COLLOQUIUM MATHEMATICUM

VOL. 92 2002 NO. 1

THE ALGEBRA OF THE SUBSPACE SEMIGROUP OF $M_2(\mathbb{F}_q)$

BY

JAN OKNIŃSKI (Warszawa)

Abstract. The semigroup $S = S(M_2(\mathbb{F}_q))$ of subspaces of the algebra $M_2(\mathbb{F}_q)$ of 2×2 matrices over a finite field $\mathbb{F}_q$ is studied. The ideal structure of S, the regular $\mathcal{J}$-classes of S and the structure of the complex semigroup algebra $\mathbb{C}[S]$ are described.

1. Introduction. Let $M_n(K)$ be the algebra of $n \times n$ matrices over a field K. By $S(M_n(K))$ we denote the subspace semigroup of $M_n(K)$, defined as the set of all K-subspaces equipped with the operation $V * W = \mathrm{lin}_K(VW)$. This semigroup arose in the context of discrete dynamical systems, [3], and was first studied in [6]. It was shown that there exists a finite ideal chain $I_1 \subset \ldots \subset I_t = S(M_n(K))$ such that I_1 and every Rees factor I_k/I_{k-1} are either nil or 0-disjoint unions of completely 0-simple ideals.

In this paper we consider the case where K is a finite field. A natural problem is to determine the complex irreducible representations of $S(M_n(K))$ and to study the structure and symmetries of the algebra $\mathbb{C}[S(M_n(K))]$. It is well known that a description of the regular $\mathcal{J}$-classes of the semigroup is needed in this context. Our aim is to deal with these problems in the case where $n = 2$. A characterization of non-regular elements of $S(M_2(K))$ is obtained and regular $\mathcal{J}$-classes are fully described. Moreover, the ideal structures of $S(M_2(K))$ and of its complex semigroup algebra are determined.

We refer to [2] for basic semigroup theory, to [5] for background on semigroups of matrices, while [4] is our reference for semigroup algebras.

2. Regular $\mathcal{J}$-classes. Let $S = S(M_2(K))$ for a finite field K. Since the idempotents of S play a crucial role, first we list unitary subalgebras of $M_2(K)$:

2000 *Mathematics Subject Classification*: Primary 20M25, 16S36; Secondary 20M20, 16P10.

Supported by KBN research grant 2P03A 030 18.

1. $A_1 = M_2(K)$.
2. $A_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \, a, b, c \in K \right\}$.

For simplicity, we write $A_2 = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$, if unambiguous.

3. $A_3 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.
4. $A_4 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.
5. $A_5 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.
6. $A_6 =$ a field extension F of dimension 2 over K.

(By the Noether–Skolem theorem any two such subfields are conjugate, as they are isomorphic).

7. $A_7 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.

Let $J(A)$ be the radical of an algebra $A \subseteq M_2(K)$. Recall that by Wedderburn's structure theorem for algebras over a perfect field [1] we know that $A = B + J(A)$ where B is a subalgebra such that $B \cong A/J(A)$. Also, every nil subalgebra of $M_2(K)$ is conjugate to $\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$. So it is easy to see that, up to conjugation, the above list exhausts all unitary subalgebras of $M_2(K)$.

From [6] we know that every non-zero regular $\mathcal{J}$-class J of $S = S(M_2(K))$ contains a unitary algebra, whence it contains one of the algebras A_i. Recall that this $\mathcal{J}$-class consists of subspaces $V \subseteq M_2(K)$ such that V and A_i generate the same ideal of $S(M_2(K))$. Clearly, A_5 is the identity of $S(M_2(K))$. Any two of the elements $A_1, A_2, A_3, A_4, A_5, A_6$ are in different $\mathcal{J}$-classes of S. This can be checked directly but it also follows from the fact that $A \mathcal{J} B$ implies that A, B are Morita equivalent [6]. Clearly A_1 and A_7 are in the same $\mathcal{J}$-class of S.

For any $n \geq 2$, let A be a subalgebra of $M_n(K)$ which is basic. That is, A has a unity and $A/J(A)$ has no non-zero nilpotents. Let $U = U(A)$ be the unit group of A and $N = N(A)$ be the normalizer of A in $\mathrm{Gl}_n(K)$. So $N = \{g \in \mathrm{Gl}_n(K) \mid gA = Ag\}$. Notice that $\mathrm{lin}_K U(A) = A$ if $K \neq \mathbb{F}_2$, the field of two elements (it is enough to assume that $A/J(A)$ has at most one copy of $\mathbb{F}_2$ as a direct summand). Therefore, in this case $N = \{g \in \mathrm{Gl}_n(K) \mid gU = Ug\}$. By H_A we denote the maximal subgroup of S containing A, treated as an idempotent of S. In other words, H_A consists of all subspaces V of $M_n(K)$ such that $V = AV = VA$ and $VW = WV = A$ for some subspace W. Let e be the identity of A. Then $eN = Ne$ is a subgroup of

$U(eM_n(K)e) \cong M_{\mathrm{rank}(e)}(K)$, which we denote by N_e. It is easy to see that

$$H_A = \{Ax \mid x \in N\} \quad \text{and} \quad H_A \cong N_e/U.$$

In particular, if A contains the identity matrix, then $U \subseteq N = N_e$ and $H_A = [N : U]$. Moreover

$$\begin{aligned}[\mathrm{Gl}_n(K) : N] &= \text{the number of } \mathcal{H}\text{-classes of } S \text{ of the form } gH_A,\ g \in \mathrm{Gl}_n(K)\\ &= \text{the number of } \mathcal{H}\text{-classes of } S \text{ of the form } H_A g,\ g \in \mathrm{Gl}_n(K).\end{aligned}$$

We shall consider the case where $K = \mathbb{F}_q$, a finite field of q elements. We count the subspaces of $M_2(\mathbb{F}_q)$ of any given dimension:

Dimension	Number of subspaces
0	1
1	$(q^4 - 1)/(q - 1) = q^3 + q^2 + q + 1$
2	$(1 + q + q^2)(1 + q^2)$
3	$q^3 + q^2 + q + 1$
4	1

It follows that $|S| = q^4 + 3q^3 + 4q^2 + 3q + 5$.

Write $G = \mathrm{Gl}_2(K)$. We have seen above that $\{gAh \mid g, h \in G\}$ yields $[N : U][G : N]^2$ elements in the $\mathcal{J}$-class of A in the subspace semigroup $S = S(M_2(K))$. We discuss the seven cases listed above.

1) $A = M_2(K)$. Then $A\mathcal{J}B$ for $B = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Every non-zero subspace $V \subseteq \left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$ is a left B-module and satisfies $VW = B$ for some right B-module W. So the $\mathcal{R}$-class of B consists of all such subspaces V, whence it has $q + 2$ elements. As the same holds for the $\mathcal{L}$-class of B, it follows that the $\mathcal{J}$-class of B has $\geq (q + 2)^2$ elements.

2) $A = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$. It is easy to see that $N = U$ and $[G : N] = q + 1$. So the $\mathcal{J}$-class of A has $\geq (q + 1)^2$ elements.

3) $A = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$. Then N consists of invertible matrices of the form $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right)$. Hence $[N : U] = 2$ and $[G : N] = (q+1)q/2$. Therefore the $\mathcal{J}$-class of A has $\geq q^2(q + 1)^2/2$ elements.

4) $A = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$. Then N consists of invertible matrices of the form $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$. So $|U| = (q - 1)q$ and $|N| = (q - 1)^2 q$. Hence $[N : U] = q - 1$ and $[G : N] = q + 1$ and therefore the $\mathcal{J}$-class of A has $\geq (q + 1)^2(q - 1)$ elements.

5) $A = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. Then $N = G$ and $U \cong K^*$. So $[N : U] = q(q^2 - 1)$ and $[G : N] = 1$. It follows that the $\mathcal{J}$-class of A has $\geq q(q^2 - 1)$ elements.

6) A is a subfield of dimension 2 over K. Now $|A| = q^2$, so that $|U| = q^2 - 1$. Let C be the centralizer of A in $M_2(K)$. Then C is a simple algebra, so it is a maximal subfield of $M_2(K)$ containing A, [1]. Hence $C = A$. The

Galois group $G(A/K)$ is $\{\mathrm{Id}, \phi\}$, where $\phi(x) = x^q$. So $n \in N$ if and only if $nan^{-1} = a$ or $nan^{-1} = a^q$ for $a \in A$ (as there are no other automorphisms). Hence $n \in C = A$ or $nan^{-1} = a^q$. By the Noether–Skolem theorem there exists an element $n \in N$ of the latter type. Then any other $y \in N$ satisfies either $y^{-1}n \in C$ or $y \in C$. So $N \subseteq C \cup Cn$ and consequently $N = U \cup Un$. Hence $[N : U] = 2$. But $[G : U] = (q^2 - q)(q^2 - 1)/(q^2 - 1) = q^2 - q$. So $[G : N] = (q^2 - q)/2$. Therefore we get $2((q^2-q)/2)^2 = q^2(q-1)^2/2$ elements in the $\mathcal{J}$-class of A.

We now add the numbers of subspaces produced in cases 1)–6) (note that they are in different $\mathcal{J}$-classes of S):

1) $(q+1)^2$ spaces of dimension 1,
   $2q+2$ spaces of dimension 2,
   1 space of dimension 4,
2) $(q+1)^2$ spaces of dimension 3,
3) $q^2(q+1)^2/2$ spaces of dimension 2,
4) $(q+1)^2(q-1)$ spaces of dimension 2,
5) $q(q^2-1)$ spaces of dimension 1,
6) $q^2(q-1)^2/2$ spaces of dimension 2.

So we have constructed

$$q^4 + q^3 + 2q^2 + q + 1 = (1 + q + q^2)(1 + q^2)$$

subspaces of dimension 2, whence these are all such subspaces. Also, we have got $q^3 + q^2 + q + 1$, hence all, subspaces of dimension 1. Moreover, there are

$$|S| - 1 - |\{\text{elements listed in 1)–6)}\}| = q^3 - q$$

remaining non-zero elements of S (all of them of dimension 3). We will show that they are all not regular. So, it will follow that the elements listed in 1)–6) cover all non-zero regular $\mathcal{J}$-classes of S, and hence they exhaust all non-zero regular elements of S. It also follows that the regular $\mathcal{J}$-classes of S consist of unit regular elements of S.

PROPOSITION 2.1. *Assume that K is any field and let $n \geq 2$. Let $V \in S = S(M_n(K))$ be a subspace of dimension $n^2 - 1$. Let V be described by a linear equation $\sum_{i,j=1}^n a_{ij}x_{ij} = 0$, $a_{ij} \in K$. If $VwV \subseteq V$ for some non-zero $w \in M_n(K)$, then the rank of the matrix $A = (a_{ij})$ is 1. Moreover, the latter is equivalent to the fact that V is a regular element of S.*

Proof. Assume that $h \in M_n(K)$ is an elementary matrix. So it is a transposition or $h = 1 + \lambda e_{pq}$ for some $p \neq q$ and $\lambda \in K^*$, where e_{pq} denotes a matrix unit. Let $B = (b_{ij})$ be the matrix determined by an equation describing the subspace hV. If h is a transposition with non-diagonal entries h_{pq}, h_{qp}, then clearly we may take $b_{qj} = a_{pj}$, $b_{pj} = a_{qj}$ and $b_{ij} = a_{ij}$ if $i \neq p, q$, for $j = 1, \ldots, n$. If $h = 1 + \lambda e_{pq}$, then it is easy to see that we may take $b_{qj} = a_{qj} - \lambda a_{pj}$ and $b_{ij} = a_{ij}$ for $i \neq q$, and for all j. It follows that $\operatorname{rank}(A) = \operatorname{rank}(B)$. Hence every hV is described by an equation with the corresponding matrix having the same rank as A. The same holds if $h = 1 + \lambda e_{pp}$ with any $p \in \{1, \ldots, n\}$ and $\lambda \neq -1$, and therefore for every $h \in \mathrm{Gl}_n(K)$. The same applies to Vh.

Suppose that $VwV \subseteq V$ for some $w \in M_n(K)$. If $g, h \in \mathrm{Gl}_n(K)$, then

$$g^{-1}Vhh^{-1}wgg^{-1}Vh \subseteq g^{-1}Vh.$$

Clearly, V is a regular element of S if and only if so is $g^{-1}Vh$. It follows that, when proving both statements, we may replace V by any $g^{-1}Vh$. Hence we may assume that the matrix A is of the form $A = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for an identity matrix I of size $r \leq n$.

First assume that $\operatorname{rank}(A) = 1$. So, let $a_{11} = 1$ be the only non-zero entry. This means that $V = \left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right)$ where b, c, d are $1 \times (n-1), (n-1) \times 1$, $(n-1) \times (n-1)$ matrices, respectively. It is easy to see that $W = \left(\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right)$ satisfies $VWV = V$. Therefore V is a regular element of S.

It is clear that if V is a regular element of S then there exists a non-zero $w \in M_n(K)$ such that $VwV \subseteq V$.

Finally, suppose that $VwV \subseteq V$ for a non-zero matrix w. Because of the diagonal idempotent form of A we have

$$V = \{x = (x_{ij}) \in M_n(K) \mid x_{11} + \ldots + x_{rr} = 0\}.$$

If $r = 1$ then $\operatorname{rank}(A) = 1$ and we are done. So suppose that $r \geq 2$. Let $w = (w_{ij})$ and suppose that $w_{kt} \neq 0$ for some k, t. If $k, t \neq 1$ then let $v = (v_{ij}), v' = (v'_{ij})$ be such that $v_{1k} = 1 = v'_{t1}$ and all the remaining entries are 0. Then $v, v' \in V$, so that $vwv' \in V$. But vwv' has only one non-zero entry and it is in position $(1,1)$. This contradicts the above description of V. It follows that $w_{ij} = 0$ if $i, j \neq 1$. The same argument applied to position $(2,2)$ implies that also $w_{ij} = 0$ if $i, j \neq 2$. So w_{12}, w_{21} can be the only non-zero entries of w. Choose a matrix $u = (u_{ij})$ whose only non-zero entry is u_{21} and let $u' = (u'_{ij})$ be such that $u_{11} = -1$ and $u_{22} = 1$ and all other entries are zero. Then $u, u' \in V$ and $uwu' \in V$. The second row of uwu' is equal to $(0, w_{12}, 0, \ldots, 0)$ and all other rows are zero. So the description of V yields $w_{12} = 0$. A similar argument applied to the product $u^t wu'$ (where u^t is the transpose of u) yields $w_{21} = 0$. Therefore $w = 0$. This contradiction shows that $r = 1$, completing the proof of the proposition. ∎

We come back to the case $K = \mathbb{F}_q$ and $n = 2$. Notice that there are

$$|\mathrm{Gl}_2(K)|/(q-1) = (q^2 - q)(q^2 - 1)/(q-1) = q^3 - q$$

subspaces of dimension 3 defined by an equation $\alpha x_{11} + \beta x_{12} + \gamma x_{21} + \delta x_{22} = 0$ such that $\det \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \neq 0$.

Let $V=\{\left(\begin{smallmatrix} a & b\\ b & c\end{smallmatrix}\right) \mid a,b,c\in K\}$. So, V is defined by the equation $x_{12}-x_{21}=0$, and is of the desired type. We determine the stabilizer C of V under the action of $\mathrm{Gl}_2(K)$ on S by left multiplication. So, let $g=(g_{ij})\in \mathrm{Gl}_2(K)$ satisfy

$$\begin{pmatrix} g_1 & g_2\\ g_3 & g_4\end{pmatrix}\begin{pmatrix} a & b\\ b & c\end{pmatrix}=\begin{pmatrix} g_1a+g_2b & g_1b+g_2c\\ g_3a+g_4b & g_3b+g_4c\end{pmatrix}\in V$$

for all $a,b,c\in K$. Then $g_3a+g_4b=g_1b+g_2c$, whence $g_3=0=g_2$ and $g_4=g_1$. So C consists of scalar matrices and

$$|\{gV \mid g\in \mathrm{Gl}_2(K)\}|=|\mathrm{Gl}_2(K)|\cdot |K^*|^{-1}=q^3-q.$$

Clearly, every element of the form gV, $g\in \mathrm{Gl}_2(K)$, satisfies $gV\mathcal{L}V$ in S, whence it is not regular by Proposition 2.1. It then follows that we have constructed q^3-q non-regular elements of the form gV. Therefore, comparing the cardinality of S and the number of regular elements constructed before, we see that the elements listed in cases 1)–6) exhaust all non-zero regular $\mathcal{J}$-classes of S and the elements gV, $g\in \mathrm{Gl}_2(K)$, exhaust all non-regular elements of S.

COROLLARY 2.2. *Let* $V=\{x=(x_{ij})\in M_2(K) \mid x_{12}=x_{21}\}$. *Then the* $\mathcal{J}$*-class of* V *in* S *is equal to* $\{gV \mid g\in \mathrm{Gl}_2(K)\}=\{Vg \mid g\in \mathrm{Gl}_2(K)\}$ *and it coincides with the* $\mathcal{H}$*-class of* V. *Moreover* S *has exactly eight* $\mathcal{J}$*-classes, namely the classes of* $A_1,\dots,A_6,V,\{0\}$.

Proof. We have seen that $\{gV \mid g\in \mathrm{Gl}_2(K)\}$ exhaust all non-regular elements in S. A symmetric argument shows that $\{Vg \mid g\in \mathrm{Gl}_2(K)\}$ also is the set of all non-regular elements of S and hence $\{gV \mid g\in \mathrm{Gl}_2(K)\}=\{Vg \mid g\in \mathrm{Gl}_2(K)\}$. Therefore non-regular elements of S form a single $\mathcal{H}$-class of S and the assertion follows. ■

3. Structure of the algebra. In this section we describe the radical of $\mathbb{C}[S]$ and we show that, for every regular principal factor T of S, the contracted semigroup algebra $\mathbb{C}_0[T]$ is semisimple. Hence $\mathbb{C}[S]/J(\mathbb{C}[S])$ is a direct product of all $\mathbb{C}_0[T]$ (see [4]). As $\mathbb{C}[S]=B+J(\mathbb{C}[S])$, a direct sum of subspaces, for a subalgebra $B\cong \mathbb{C}[S]/J(\mathbb{C}[S])$, this yields a description of the structure of the algebra.

LEMMA 3.1. *Let* $A\subseteq M_n(K)$ *be a subalgebra with* $1\in A$. *Then* $\mathcal{A}=\{gAh \mid g,h\in G\}$ *with zero adjoined is a completely* 0*-simple inverse subsemigroup of the principal factor* J_A *of* A *in* $S(M_n(K))$. *Moreover*, $\mathcal{A}$ *is a union of* $\mathcal{H}$*-classes of* J_A.

Proof. We know that $H_A=\{Ax \mid x\in N\}$, where N is the normalizer of A in $\mathrm{Gl}_n(K)$. It follows that $\mathcal{A}$ is a union of $\mathcal{H}$-classes of J_A. Moreover every non-empty intersection R of $\mathcal{A}$ with an $\mathcal{R}$-class of S contains an idempotent. Namely, if $uAv\in R$ for some $u,v\in G$, then $uAu^{-1}\in R$.

Suppose that $B \in \mathcal{A}$ is an idempotent from the $\mathcal{R}$-class of A in S. Since $B \cap G \neq \emptyset$ and B is a subalgebra of $M_n(K)$, we must have $1 \in B$. But $AB = B$ and $BA = A$. It follows that $A = B$. Now, if gAh is an idempotent, where $g, h \in G$, then Ahg is also an idempotent and $A\mathcal{R}Ahg$. So $Ahg = A$ by the preceding part of the proof. Then $gAh = gAg^{-1}$. Now, suppose that two idempotents gAg^{-1}, fAf^{-1} $(g, f \in G)$ are in the same $\mathcal{R}$-class of S. Then $A\mathcal{R}g^{-1}fAf^{-1}g$ and again we get $A = g^{-1}fAf^{-1}g$. Hence $gAg^{-1} = fAf^{-1}$. Similarly one proves that every non-empty intersection of $\mathcal{A}$ with an $\mathcal{L}$-class of S contains exactly one idempotent. The assertion follows. ∎

We have seen that the regular $\mathcal{J}$-classes of S described in cases 2)–6) are of the form $\mathcal{A}$, where $\mathcal{A}$ is a subalgebra containing 1. So, the lemma above applies to these $\mathcal{J}$-classes.

PROPOSITION 3.2. *Let J be a completely 0-simple principal factor of the semigroup $S = S(M_2(\mathbb{F}_q))$. Then $\mathbb{C}_0[J]$ is a semisimple algebra.*

Proof. Let J be one of the regular $\mathcal{J}$-classes of S described in 2)–6), with zero adjoined. Then by Lemma 3.1, $\mathbb{C}_0[J] \cong M_k(\mathbb{C}[H])$ for the maximal subgroup H of J and some k (see [4], Corollary 5.26). It remains to consider the $\mathcal{J}$-class J containing $A = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The maximal subgroup of J is trivial. So, to consider the Rees presentation of J (see [2]) in the coordinate system corresponding to the maximal subgroup $\{A\}$ of J, we list the elements of the $\mathcal{R}$-class of A (in the leading column) and of the $\mathcal{L}$-class of A (in the leading row). This yields the following form of the sandwich matrix P of J:

	$\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ \alpha a & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$
$\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$	0	$1 \dots 1$	1	1
$\begin{pmatrix} b & -\alpha^{-1}b \\ 0 & 0 \end{pmatrix}$	$\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}$	$\begin{matrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{matrix}$	$\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}$	$\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}$
$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$	1	$1 \dots 1$	0	1
$\begin{pmatrix} d & b \\ 0 & 0 \end{pmatrix}$	1	$1 \dots 1$	1	1

Here the second row (column, respectively) represents $q-1$ different rows (columns) of P corresponding to different $q-1$ elements α of $\mathbb{F}_q^*$. Performing elementary operations on rows and columns of P, one brings P to the identity matrix. So, P is invertible as a matrix over $\mathbb{C}$ and consequently $\mathbb{C}_0[J] \cong M_{q+2}(\mathbb{C})$, again by Corollary 5.26 of [4]. The assertion follows. ∎

It is easy to verify that the inverse of the above sandwich matrix is

$$P^{-1} = \begin{pmatrix} -1 & 0 & \dots & 0 & 1 \\ 0 & -1 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 1 \\ 1 & 1 & \dots & 1 & -q \end{pmatrix}.$$

Finally, we describe the radical of the algebra $\mathbb{C}_0[S]$. Let J be the $\mathcal{J}$-class containing $M_n(K)$ and $A = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, together with the zero subspace. Notice that J is an ideal of S. Since $\mathbb{C}_0[J]$ is semisimple, it has an identity E, which can be effectively determined. Namely, in the Munn algebra notation for $\mathbb{C}_0[J]$ (see [4]), E can be identified with P^{-1}. Therefore E can be expressed as a linear combination of elements of J with coefficients $1, -1$ and q as follows:

$$\begin{aligned} E = {} & \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b & d \end{pmatrix} + \sum_{\alpha \in K} \left(\begin{pmatrix} a & \alpha a \\ c & \alpha c \end{pmatrix} + \begin{pmatrix} a & b \\ \alpha a & \alpha b \end{pmatrix} \right) \\ & - \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} - \sum_{\alpha \in K^*} \begin{pmatrix} a & -\alpha^{-1} a \\ \alpha a & -a \end{pmatrix} - q M_2(K). \end{aligned}$$

PROPOSITION 3.3. *Let* $V = \{x \in M_2(K) \mid x_{12} = x_{21}\}$. *Then*

$$J(\mathbb{C}_0[S]) = \mathrm{lin}_{\mathbb{C}} \{gV - EgV \mid g \in \mathrm{Gl}_2(K)\}$$

and $J(\mathbb{C}_0[S])^2 = 0$.

Proof. Denote the right hand side by I. Let J be the $\mathcal{J}$-class of S containing $M_n(K)$ with zero adjoined. Then $\mathbb{C}_0[J]$ is an ideal of $\mathbb{C}_0[S]$ since J is an ideal of S. We have $\mathbb{C}_0[J] I = I \mathbb{C}_0[J] = 0$ because E is a central idempotent in $\mathbb{C}_0[S]$. Moreover $I^2 = 0$. Indeed, if $g \in \mathrm{Gl}_2(K)$ then $gV = Vh$ for some $h \in \mathrm{Gl}_2(K)$ by Corollary 2.2. Therefore

$$(V - EV)(gV - EgV) = (V - EV)(Vh - VhE).$$

Since $V^2 = M_2(K) \in J$, we get $(V - EV)(Vh - VhE) = 0$, so $I^2 = 0$, as desired.

We know that the set $H_V = \{gV \mid g \in \mathrm{Gl}_2(K)\}$ has cardinality $q^3 - q$, so the dimension of I is at most $q^3 - q$. Since the image of I modulo $\mathbb{C}_0[J]$ is spanned by H_V, it has dimension $q^3 - q$. Hence, this is the dimension of I as well.

We claim that I is an ideal of $\mathbb{C}_0[S]$. By symmetry of H_V and since E is central, it is enough to show that I is a left ideal. Let $X \in S$, $X \neq 0$. If X is not regular in S, then $X = gV$ for some $g \in \mathrm{Gl}_2(K)$ and $XV = M_2(K) \in J$. If X is in one of the regular $\mathcal{J}$-classes listed in cases 2)–6), then X contains an invertible matrix u. Thus, XV is either of the form uV or it is equal to $M_2(K)$. So $X(V - EV) \in I$ in the former case and $X(V - EV) = 0$ in the latter. Finally, if $X \in J$, then we also get $X(V - EV) = 0$ because E is the identity of $\mathbb{C}_0[J]$. So I is a left ideal, as claimed.

It follows that $I \subseteq J(\mathbb{C}_0[S])$. By Proposition 3.2, the dimension of $\mathbb{C}_0[S]$ modulo its radical is $q^3 - q$. Comparing dimensions we get $J(\mathbb{C}_0[S]) = I$. ■

Notice that we have in fact shown that the $\mathcal{H}$-class H_V of V in S, with zero adjoined, is a minimal non-zero ideal of the Rees factor S/J.

REFERENCES

[1] Yu. A. Drozd and V. V. Kirichenko, *Finite Dimensional Algebras*, Springer, Berlin, 1994.
[2] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford Univ. Press, 1995.
[3] J. Kwapisz, *Cocyclic subshifts*, Math. Z. 234 (2000), 255–290.
[4] J. Okniński, *Semigroup Algebras*, Dekker, New York, 1991.
[5] —, *Semigroups of Matrices*, World Sci., Singapore, 1998.
[6] J. Okniński and M. S. Putcha, *Subspace semigroups*, J. Algebra 233 (2000), 87–104.

Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: okninski@mimuw.edu.pl

Received 4 June 2001;
revised 6 September 2001 (4075)