$$\Big\|\int_0^1 |K_n(\cdot,t)|\,dt\Big\|_\infty \le A.$$

According to a theorem of Orlicz [8], if the individual members of an orthonormal system are bounded, and if, for every n,

$$\Big\|\int_0^1 |K_n(\cdot,t)|\,dt\Big\|_\infty \le A,$$

where A is a constant independent of n, then

$$\|S_n f\|_p \le A\|f\|_p \qquad \forall f \in L^p[0,1],\ \forall p \in [1,\infty).$$

Theorem 2. *If one deletes any finite number of elements from a regular Schauder system Φ, then the Gram–Schmidt orthonormalization of the residual system is a Schauder basis for each of the spaces $L^p[0,1], 1 \le p < \infty$.*

Proof. The residual system, Φ_ϱ, satisfies condition (i) of Theorem A; thus, for each p in $[1,\infty)$, it follows that the finite linear combinations of the elements of $\Psi = \mathrm{GS}\,\Phi_\varrho$ form a dense subset of $L^p[0,1]$. (See, for example, [14].) As a consequence of Theorem 1, there is, for each $p \in [1,\infty)$, a constant A_p such that $\|S_n\|_p \le A_p$ for all n, where S_n is the nth partial-sum operator on $L^p[0,1]$ associated with Ψ. These two conditions yield the desideratum.

4. As one sees from Theorem A, it is possible to delete from a Schauder system infinitely many of its members in such a way that the G-S orthonormalization of the residual system will be a Schauder basis for each of the spaces $L^p[0,1]$, $1 \le p < \infty$. Given the extreme redundancy of a Schauder system, as a set of functions whose span is a dense subset of each of the L^p-spaces, it is at least conceivable that the condition (i), of Theorem A, might be all that one needs for the conclusion of that theorem.

Suppose that the subset $\Phi_\delta = \{\varphi_{n_k} : k = 1, 2, \ldots\}$ were deleted from Φ. Let

$$\Phi_\delta^{(N)} = \{\varphi_{n_k} : k = 1, \ldots, N\},$$

and let $\mathcal{A}_N$ be the Banach subspace of $C[0,1]$ whose elements are obliged to satisfy the convex linear constraints associated with $\Phi_\delta^{(N)}$. Then

$$\mathcal{A}_1 \supset \mathcal{A}_2 \supset \ldots \quad \text{and} \quad C_\varrho[0,1] = \bigcap_{N=1}^{\infty} \mathcal{A}_N,$$

where $C_\varrho[0,1]$ is the (Banach) subspace of $C[0,1]$ whose elements satisfy all of the constraints that arise by virtue of the removal of Φ_δ from Φ.

In the case of a regular system, Theorem 1 shows that $\Psi_N = \mathrm{GS}(\Phi \backslash \Phi_\delta^{(N)})$ is a Schauder basis for $\mathcal{A}_N$; thus, there is a positive constant A_N such that

$$\|S_n^{(N)}\|_\infty \le A_N \quad \forall n \in \mathbb{N},\ \forall N \in \mathbb{N},$$

where $S_n^{(N)}$ is the nth partial-sum operator on $\mathcal{A}_N$ associated with Ψ_N.

If, in turn, the sequence $\{A_N\}_{N=1}^\infty$ were bounded, then $\Psi = \mathrm{GS}\,\Phi_\varrho$ would be a Schauder basis for $C_\varrho[0,1]$, and by virtue of an argument similar to the one employed in the proof of Theorem 2, Ψ would be a Schauder basis for each space $L^p[0,1]$, $1 \le p < \infty$.

There is contained in the proof of Theorem 1, however, a hint that this approach to the problem will not be fruitful. There one finds, in the estimate for $\|s_n - f\|_\infty$, a parameter related to the order of Φ_δ, which suggests that $\{A_N\}_{N=1}^\infty$ might be an increasing, unbounded sequence. In any event, if Ψ is to be a Schauder basis for $C_\varrho[0,1]$, $\{A_n\}_{n=1}^\infty$ must be bounded. Indeed, an even more restrictive condition must be satisfied.

THEOREM 3. *Let Φ_ϱ satisfy the condition* (i), *of Theorem A*; *let $\Psi = \mathrm{GS}\,\Phi_\varrho$*; *and let S_n be the nth partial-sum operator associated with Ψ. The following propositions are equivalent*:

(1) *Ψ is a Schauder basis for $C_\varrho[0,1]$*;

(2) *Ψ is a Schauder basis for each space $L^p[0,1]$, $1 \le p < \infty$, and there is a positive constant, A, such that $\|S_n\|_p \le A$ for all $n \in \mathbb{N}$ and $1 \le p < \infty$*;

(3) *$\{S_n : n \in \mathbb{N}\}$ is a bounded set of operators on $C[0,1]$.*

Proof. Since $\varPhi_\varrho$ satisfies the condition (i), it follows, essentially from the work of Goffman [6] (see also [14]), that span $\varPhi_\varrho$ is dense in each space $L^p[0,1], 1 \le p < \infty$. As in the proof of Theorem 1, $\varPsi$ being a basis for $C_\varrho[0,1]$ implies the uniform boundedness of the sequences $\{\|S_n\|_p\}_{n=1}^\infty, 1 \le p < \infty$, and the remainder of the proof of (1)⇒(2) follows along the same line as the argument given in the demonstration of Theorem 2. (Indeed, the same argument, *mutatis mutandis*, leads to the conclusion that $\varPsi$ is a Schauder basis for each separable Orlicz space on $[0,1]$; see [15].)

If f is any continuous function with $\|f\|_\infty \le 1$, then, for each $n \in \mathbb{N}$ and for each $p \in [1,\infty)$,

$$\|S_n f\|_p \le \|S_n\|_p \|f\|_p \le A\|f\|_\infty.$$

Since $\|S_n f\|_\infty = \lim_{p\to\infty} \|S_n f\|_p$, it follows that

$$\|S_n\|_\infty \le A \qquad \forall f \in C[0,1],$$

which suffices to complete the demonstration of (2)⇒(3).

Finally, $\|S_n f\|_\infty \le A\|f\|_\infty$ for all $f \in C[0,1]$, implies the corresponding result for the members of $C_\varrho[0,1]$. Since Span $\varPhi_\varrho$ is dense in $C_\varrho[0,1]$, the cycle is complete.

From these equivalences follows the insufficiency of the condition (i) for the conclusion of Theorem A.

Example 4. Let $\varPhi_\varrho$ be the subsystem of the Faber–Schauder system constructed in the following manner. Let $m_1 = 3$, $n_1 = 2m_1+2$, $n_1^* = 2n_1+6$,

$$E_1 = (2\cdot 2^{-m_1} + 2^{-n_1}, 3\cdot 2^{-m_1} - 2^{-n_1^*}) \cup (3\cdot 2^{-m_1}, 1),$$

and

$$\varPhi_1 = \{\varphi_{nj} : E_{nj} \subset E_1\} \cup \{\varphi_{m_1,1}\},$$

where E_{nj} denotes the support of φ_{nj}.

If G_1 is the lim sup of the sequence of supports of the elements of $\varPhi_1$, arranged in their usual, lexicographical, order, then $\mu(G_1) = \mu(E_1)$.

Let $\varphi^{(m_1)} = \varphi_{n_1+1,2^{n_1-m_1+1}}$, and let

$$\begin{aligned} F(a) &= \int_{2^{-m_1+1}}^{2^{-m_1+1}+2^{-n_1}} [\varphi^{(m_1)} - a(t-2^{-m_1+1})]^2\,dt \\ &= \frac{2^{-n_1}}{3}[1 - 3\cdot 2^{-n_1-1}a + 2^{-2n_1}a^2]. \end{aligned}$$

Then

$$\min F(\cdot) = \tfrac{7}{48}\cdot 2^{-n_1};$$

this minimum value is attained when $a = a_{\min} = \frac{3}{4}\cdot 2^{n_1}$, and

$$F\big(\tfrac12 a_{\min}\big) = F(a_{\min}) + \tfrac32 \cdot 2^{-(n_1+5)}.$$

Let the function f be defined on $[0,1]$ by the conditions

$$f(t) = \begin{cases} 2^{-m_1} a_{\min} \varphi_{m_1,1}(t), & \text{if } t \in E_1 \cup [2^{-m_1+1}, 2^{-m_1+1} + 2^{-n_1}]; \\ 0, & \text{otherwise.} \end{cases}$$

(On the interval $[2^{-m_1+1}, 2^{-m_1+1} + 2^{-n_1}]$, one has $f(t) = a_{\min}(t - 2^{-m_1+1})$.)

By virtue of the work of Goffman *et al.*, to which reference has been made above, there exists a function $\ell \in \operatorname{Span}(\Phi_1 \setminus \{\varphi_{m_1,1}\})$ such that

$$\int_{E_1} [f(t) - \ell(t)]^2 \, dt < \tfrac{3}{2} \cdot 2^{-(n_1+7)}.$$

Choose $n_1^{**} > n_1^*$ so large that the elements of Φ_1 used to compose ℓ have ranks that do not exceed n_1^{**}, and let

$$\Phi_1^* = \{\varphi_{nk} \in \Phi_1 : n \le n_1^{**}\}, \qquad N_1 = |\Phi_1^*|.$$

The initial segment of Φ_ϱ will be the subfamily Φ_1^*, with its elements arranged in accord with the original ordering of Φ.

By virtue of the triangular nature of the Gram–Schmidt process, the N_1th partial sum of the $\mathrm{GS}\,\Phi_\varrho$ Fourier series for $\varphi^{(m_1)}$ can be written in the form

$$S_{N_1} \varphi^{(m_1)} = c\varphi_{m_1,1} + \sum_{(n,k) \in I_1} c_{nk} \varphi_{nk},$$

where $I_1 = \{(n,k) : \varphi_{nk} \in \Phi_1^* \setminus \{\varphi_{m_1,1}\}\}$.

One must have $c \ge 2^{-m_1-1} a_{\min}$, since, for $c < 2^{-m_1-1} a_{\min}$,

$$\begin{aligned} &\int_0^1 [\varphi^{(m_1)}(t) - S_{N_1}\varphi^{(m_1)}(t)]^2 \, dt \\ &\qquad = \int_{2^{-m_1+1}}^{2^{-m_1+1}+2^{-n_1}} [\varphi^{(m_1)}(t) - c\varphi_{m_1,1}(t)]^2 \, dt + \int_{E_1} [S_{N_1}\varphi^{(m_1)}(t)]^2 \, dt \\ &\qquad \ge \tfrac{7}{48} \cdot 2^{-n_1} + \tfrac{3}{2} \cdot 2^{-(n_1+5)}, \end{aligned}$$

while

$$\begin{aligned} &\int_0^1 [\varphi^{(m_1)}(t) - (2^{-m_1} a_{\min} \varphi_{m_1,1}(t) - \ell(t))]^2 \, dt \\ &\quad = \int_{2^{-m_1+1}}^{2^{-m_1+1}+2^{-n_1}} [\varphi^{(m_1)}(t) - 2^{-m_1} a_{\min} \varphi_{m_1,1}(t)]^2 \, dt \\ &\qquad + \int_{E_1} [f(t) - \ell(t)]^2 \, dt + \int_{[3\cdot 2^{-m_1} - 2^{-n^*}, 3\cdot 2^{-m_1}]} [2^{-m_1} a_{\min} \varphi_{m_1,1}(t)]^2 \, dt \end{aligned}$$

$$\begin{aligned}&< \tfrac{7}{48}\cdot 2^{-n_1} + \tfrac{3}{2}\cdot 2^{-(n_1+7)} + \tfrac{9}{16}\cdot 2^{2(n_1-m_1)-n_1^*}\\ &= \tfrac{7}{48}\cdot 2^{-n_1} + \tfrac{3}{8}\cdot 2^{-(n_1+5)} + \tfrac{9}{8}\cdot 2^{-(n_1+5)}\\ &= \tfrac{7}{48}\cdot 2^{-n_1} + \tfrac{3}{2}\cdot 2^{-(n_1+5)}.\end{aligned}$$

It follows that

$$\|S_{N_1}\varphi^{(m_1)}\|_\infty \ge \|c\varphi_{m_1,1}\|_\infty \ge 2^{-m_1-1}a_{\min} > 2^{m_1}.$$

Moreover, for any $q > 2$,

$$\begin{aligned}\Big(\int_0^1 |S_{N_1}\varphi^{(m_1)}(t)|^q\,dt\Big)^{1/q} &\ge \Big(\int_{3\cdot 2^{-m_1}-2^{-n_1^*}}^{3\cdot 2^{-m_1}} (c\varphi_{m_1,1}(t))^q\,dt\Big)^{1/q}\\ &\ge \frac{c}{2}\cdot 2^{-n_1^*/q} \ge \tfrac{3}{16}\cdot 2^{n_1-m_1-n_1^*/q};\end{aligned}$$

thus,

$$\begin{aligned}\frac{\|S_{N_1}\varphi^{(m_1)}\|_q}{\|\varphi^{(m_1)}\|_q} &\ge \big[\tfrac{3}{16}\cdot 2^{n_1-m_1-n_1^*/q}\big]\big[(q+1)^{1/q}2^{n_1/q}\big]\\ &= \tfrac{3}{4}\cdot 2^{-8/q}(q+1)^{1/q}2^{m_1(1-2/q)}.\end{aligned}$$

Hence, regarding S_{N_1} as an operator on $L^q[0,1]$, one concludes that

$$\|S_{N_1}\|_q \ge C_q 2^{m_1(1-2/q)},$$

where C_q is a constant determined by q only.

Proceeding inductively, let $m_2 = n_1^{**}+1$, $n_2 = 2m_2+2$, $n_2^* = 2n_2+6$,

$$E_2 = (2\cdot 2^{-m_2} + 2^{-n_2}, 3\cdot 2^{-m_2} - 2^{-n_2^*}) \cup (3\cdot 2^{-m_2}, 1),$$

and

$$\varPhi_2 = \{\varphi_{nk} : n \ge m_2,\ E_{nk} \subset E_2\} \cup \{\varphi_{m_2,1}\}.$$

By essentially duplicating the argument presented above, one finds an $n_2^{**} \ge n_2^*$ and a corresponding subfamily

$$\varPhi_2^* = \{\varphi_{nk} \in \varPhi_2 : n \le n_2^{**}\}$$

such that, for $N_2 = |\varPhi_1^* \cup \varPhi_2^*|$,

$$\|S_{N_2}\varphi_{n_2+1,2^{n_2-m_2+1}}\|_\infty > 2^{m_2},$$

and, for each $q > 2$,

$$\|S_{N_2}\varphi_{n_2+1,2^{n_2-m_2+1}}\|_q \ge \tfrac{3}{16}\cdot 2^{n_2-m_2-n_2^*/q},$$

so that both $\|S_{N_2}\|_\infty > 2^{m_2}$ and $\|S_{N_2}\|_q \ge C_q 2^{m_2(1-2/q)}$.

In the end, one will have a subfamily of $\varPhi$ that satisfies the condition (i),

$$\varPhi_\varrho = \bigcup_{n=1}^\infty \varPhi_n^*,$$

and an increasing sequence of natural numbers, $\{N_k\}_{k=1}^{\infty}$, such that each of the sequences $\{\|S_{N_k}\|\}_{k=1}^{\infty}$ and $\{\|S_{N_k}\|_q\}_{k=1}^{\infty}$, $q > 2$, is unbounded.

Consequently, the Gram–Schmidt orthonormalization of this Φ_ϱ can be a basis neither for $C_\varrho[0,1]$ nor for any of the L^p-spaces other than $L^2[0,1]$.

5. The following remarks are, perhaps, worthy of mention.

The proof of Theorem 1 depends upon the regularity of the underlying partitions, and, although it is not explicitly mentioned in their work, it appears that Kaczmarz and Steinhaus require a similar regularity hypothesis to ensure the validity of their proof that a Franklin system is a Schauder basis for $C[0,1]$. On the other hand, Ciesielski's proof of the latter result is independent of any such condition, and this suggests the possible existence of a stronger version of Theorem 1. The search for such a proposition thus far has been unsuccessful.

Veselov [12] has observed that if a system $\Phi = \{\varphi_n : n = 1, 2, \ldots\}$ whose elements are continuous functions is a Schauder basis for $C[0,1]$, then one of the following four situations must obtain:

(1) $\mathrm{GS}\,\Phi$ is a Schauder basis for $C[0,1]$ and for each space $L^p[0,1]$, $1 \le p < \infty$.

(2) $\mathrm{GS}\,\Phi$ is a Schauder basis for each space $L^p[0,1]$, $1 < p < \infty$, but neither for $C[0,1]$ nor for $L^1[0,1]$.

(3) There is an $\alpha \ge 2$ such that $\mathrm{GS}\,\Phi$ is a basis for, precisely, those spaces $L^p[0,1]$ with $p \in [\alpha/(\alpha-1), \alpha]$.

(4) There is an $\alpha > 2$ such that $\mathrm{GS}\,\Phi$ is a Schauder basis for, precisely, those spaces $L^p[0,1]$ with $p \in (\alpha/(\alpha-1), \alpha)$.

For sufficiently thick subsystems of a Schauder system, there is a corresponding proposition, nearly identical to the above.

THEOREM 5. *Let Φ be a Schauder system, let Φ_ϱ be a subsystem of Φ for which the condition* (i) *is satisfied, and let $C_\varrho[0,1]$ be the subspace of $C[0,1]$ generated by Φ_ϱ. Exactly one of the following propositions is valid*:

(1*) $\mathrm{GS}\,\Phi_\varrho$ *is a Schauder basis for $C_\varrho[0,1]$ and for each space $L^p[0,1]$, $1 \le p < \infty$.*

(2*) $\mathrm{GS}\,\Phi_\varrho$ *is a Schauder basis for each space $L^p[0,1]$, $1 < p < \infty$.*

(3*) *There is an $\alpha \ge 2$ such that $\mathrm{GS}\,\Phi_\varrho$ is a Schauder basis for*, *precisely*, *those spaces $L^p[0,1]$ with $p \in [\alpha/(\alpha-1), \alpha]$.*

(4*) *There is an $\alpha > 2$ such that $\mathrm{GS}\,\Phi_\varrho$ is a Schauder basis for*, *precisely*, *those spaces $L^p[0,1]$ with $p \in (\alpha/(\alpha-1), \alpha)$.*

Proof. If $\Psi = \mathrm{GS}\,\Phi_\varrho$ is a Schauder basis for $L^1[0,1]$, then Ψ is a basis for each space $L^p[0,1]$ with $1 \le p \le 2$ by virtue of the Reisz–Thorin interpolation theorem [16]. Because (Ψ, Ψ) is a biorthogonal system, Ψ is a basis for each space $L^q[0,1]$ with $2 \le q < \infty$, since these spaces are the conjugates of the L^p-spaces with $p \in (1,2]$ (see [1]). Moreover, if S_n is the nth partial-sum operator associated with Ψ, then the set $\{\|S_n\|_p : 1 \le p < \infty,\ n \in \mathbb{N}\}$ is bounded, by $\max\{A_1, A_2\}$ for example, where $A_p = \sup_n \|S_n\|_p$, $p = 1, 2$. By virtue of Theorem 3, Ψ is a basis for $C_\varrho[0,1]$ as well.

In the contrary case, let

$$B = \{r : \Psi \text{ is a Schauder basis for } L^r[0,1]\}, \quad \text{and} \quad \beta = \inf B.$$

If $\beta \in B$, then $\beta > 1$, and (3^*) obtains, with $\alpha = \beta/(\beta - 1)$. If $\beta \notin B$, then either (2^*) or (4^*) obtains according as $\beta = 1$ or $\beta > 1$.

Veselov also provides examples of systems of each of the types (1)–(4), and it is reasonable to expect that examples of Schauder subsystems of types (1^*)–(4^*) abound.

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