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POSITIVE SOLUTIONS FOR SUBLINEAR ELLIPTIC EQUATIONS

BY

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Abstract. The existence of a positive radial solution for a sublinear elliptic boundary value problem in an exterior domain is proved, by the use of a cone compression fixed point theorem. The existence of a nonradial, positive solution for the corresponding nonradial problem is obtained by the sub- and supersolution method, under an additional monotonicity assumption.

1. Introduction. In the first part of the paper we consider the problem

$$(1)\qquad \begin{aligned} -\Delta u &= f(\|x\|, u) && \text{for } \|x\| > 1,\ x \in \mathbb{R}^n,\ n \geq 3,\\ u &= 0 && \text{for } \|x\| = 1,\\ u &\to 0 && \text{as } \|x\| \to \infty. \end{aligned}$$

Looking for its radial solutions $u(x) = z(\|x\|)$, where $z : [1,\infty) \to \mathbb{R}$, one can substitute $v(t) = z((1-t)^{1/(2-n)})$, thus reducing the elliptic BVP (1) to the following BVP for ODE, which is singular at 1:

$$(2)\qquad \begin{aligned} &v''(t) + g(t, v(t)) = 0 \quad \text{for } t \in (0,1),\\ &v(0) = v(1) = 0, \end{aligned}$$

where

$$(3)\qquad g(t,v) = \frac{1}{(n-2)^2}(1-t)^{(2n-2)/(2-n)} f((1-t)^{1/(2-n)}, v(t)).$$

Using some fixed point theorem in a cone [6] we obtain the existence of at least one positive solution for (2) and therefore a radial positive solution for BVP (1). The nonlinearity g (or f) is assumed to be sublinear with respect to the second variable both at 0 and ∞. We relax the sublinearity assumption on g at 0 used in [8], where some results for BVP (2) were obtained by means of lower and upper solutions. BVP (2) generalizes the Emden–Fowler equation considered in [23]. Related problems were considered in [3], [7], [10], [11], [21]. A similar method (with another cone) has been used in [4]. Problems of the form (1) but with superlinear nonlinearity were considered in [9], [22]. In problem (1) we have used the exterior of the unit ball only

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for the sake of convenient notation (one could replace it with the exterior of a ball with an arbitrary radius).

In the second part using the existence result for the radial case together with the existence theorem of Noussair [13] we obtain the existence of a nonradial solution for the following nonradial BVP:

$$(4)\qquad \begin{aligned} -\Delta u &= f(x,u) && \text{for } \|x\|>1,\ x\in\mathbb{R}^n,\ n\ge 3,\\ u(x)&=0 && \text{for } \|x\|=1,\\ u(x)&\to 0 && \text{as } \|x\|\to\infty. \end{aligned}$$

The method of sub- and supersolutions developed in [13] cannot be applied directly, since we do not know any positive subsolution. If we allow $f(x,\cdot) \equiv 0$ then we have the trivial solution which cannot be used as a subsolution to produce a new one (the theorem of Noussair does not exclude that a solution is different from a subsolution). The existence of a nonnegative subsolution (which is neither zero nor positive) was used in [14] to obtain a nonnegative solution; in our approach we obtain a positive one. The problem of symmetry for BVP (4) was studied in the autonomous case in [19], while the multiplicity result was obtained in [12], [16]. Related problems were considered in [15], [17].

In our paper we treat the case of f decaying as x tends to infinity, therefore the autonomous case cannot be considered in this framework.

2. Radial case. First we establish the existence result for the following BVP (possibly singular at 0 and 1):

$$(5)\qquad \begin{aligned} &v''(t)+g(t,v(t))=0 \quad \text{for } t\in(0,1),\\ &v(0)=v(1)=0, \end{aligned}$$

where $g:(0,1)\times[0,\infty)\to[0,\infty)$. We shall use the following theorem [6, Theorem 2.3.4]:

THEOREM 2.1. *Let E be a Banach space, and let $P\subset E$ be a cone in E. Let Ω_1 and Ω_2 be two bounded open sets in E such that $0\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$. Let $A:P\cap(\overline{\Omega}_2\setminus\Omega_1)\to P$ be a completely continuous operator. Suppose that either*

$$(6)\qquad \begin{aligned} \|Ax\|&\le\|x\| \quad \textit{for any } x\in P\cap\partial\Omega_1,\\ \|Ax\|&\ge\|x\| \quad \textit{for any } x\in P\cap\partial\Omega_2, \end{aligned}$$

or

$$(7)\qquad \begin{aligned} \|Ax\|&\ge\|x\| \quad \textit{for any } x\in P\cap\partial\Omega_1,\\ \|Ax\|&\le\|x\| \quad \textit{for any } x\in P\cap\partial\Omega_2. \end{aligned}$$

Then A has at least one fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Let E be the space $C([0,1])$ of continuous functions with the norm $\|v\|_\infty = \sup_{t\in[0,1]} |v(t)|$. Define

$$H = \Big\{ h \in C((0,1)) : h > 0,\ \int_0^1 t(1-t)h(t)\,dt < \infty \Big\}.$$

THEOREM 2.2. *Let $g : (0,1)\times[0,\infty) \to \mathbb{R}$ be a continuous function. Assume that*:

(A1) *for any $M > 0$ there exists a function $h_M \in H$ such that for any $0 \le v \le M$, $t \in (0,1)$ we have*

$$0 \le g(t,v) \le h_M(t), \qquad \limsup_{M\to\infty} \frac{\int_0^1 s(1-s)h_M(s)\,ds}{M} < 1,$$

(A2) *there exists a set $A \subset (0,1)$ of positive measure such that*

$$\liminf_{v\to 0^+} \frac{g(t,v)}{v} = \infty \qquad \textit{uniformly w.r.t. } t \in A.$$

Then BVP (5) *has at least one positive solution.*

Proof. The Green function corresponding to the linear homogeneous problem has the form

$$G(t,s) = \begin{cases} s(1-t) & \text{for } 0 \le s \le t, \\ t(1-s) & \text{for } t \le s \le 1, \end{cases} \tag{8}$$

and satisfies the following estimate:

$$|G(t,s)| \le s(1-s) \quad \text{for all } t,s \in [0,1]. \tag{9}$$

Taking any set of positive measure $B \subset A \cap (\delta, 1-\delta)$, for some positive δ, where A is the set from assumption (A2), we can define a cone P in E by

$$P = \{ v \in E : v(t) \ge 0,\ t \in [0,1],\ \inf_{t\in B} v(t) \ge \min\{a, 1-b\}\|v\|_\infty \}, \tag{10}$$

where $a = \inf B$ and $b = \sup B$. Then BVP (5) can be restated as an equation in E:

$$Sv = v, \tag{11}$$

where $S : P \to E$ is defined by

$$Sv(t) := \int_0^1 G(t,s)g(s,v(s))\,ds. \tag{12}$$

By (A1) and (9) one can see that S is well defined on the set of all nonnegative, continuous functions and maps it into E. Moreover, it maps all nonnegative functions into the cone P. Indeed, for $v \ge 0$ we have $g(t,v) \ge 0$ by (A1), therefore

$$\inf_{t\in B} Sv(t) = \inf_{t\in B} \int_0^t (1-t)sg(s,v(s))\,ds + \int_t^1 t(1-s)g(s,v(s))\,ds$$

$$
\begin{aligned}
&\geq \min\{1-b,a\} \inf_{t\in B} \Big(\int_0^t sg(s,v(s))ds + \int_t^1 (1-s)g(s,v(s))\,ds\Big)\\
&\geq \min\{1-b,a\} \int_0^1 s(1-s)g(s,v(s))\,ds\\
&= \min\{1-b,a\} \sup_{t\in[0,1]} \Big(\int_0^t s(1-s)g(s,v(s))\,ds + \int_t^1 s(1-s)g(s,v(s))\,ds\Big)\\
&\geq \min\{1-b,a\} \sup_{t\in[0,1]} \Big(\int_0^t s(1-t)g(s,v(s))\,ds + \int_t^1 t(1-s)g(s,v(s))\,ds\Big)\\
&= \min\{1-b,a\}\|Sv\|_\infty.
\end{aligned}
$$

By standard reasoning one can show that assumption (A1) guarantees that S maps E into itself and is also continuous. To prove that S is compact take any closed ball $B(0,M)$ in E. We shall show that the functions from $S(B(0,M)) = \{Sv : \|v\|_\infty \leq M,\ v \in P\}$ are equicontinuous and equibounded.

To this end take $\varepsilon > 0$ and notice that by the integrability of the function $s \mapsto s(1-s)h_M(s)$, there exists $\delta > 0$ such that

$$\int_t^{t'} s(1-s)h_M(s)\,ds < \varepsilon \quad \text{if } |t'-t| < \delta.$$

Thus, for such t and t', we have

$$|Sv(t) - Sv(t')| \leq \int_{\min\{t,t'\}}^{\max\{t,t'\}} s(1-s)h_M(s)\,ds < \varepsilon$$

where we have used (9). This proves that the Sv are equicontinuous. Similarly, one can show they are equibounded. Hence the functions from the set $\{Sv : \|v\|_\infty \leq M\}$ satisfy the assumptions of the Ascoli–Arzelà Theorem and in consequence this set must be compact in E. Since $M > 0$ was arbitrary we get compactness of the operator $S : P \to E$.

Now we shall show that assumptions (7) from Theorem 2.1 are satisfied. By (A1) we can choose $M > 0$ large enough so that $\int_0^1 t(1-t)h_M(t)\,dt \leq M$, whence for any $0 \leq v \leq M$ and $t \in (0,1)$,

$$0 \leq g(t,v) \leq h_M(t). \tag{13}$$

Then for any $v \in P$ such that $\|v\|_\infty = M$, by (9) one obtains

$$Sv(t) = \int_0^1 G(t,s)g(s,v(s))\,ds \leq \int_0^1 s(1-s)g(s,v(s))\,ds$$

$$\leq \int_0^1 s(1-s)h_M(s)\,ds \leq M = \|v\|_\infty$$

for any $t \in [0,1]$. Therefore $\|Sv\|_\infty \leq \|v\|_\infty$ for any $v \in \partial\Omega_2 \cap P$, where $\Omega_2 := \{v \in E : \|v\|_\infty < M\}$.

Finally, choose $\mu > 0$ such that $\mu \min\{a, 1-b\} \int_B G((a+b)/2, s)\,ds \geq 1$ ($B \subset A \cap (\delta, 1-\delta)$ for some positive constant δ as in the definition of the cone P). By (A2), there exists $R < M$ such that for $0 \leq v \leq R$,

$$\inf_{t\in B} g(t,v) \geq \mu v.$$

Let $\Omega_1 = \{v \in E : \|v\|_\infty < R\}$. If $v \in P \cap \partial\Omega_1$, then $g(t, v(t)) \geq \mu v(t)$ for all $t \in B$. Then

$$\begin{aligned} Sv\bigg(\frac{a+b}{2}\bigg) &= \int_0^1 G\bigg(\frac{a+b}{2}, s\bigg) g(s, v(s))\,ds \geq \int_B G\bigg(\frac{a+b}{2}, s\bigg) g(s, v(s))\,ds \\ &\geq \mu \int_B G\bigg(\frac{a+b}{2}, s\bigg) v(s)\,ds \\ &\geq \|v\|_\infty \mu \min\{a, 1-b\} \int_B G\bigg(\frac{a+b}{2}, s\bigg)\,ds \geq \|v\|_\infty. \end{aligned}$$

Therefore $\|Sv\|_\infty \geq \|v\|_\infty$ for $v \in P \cap \partial\Omega_1$. Applying Theorem 2.1 to S one obtains a fixed point v_0 in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Moreover, since v_0 is nonnegative and $\|v_0\|_\infty \geq R$ we have $v_0(t_0) > 0$ at some point t_0. Since v_0 satisfies integral equation (11), the function $g(\cdot, v_0(\cdot))$ cannot vanish on the whole interval $(0,1)$ (otherwise $v_0 \equiv 0$ by (11)) so once again by (11), v_0 must be positive in $(0,1)$. ■

Remark 1. One can see from the proof of the theorem that the assumption (A2) could be replaced by

(A2′) there exists a set $B \subset (0,1)$ of positive measure and $\varepsilon > 0$ such that

$$g(t,v) \geq \mu_B v \quad \text{for } t \in B,\ 0 < v < \varepsilon,$$

where

$$\mu_B := \inf_{C \subset B} \Big(\min\{\inf C, 1 - \sup C\} \sup_{t\in(0,1)} \int_C G(t,s)\,ds \Big)^{-1}.$$

Obviously, if $C \subset B$ then the inequality holds on C as well, but the constant in round brackets can be less than the one for B.

Remark 2. For example, if $B = (0,1)$, then $\mu_B = 24\sqrt{3}$, which is obtained for $C = (1/(2\sqrt{3}), 1 - 1/(2\sqrt{3}))$. In this case assumption (A2′) can be compared with the assumption

(A) there exists $k > 1$ and for any compact set $K \subset (0,1)$, there is $\varepsilon > 0$ such that

$$g(t,v) \geq k^2 v \quad \text{for all } t \in K,\ u \in (0,\varepsilon]$$

from [8]. Obviously, our constant is worse than this one, but our theorem also works for nonlinearities that satisfy the inequality on smaller sets than $t \in (0,1)$.

Define $K := \{p \in C((1,\infty)) : \int_1^\infty s(1-s^{2-n})p(s)\,ds < \infty\}$. Now we are ready to formulate the main result for elliptic BVP (1) in the radial case, which is an immediate consequence of Theorem 2.2:

THEOREM 2.3. *Let* $f : (1,\infty)\times[0,\infty) \to [0,\infty)$ *be a continuous function satisfying*

(B1) *for any* $M > 0$ *there exists a function* $p_M \in K$ *such that, for any* $0 \leq u \leq M$, $s > 1$,

$$0 \leq f(s,u) \leq p_M(s), \qquad \limsup_{M\to\infty} \frac{\int_1^\infty s(1-s^{2-n})p_M(s)\,ds}{M} < 1,$$

(B2) *there exists a set* B *of positive Lebesgue measure such that*

$$\lim_{u\to 0^+} \frac{f(s,u)}{u} = \infty \qquad \textit{uniformly w.r.t. } s \in B.$$

Then BVP (1) *has at least one positive solution.*

REMARK 3. It is worth noticing that even if $f(s,0) \equiv 0$ then by the above theorem we obtain additionally a positive solution.

REMARK 4. One can see from the proof of the theorem that the assumption (B2) could be replaced by

(B2′) there exists a set $B \subset (1,\infty)$ of positive measure and a sufficiently large constant L_B such that

$$f(s,u) \geq L_B u \quad \text{for } s \in B, u \geq R.$$

REMARK 5. The assumption (B1) excludes the case of the nonlinearity f depending only on u. In fact f not only has to depend on s but also to decay sufficiently fast as $s \to \infty$. The case of slower decay was considered in [2].

COROLLARY 1. *If* $f(s,u) = p(s)h(u)$, *where* $p : (1,\infty) \to (0,\infty)$ *and* $h : [0,\infty) \to [0,\infty)$ *are continuous functions, then the assumptions of Theorem* 2.3 *reduce to*:

(C1) $p_0 := \int_1^\infty s(1-s^{2-n})p(s)\,ds < \infty$, (C2) $\limsup_{u\to\infty} \frac{h(u)}{u} < \frac{1}{p_0}$,

(C3) $\liminf_{u\to 0^+} \frac{h(u)}{u} = \infty$.

This includes the following

EXAMPLE 1. The equation

$$-\Delta u = \frac{u^\alpha}{\|x\|^\beta} \quad \text{for } \|x\| \geq 1, \tag{14}$$

where $\alpha < 1$, $\beta > 2$, has a positive solution satisfying the boundary conditions as in (1).

Theorem 2.3 also applies to the case where one cannot separate the variables of the function f, as in the following:

EXAMPLE 2. Consider the equation

$$-\Delta u = \frac{u^{\alpha(\|x\|)}}{\|x\|^\beta} \quad \text{for } \|x\| \geq 1, \tag{15}$$

where $\beta > 2$ and $\alpha : [1,\infty) \to \mathbb{R}$ satisfies $\sup_{s\in[1,\infty)} \alpha(s) < 1$. Then (15) with the boundary conditions as in (1) has a positive solution.

REMARK 6. Considering $f : (1,\infty) \times \mathbb{R} \to \mathbb{R}$ and assuming only (B1), by the Schauder Theorem, one can obtain the existence of at least one solution to BVP (1). If $f(\cdot,0) \not\equiv 0$ then the solution is not trivial, otherwise ($f(\cdot,0) \equiv 0$) we do not know whether a nonzero solution exists. Therefore the situation is quite different from the one considered above.

3. Nonradial case. Consider the following BVP in an exterior domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$):

$$\begin{aligned} -\Delta u &= f(x,u,\nabla u) && \text{for } x \in \Omega, \\ u(x) &= 0 && \text{for } x \in \partial\Omega. \end{aligned} \tag{16}$$

We look for classical solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$. We recall a well known notion of sub- and supersolution for BVP (16). We shall call $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ a *subsolution* for (16) if it satisfies

$$\begin{aligned} -\Delta \underline{u} &\leq f(x,\underline{u},\nabla \underline{u}) && \text{for } x \in \Omega, \\ \underline{u}(x) &\leq 0 && \text{for } x \in \partial\Omega, \end{aligned} \tag{17}$$

In a similar way we define a *supersolution* $\overline{u}$ for (16)—it suffices to reverse the inequalities in (17). A good survey of results obtained by the sub- and supersolution method is the book of Pao [18].

Now we recall the existence result for a nonradial elliptic BVP in an exterior domain due to Noussair [13]:

THEOREM 3.1. *Let Ω be an exterior domain. Assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following conditions*:

(D1) *for each bounded domain $M \subset \Omega$ there exists a continuous function $\varrho_M : \mathbb{R} \to \mathbb{R}$ such that for all $x \in M$, $u \in \mathbb{R}$ and $p \in \mathbb{R}^n$,*

$$|f(x,u,p)| \le \varrho_M(u)(1+|p|^2),$$

(D2) *f is Hölder continuous ($C^{0,r}$) with respect to (x,u,p) and C^1 with respect to u,p,*

(D3) *there exist sub- and supersolutions for* (16), *$\underline{u}$ and $\overline{u}$ respectively, such that $\underline{u}(x) \le \overline{u}(x)$ for any $x \in \Omega$.*

Then there exists at least one solution u of (16) *such that $\underline{u}(x) \le u(x) \le \overline{u}(x)$ for any $x \in \Omega$.*

Let Ω be the exterior of the closed ball $B(0,1)$ of radius 1 in $\mathbb{R}^n$ and let $n \ge 3$. We consider the following BVP in Ω with a not necessarily radial nonlinearity f:

$$(18)\qquad \begin{aligned} -\Delta u &= f(x,u) && \text{in } \Omega,\\ u(x) &= 0 && \text{on } \partial\Omega,\\ u(x) &\to 0 && \text{as } \|x\| \to \infty. \end{aligned}$$

THEOREM 3.2. *Assume that a function $f : \Omega \times [0,\infty) \to [0,\infty)$ satisfies*:

(E1) *f is Hölder continuous ($C^{0,r}$) with respect to (x,u) and continuously differentiable and nonincreasing with respect to the second variable,*

(E2) *for any $M > 0$ there exists a function $p_M : (1,\infty) \to (0,\infty)$ such that $\int_1^\infty s(1-s^{2-n})p_M(s)\,ds < \infty$ and*

$$0 \le f(x,u) \le p_M(\|x\|) \quad \textit{for any } 0 \le u \le M,\ 1 < \|x\|,$$

(E3) *there exists a set $B \subset (1,\infty)$ of positive measure such that*

$$\liminf_{u\to 0^+} \frac{f(x,u)}{u} = \infty \quad \textit{uniformly w.r.t. } \|x\| \in B.$$

Then there exists at least one positive solution to BVP (18).

Proof. Define $f_1 : (1,\infty) \times [0,\infty) \to [0,\infty)$ and $f_2 : (1,\infty) \times [0,\infty) \to [0,\infty)$ by

$$f_1(r,u) = \inf_{\|x\|=r} f(x,u), \qquad f_2(r,u) = \sup_{\|x\|=r} f(x,u).$$

Since f_1 and f_2 satisfy the assumptions of Theorem 2.3 we obtain the existence of two radial solutions u_1 and u_2 to BVP (1) with $f = f_1$ and $f = f_2$ respectively. But then

$$-\Delta u_1 = f_1(\|x\|, u_1) \le f(x,u_1)$$

so u_1 is a subsolution to BVP (18). By a similar reasoning u_2 is a supersolution to BVP (18). Moreover if $u_1(x_0) > u_2(x_0)$ for some $x_0 \in \Omega$ then taking the connected component U of the set $\{x \in \Omega : u_1(x) > u_2(x)\}$ such that $x_0 \in U$ and considering the function $h(x) = u_2(x) - u_1(x)$ we arrive at

$$\begin{aligned} -\Delta h(x) &= f_2(\|x\|, u_2(x)) - f_1(\|x\|, u_1(x)) \\ &\geq f_1(\|x\|, u_2(x)) - f_1(\|x\|, u_1(x)), \end{aligned}$$

which together with the monotonicity (by (E1) the function $f_1(r, \cdot)$ is nonincreasing) implies $-\Delta h(x) \geq f_1(\|x\|, u_1(x)) - f_1(\|x\|, u_1(x)) = 0$ for any $x \in U$ (since $u_1(x) > u_2(x)$ for those x). Consequently,

$$\begin{aligned} &-\Delta h(x) \geq 0 && \text{for } x \in U, \\ &h(x) = 0 && \text{for } x \in \partial U, \\ &h(x) \to 0 && \text{as } \|x\| \to \infty,\ x \in U, \end{aligned}$$

so if h is negative at some point it has to attain a negative minimum at some x_0. Then take $r_0 > 0$ such that $\|x_0\| < r_0$ and $h(x) \geq \frac{1}{2}h(x_0)$ for $\|x\| \geq r_0$. Consequently, by the maximum principle (see [5] or [20]) applied in the set $U_1 = \{x \in U : 1 < \|x\| < r_0\}$, the function h attains its minimum on the boundary of U_1 so we get $h(x_0) \geq \inf_{\partial U_1} h(x) \geq \frac{1}{2}h(x_0)$, which contradicts the fact that $x_0 \in U$.

Therefore by Theorem 3.1 we obtain the existence of a solution u_0 for (18) such that $u_1 \leq u_0 \leq u_2$. Thus if f is not radial with respect to x then obviously neither is u_0. ■

Remark 7. The above theorem provides a nonradial solution only if the nonlinearity f is nonradial with respect to x.

Example 3. Consider the following BVP:

$$\begin{aligned} &-\Delta u = g(x)u^{\sigma} && \text{in } \Omega, \\ &u(x) = 0 && \text{on } \partial\Omega, \\ &u(x) \to 0 && \text{as } \|x\| \to \infty, \end{aligned} \tag{19}$$

where $\sigma < 1$ and $g : \Omega \to (0, \infty)$ is a continuous, nonnegative function, strictly positive on some annulus $\Omega_1 = \{x \in \Omega : a \leq \|x\| \leq b\} \subset \Omega$ ($1 < a < b$ some positive constants) satisfying

$$\int_1^{\infty} r(1 - r^{2-n}) \sup_{\|x\|=r} g(x)\, dr < \infty.$$

Then BVP (19) has a positive solution (nonradial if g is such).

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