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ON INDECOMPOSABLE PROJECTIVE REPRESENTATIONS OF FINITE GROUPS OVER FIELDS OF CHARACTERISTIC $p > 0$

BY

LEONID F. BARANNYK and KAMILA SOBOLEWSKA (Słupsk)

Abstract. Let G be a finite group, F a field of characteristic p with $p \mid |G|$, and $F^\lambda G$ the twisted group algebra of the group G and the field F with a 2-cocycle $\lambda \in Z^2(G, F^*)$. We give necessary and sufficient conditions for $F^\lambda G$ to be of finite representation type. We also introduce the concept of projective F-representation type for the group G (finite, infinite, mixed) and we exhibit finite groups of each type.

Introduction. Let F be a field of characteristic $p > 0$, F^* the multiplicative group of the field F, $F^p = \{a^p : a \in F\}$, G a finite group of order $|G|$, where $p \mid |G|$, and G_p a Sylow p-subgroup of G. Let G' be the commutant of G, C_p a Sylow p-subgroup of G', $C_p \subset G_p$, G'_p the commutant of G_p, and $Z^2(G, F^*)$ the group of all F^*-valued normalized 2-cocycles of the group G, where we assume that G acts trivially on F^* (see [26, Chapter 1]). Denote by $F^\lambda G$ the twisted group algebra of the group G and the field F with a cocycle $\lambda \in Z^2(G, F^*)$ and by $\operatorname{rad} F^\lambda G$ the radical of $F^\lambda G$. An F-basis $\{u_g : g \in G\}$ of $F^\lambda G$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in G$ is called *natural*. By an $F^\lambda G$-*module* we mean a finitely generated left $F^\lambda G$-module. If H is a subgroup of G, then the restriction of $\lambda \in Z^2(G, F^*)$ to $H \times H$ will also be denoted by λ. In this case, $F^\lambda H$ is a subalgebra of $F^\lambda G$.

Higman [21] proved that a group algebra FG is of finite representation type if and only if G_p is a cyclic group. In this case Kasch, Kneser and Kupisch [27] gave a sharper upper bound of the number of indecomposable FG-modules. They also obtained conditions on G under which the bound is attained. Later Janusz [22] gave a formula for the exact number of indecomposable FG-modules for the case when F is an algebraically closed field. In [23] he determined the structure of indecomposable modules in more detail. Indecomposable FG-modules with G_p being cyclic are also investigated in [5], [11], [24], [25], [28], [29] (see as well [16, Chapter VII]). The representation type of group rings SG, where S is an arbitrary commutative artinian ring or a local artinian ring whose quotient ring $S/\operatorname{rad} S$ is finitely generated over its center, is determined by Gustafson [20] and Dowbor and Simson [14].

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Generalizations to the case when S is an arbitrary finite-dimensional algebra over a field F and G is a finite group have been found by Meltzer and Skowroński [30], [31] and Skowroński [35], [36]. Representation-infinite group algebras SG of polynomial growth are classified in [36]. Gudivok [18] and Janusz [24], [25] showed that if F is an infinite field and G is an abelian p-group which is neither cyclic nor of order 4, then there exist infinitely many non-isomorphic indecomposable FG-modules of F-dimension n for every natural number $n > 1$. If G is the non-cyclic group of order 4, then the preceding result is valid for even natural numbers n.

Higman [21] proved, in fact, that the first Brauer–Thrall conjecture holds for group algebras of finite groups. Results by Gudivok [18] and Janusz [24], [25] give the solution of the second Brauer–Thrall conjecture for group algebras of finite groups. As is well known, the first Brauer–Thrall conjecture for finite-dimensional algebras over an arbitrary field was solved by Roĭter [34]. The second Brauer–Thrall conjecture was proved by Nazarova and Roĭter [32], Bautista [3], Bongartz [6], Bautista, Gabriel, Roĭter and Salmerón [4].

In [7], Conlon developed the theory of twisted group algebras $F^\lambda G$ by exploiting their analogy with group algebras FG assuming that F is large enough. In this case $F^\lambda G_p$ is a group algebra and therefore $F^\lambda G$ is of finite representation type if and only if G_p is cyclic. Moreover, in the same paper Conlon established that if G_p is a cyclic group then a rough upper bound for the number of indecomposable FG-modules which was found in [21] also holds for the number of indecomposable $F^\lambda G$-modules. It should be noted that Reynolds [33] computed the number of non-isomorphic simple $K^\mu G$-modules where K is an arbitrary field, G is a finite group and $\mu \in Z^2(G, K^*)$. We also remark that if the characteristic of K does not divide the order of the group G, then $K^\mu G$ is a semisimple algebra for any $\mu \in Z^2(G, K^*)$, and hence is of finite representation type. Using Green's results [17], for the case when G is a finite abelian p-group and the radical of $F^\lambda G$ is not cyclic, Sobolewska [37] constructed increasing functions $f_\lambda : \mathbb{N} \to \mathbb{N}$ such that there exist infinitely many isomorphism classes of indecomposable $F^\lambda G$-modules of F-dimension $f_\lambda(n)$ for every natural number $n > 1$.

In the present paper we shall characterize twisted group algebras $F^\lambda G$ of finite representation type. We shall also describe finite groups depending on a projective representation type over the field F.

Let us briefly present the main results of the paper. In Section 1, we prove that an algebra $F^\lambda G$ is of finite representation type if and only if $F^\lambda G_p$ is a uniserial algebra (Theorem 1.1; we use the terminology introduced in [15]). We also establish (Theorem 1.2) that if $p \neq 2$, then $F^\lambda G_p$ is a uniserial algebra if and only if C_p is cyclic and one of the following conditions holds:

(1) the quotient algebra $F^\lambda G_p/F^\lambda G_p{\cdot}\mathrm{rad}F^\lambda C_p$ is a field;

(2) $C_p=\{e\}$ and there exists a decomposition $G_p=H\times N$ such that H is cyclic and $F^\lambda N$ is a field;

(3) $C_p\neq\{e\}$ and there exists a decomposition $G_p/C_p=\langle a_1C_p\rangle\times\ldots\times\langle a_sC_p\rangle$ such that $C_p\subset\langle a_1\rangle$, $C_p\not\subset\langle a_j\rangle$ for every $j=2,\ldots,s$ and $F^\lambda D/F^\lambda D\cdot\mathrm{rad}\,F^\lambda C_p$ is a field, where D is the subgroup of G_p generated by $C_p,a_2,\ldots,a_s$.

The proofs of these theorems are based on the characterization of local rings of finite representation type which was obtained in [12]–[14]. A special case of such rings was investigated in [19]. In Section 1 of this paper, we also obtain indecomposable $F^\lambda G$-modules for the case when G_p is a normal subgroup of G and $F^\lambda G_p$ is a uniserial algebra (Theorems 1.3 and 1.4).

We say that a group G is of *finite* (resp. *infinite*) *PFR-type* (Projective F-Representation type) if the algebra $F^\lambda G$ is of finite (resp. infinite) representation type for every cocycle $\lambda\in Z^2(G,F^*)$. Otherwise, G is said to be of *mixed PFR-type.*

In Section 2, we classify finite groups depending on their PFR-type (Theorems 2.1 and 2.2, Proposition 2.1). We also state necessary and sufficient conditions for G and G_p to be of the same PFR-type (Propositions 2.2–2.3).

1. Twisted group algebras of finite representation type and their representations

LEMMA 1.1. *Let $\lambda\in Z^2(G,F^*)$. Every $F^\lambda G$-module is isomorphic to an $F^\lambda G$-component of an induced $F^\lambda G$-module $F^\lambda G\otimes_{F^\lambda G_p}V$, where V is some $F^\lambda G_p$-module.*

LEMMA 1.2. *Let H be a subgroup of G and $\lambda\in Z^2(G,F^*)$. If $F^\lambda H$ is of infinite representation type, then $F^\lambda G$ is also of infinite representation type.*

LEMMA 1.3. *An algebra $F^\lambda G$ is of finite representation type if and only if $F^\lambda G_p$ is of finite representation type.*

The proofs of Lemmas 1.1–1.3 are similar to those of the corresponding propositions about group algebras (see [8, §63]).

LEMMA 1.4 ([21]). *A group algebra FG is of finite representation type if and only if G_p is a cyclic group.*

LEMMA 1.5. *Suppose $p\,|\,|G'|$, $C_p\subset G_p$ and $\lambda\in Z^2(G,F^*)$. Then*:

(1) *Up to cohomology*

$$\lambda_{g,h}=\lambda_{h,g}=1 \tag{1.1}$$

for any $g\in G_p$ and any $h\in C_p$.

(2) *Suppose* λ *satisfies condition* (1.1), $\overline{G}_p = G_p/C_p$, $\overline{g} = gC_p$ *for* $g \in G_p$, *and* $\overline{\lambda}_{\overline{a},\overline{b}} = \lambda_{a,b}$ *for any* $a, b \in G_p$. *Then* $\overline{\lambda} \in Z^2(\overline{G}_p, F^*)$ *and*

$$F^{\overline{\lambda}}\overline{G}_p \cong F^{\lambda}G_p/F^{\lambda}G_p \cdot \operatorname{rad} F^{\lambda}C_p.$$

Proof. In view of [26, Proposition 5.17, p. 48] the restriction of every cocycle $\lambda \in Z^2(G, F^*)$ to $C_p \times C_p$ is a coboundary. Therefore, statements (1) and (2) follow from the properties of natural homomorphisms of twisted group algebras ([26, pp. 87–93]). ∎

In what follows, we assume that every cocycle $\lambda \in Z^2(G, F^*)$ under consideration satisfies condition (1.1). In particular, $F^{\lambda}C_p$ will always be the group algebra FC_p.

The number $i_F = \sup\{0, m\}$ is important in describing twisted group algebras of abelian p-groups which are of finite representation type, where m is a natural number such that for some $\gamma_1, \ldots, \gamma_m \in F^*$ the algebra

$$F[x]/(x^p - \gamma_1) \otimes_F \ldots \otimes_F F[x]/(x^p - \gamma_m)$$

is a field. If F is a perfect field, then $i_F = 0$, otherwise $i_F \neq 0$.

Proposition 1.1. *Let* K *be a perfect field of characteristic* p *and* $F = K(x_1, \ldots, x_n)$ *the quotient field of the polynomial ring* $K[x_1, \ldots, x_n]$. *Then* $i_F = n$.

Proof. By induction on i we prove that the algebra

$$A_i = F[y]/(y^p - x_1) \otimes_F \ldots \otimes_F F[y]/(y^p - x_i)$$

is a field for every $i \in \{1, \ldots, n\}$. From this it follows that $i_F \geq n$. Suppose that for some $\lambda_1, \ldots, \lambda_m \in F^*$ the algebra

$$B = F[y]/(y^p - \lambda_1) \otimes_F \ldots \otimes_F F[y]/(y^p - \lambda_m)$$

is a field. Let $C = B \otimes_F A_n$. The algebra A_n is isomorphic to the field $K(y_1, \ldots, y_n)$, where $y_j^p = x_j$ ($j = 1, \ldots, n$). Every element of F is the pth power of some element of A_n. It follows that

$$C \cong A_n[y]/(y^p - 1) \otimes_{A_n} \ldots \otimes_{A_n} A_n[y]/(y^p - 1) \quad (m \text{ factors}).$$

Consequently, $C/\operatorname{rad} C \cong A_n$. On the other hand, C can be viewed as a twisted group algebra of an elementary abelian p-group of order p^n over the field B. Therefore, $C/\operatorname{rad} C$ is isomorphic to a purely inseparable extension of the field B of degree p^s, where $s \leq n$. It follows that $p^n = p^s \cdot [B : F]$ or $p^n = p^s \cdot p^m$, whence $m \leq n$. Hence $i_F \leq n$, and the proof is complete. ∎

Proposition 1.2. *Let* K *be a field of characteristic* p, $X = \{x_i : i = 1, 2, \ldots\}$, *and* F *the quotient field of the polynomial ring* $K[X]$. *Then* $i_F = \infty$.

THEOREM 1.1. *Let G be a finite group, $p \,|\, |G|$ and $\lambda \in Z^2(G, F^*)$. The algebra $F^\lambda G$ is of finite representation type if and only if $F^\lambda G_p$ is a uniserial algebra.*

Proof. By Lemma 1.3, we may assume that G is a p-group. Let $\{u_g : g \in G\}$ be a natural F-basis of the algebra $F^\lambda G$ and e be the identity element of G. It is known (see [26, p. 74]) that $F^\lambda G/\operatorname{rad} F^\lambda G \cong K$, where K is a purely inseparable extension of the field F. Suppose $F^\lambda G$ is of finite representation type. Then by Lemmas 1.2, 1.4 and 1.5, G' is a cyclic group and $F^\lambda G'$ is a group algebra. Let $G' = \langle c \rangle$, $A = F^\lambda G$, $V = \operatorname{rad} A/(\operatorname{rad} A)^2$, $m = \dim_K V$ and $m' = \dim V_K$. We know (see [12]–[14]) that in the case under consideration we have $m \cdot m' \le 3$.

Suppose $m = 1$. If $u_c - u_e \notin (\operatorname{rad} A)^2$, then $\{u_c - u_e + (\operatorname{rad} A)^2\}$ is a basis of the left vector space V over the field K. It follows that any element of V is of the form

$$\overline{x}(u_c - u_e + (\operatorname{rad} A)^2) = x(u_c - u_e) + (\operatorname{rad} A)^2,$$

where $x \in A$, $\overline{x} = x + \operatorname{rad} A$. Since for each $x \in A$ there exists $y \in A$ such that $x(u_c - u_e) = (u_c - u_e)y$, we have

$$\overline{x}(u_c - u_e + (\operatorname{rad} A)^2) = (u_c - u_e + (\operatorname{rad} A)^2)\overline{y}.$$

Hence, $m' = 1$. Suppose now that $u_c - u_e \in (\operatorname{rad} A)^2$. Since for arbitrary $x, y \in A$ there exists $z \in A$ such that $xy - yx = (u_c - u_e)z$, we obtain

$$\overline{x}(y + (\operatorname{rad} A)^2) = (y + (\operatorname{rad} A)^2)\overline{x}$$

for any $x, y \in A$. In this case $m' = 1$. By the same arguments we can establish that if $m' = 1$ then $m = 1$.

Therefore, if $F^\lambda G$ is of finite representation type, then $F^\lambda G$ is a uniserial algebra. Conversely, every uniserial algebra is of finite representation type ([15, p. 171]). ∎

PROPOSITION 1.3. *Let F be a field of characteristic p, G a finite abelian p-group and $\lambda \in Z^2(G, F^*)$. The algebra $F^\lambda G$ is of finite representation type if and only if $G = H \times N$, where H is a cyclic group and $F^\lambda N$ is a field.*

Proof. Let $G = H \times N$, where H is cyclic and $F^\lambda N$ is a field. Then $F^\lambda G$ is a uniserial algebra, and hence it is of finite representation type. Now we suppose that there is no decomposition $G = H \times N$ such that H is a cyclic group and $F^\lambda N$ is a field. Let $\overline{G}$ be the socle of G. Then $F^\lambda \overline{G} \cong F^\mu B$, where B is an elementary abelian p-group of order $|\overline{G}|$ and the following conditions are satisfied: $B = L \times M$, L is a non-cyclic group of order p^2 and $F^\mu L$ is the group algebra of the group L over the field F. By Lemmas 1.2 and 1.4, the algebra $F^\mu B$ is of infinite representation type. Applying again Lemma 1.2 to $F^\lambda \overline{G}$ and $F^\lambda G$, we conclude that the algebra $F^\lambda G$ is of infinite representation type. ∎

Corollary 1. *Let G be a finite abelian p-group and $\lambda \in Z^2(G, F^*)$. Assume that $G = H \times N$, where H is a cyclic group and $F^\lambda H$ is not a field. The algebra $F^\lambda G$ is of finite representation type if and only if $F^\lambda N$ is a field.*

Corollary 2. *Let G be a finite abelian p-group, $\overline{G}$ the socle of G, and $\lambda \in Z^2(G, F^*)$. The algebra $F^\lambda G$ is of infinite representation type if and only if $F^\lambda \overline{G} \cong F^\mu H \otimes_F F^\mu N$, where $\overline{G} \cong H \times N$, H is a non-cyclic group of order p^2 and $F^\mu H$ is the group algebra.*

Corollary 3. *Let $G = \langle a_1 \rangle \times \ldots \times \langle a_s \rangle$ be an abelian p-group. If $s \geq i_F + 2$ then $F^\lambda G$ is of infinite representation type for every $\lambda \in Z^2(G, F^*)$. If $s \leq i_F + 1$ then there exists an algebra $F^\lambda G$ which is of finite representation type. If $s = 1$ then $F^\lambda G$ is of finite representation type for every $\lambda \in Z^2(G, F^*)$.*

Lemma 1.6. *Let $p \neq 2$, G be a non-abelian p-group with $G' = \langle c \rangle$ of order p, and $\{u_g : g \in G\}$ be a natural F-basis of $F^\lambda G$. Then:*

(1) $(u_a u_b)^p = u_a^p u_b^p$ for any $a, b \in G$.

(2) If $y \in F^\lambda G$, $g \in G$, then

$$u_g y = y u_g + (u_c - u_e) y' u_g, \tag{1.2}$$

$$(y u_g)^p = y^p u_g^p + (u_c - u_e)^2 z \tag{1.3}$$

for some $y', z \in F^\lambda G$.

(3) If

$$x = \sum_{g \in G} \alpha_g u_g$$

is an element of $F^\lambda G$, then

$$x^p = \sum_{g \in G} \alpha_g^p u_g^p + (u_c - u_e)^2 z, \qquad z \in F^\lambda G.$$

Proof. We remark that u_c belongs to the center of $F^\lambda G$ and if $ab = c^j ba$, then $u_a u_b = u_c^j u_b u_a$. From this we obtain (1) and formula (1.2). Then

$$\begin{aligned}(y u_g)^p &= y[y + (u_c - u_e)y'][y + 2(u_c - u_e)y'] \ldots \\ &\qquad \ldots [y + (p-1)(u_c - u_e)y'] u_g^p + (u_c - u_e)^2 z' \\ &= y^p u_g^p + (u_c - u_e)^2 z, \qquad z \in F^\lambda G.\end{aligned}$$

Hence, formula (1.3) holds.

It remains to prove (3). Suppose $\alpha_b \neq 0$. Applying (1.3) and induction on the number of non-zero summands of x, we obtain

$$\begin{aligned}
x^p &= \Big\{\Big[\alpha_b u_e + \sum_{g\neq b}\alpha_g(u_g u_b^{-1})\Big]u_b\Big\}^p \\
&= \Big[\alpha_b u_e + \sum_{g\neq b}\alpha_g(u_g u_b^{-1})\Big]^p u_b^p + (u_c - u_e)^2 z' \\
&= \Big[\alpha_b^p u_e + \sum_{g\neq b}\alpha_g^p(u_g u_b^{-1})^p + (u_c - u_e)^2 z''\Big]u_b^p + (u_c - u_e)^2 z' \\
&= \sum_{g\in G}\alpha_g^p u_g^p + (u_c - u_e)^2 z. \ \blacksquare
\end{aligned}$$

LEMMA 1.7. *Suppose* $p \neq 2$, $i_F \neq 0$, $p \,|\, |G'|$, *and* $\lambda \in Z^2(G, F^*)$. *Assume that* C_p *is cyclic*, $G_p/C_p = \langle a_1 C_p\rangle \times \ldots \times \langle a_m C_p\rangle$ *and* $C_p \not\subset \langle a_i\rangle$ *for all* $i \in \{1, \ldots, m\}$. *The algebra* $F^\lambda G$ *is of finite representation type if and only if* $F^\lambda G_p / F^\lambda G_p \cdot \operatorname{rad} FC_p$ *is a field.*

Proof. Necessity. I. First we examine the case when G_p is a group of exponent p. Taking into consideration Corollary 1 to Proposition 1.3 we may assume that G_p is non-abelian. Let $C_p = \langle c\rangle$ and suppose $F^\lambda G_p$ is of finite representation type. We prove that $V = F^\lambda G_p(u_c - u_e)$ is the radical of the algebra $F^\lambda G_p$.

Any element $g \in G_p$ can be uniquely represented in the form

$$g = a_1^{i_1} \ldots a_m^{i_m} c^j,$$

where $0 \le i_r$, $j < p$. Up to cocycle cohomology we may suppose

$$u_g = u_{a_1}^{i_1} \ldots u_{a_m}^{i_m} u_c^j, \tag{1.4}$$

where

$$u_{a_r}^p = \gamma_r u_e, \qquad u_c^p = u_e \qquad (\gamma_r \in F^*,\ 1 \le r \le m).$$

Let $\overline{F^\lambda G_p} = F^\lambda G_p / V$ and $\overline{x} = x + V$ for every $x \in F^\lambda G_p$. The algebra $\overline{F^\lambda G_p}$ is the commutative twisted group algebra $F^{\overline{\lambda}}\overline{G}_p$ of the group $\overline{G}_p = G_p/C_p$ and the field F with the cocycle $\overline{\lambda}$, where $\overline{\lambda}_{\overline{g}_1,\overline{g}_2} = \lambda_{g_1,g_2}$ for any $g_1, g_2 \in G_p$. Here $\overline{g} = gC_p$ for every $g \in G_p$. A natural F-basis of $F^{\overline{\lambda}}\overline{G}_p$ is formed by elements $\overline{u}_g$ $(g \in G_p)$ which by (1.4) can be uniquely represented in the form

$$\overline{u}_g = \overline{u}_{a_1}^{i_1} \ldots \overline{u}_{a_m}^{i_m},$$

where $\overline{u}_{a_r}^p = \gamma_r \overline{u}_e$, $1 \le r \le m$.

Suppose that V is not the radical of the algebra $F^\lambda G_p$. From Proposition 1.3 we conclude that up to reindexing $a_1, \ldots, a_m$ the algebra $F[\overline{u}_{a_1}, \ldots, \overline{u}_{a_{m-1}}]$ is a field and $F[\overline{u}_{a_1}, \ldots, \overline{u}_{a_{m-1}}, \overline{u}_{a_m}]$ is not. In this case

$$\gamma_m^{-1}\overline{u}_e = \overline{x}^p$$

for some

$$x = \sum_{i_1,\dots,i_{m-1}} \alpha_{i_1,\dots,i_{m-1}} u_{a_1}^{i_1} \dots u_{a_{m-1}}^{i_{m-1}},$$

where $\alpha_{i_1,\dots,i_{m-1}} \in F$, $0 \le i_j < p$ for $j = 1, \dots, m-1$. In view of Lemma 1.6,

$$x^p = \gamma_m^{-1} u_e + (u_c - u_e)^2 z', \qquad z' \in F^\lambda G_p,$$

and consequently

$$(xu_{a_m})^p = x^p u_{a_m}^p + (u_c - u_e)^2 z'' = u_e + (u_c - u_e)^2 z,$$

where $z'' \in F^\lambda G_p$, $z = \gamma_m z' + z''$. Let $w = xu_{a_m} - u_e$. Then $w^p = (u_c - u_e)^2 z$. We also have $\operatorname{rad} \overline{F^\lambda G_p} = \overline{F^\lambda G_p} \cdot \overline{w}$.

By Theorem 1.1 the algebra $F^\lambda G_p$ is uniserial. Applying the Morita Theorem (see [10, p. 507]) and [10, Corollary 62.31, p. 510] we conclude that $\operatorname{rad} F^\lambda G_p = F^\lambda G_p \cdot \theta = \theta \cdot F^\lambda G_p$, where $\theta^{p^2} = 0$ and $\theta^l \ne 0$ for every $l < p^2$. We also obtain $\operatorname{rad} \overline{F^\lambda G_p} = \overline{F^\lambda G_p} \cdot \overline{\theta}$. It follows that $\overline{w} = \overline{\theta} \cdot \overline{y'}$, where y' is an invertible element of $F^\lambda G_p$. The equality $u_c - u_e = \theta^p y''$, $y'' \in F^\lambda G_p$, now shows that $w = \theta y = z\theta$, where y and z are invertible in $F^\lambda G_p$. This makes it possible to take $\theta = w$. However,

$$w^{p(p+1)/2} = (u_c - u_e)^{p+1} \widetilde{z} = 0 \quad \text{and} \quad \frac{p+1}{2} < p.$$

This contradiction shows that V is the radical of $F^\lambda G_p$.

II. Now we examine the general case. Let $C_p = \langle c \rangle$, $\widetilde{G}_p = G_p/\langle c^p \rangle$, $\widetilde{C}_p = C_p/\langle c^p \rangle$, $\widetilde{g} = g\langle c^p \rangle$ for every $g \in G_p$, and $\widetilde{\lambda}_{\widetilde{a},\widetilde{b}} = \lambda_{a,b}$ for any $a, b \in G_p$. Then $\widetilde{\lambda} \in Z^2(\widetilde{G}_p, F^*)$, $F^{\widetilde{\lambda}} \widetilde{C}_p$ is the group algebra, $F^{\widetilde{\lambda}} \widetilde{G}_p$ is a quotient algebra of $F^\lambda G_p$ and $F^{\widetilde{\lambda}} \widetilde{G}_p / F^{\widetilde{\lambda}} \widetilde{G}_p \cdot \operatorname{rad} F^{\widetilde{\lambda}} C_p \cong F^\lambda G_p / F^\lambda G_p \cdot \operatorname{rad} F^\lambda C_p$. Suppose that $F^\lambda G_p$ is of finite representation type. Then so is $F^{\widetilde{\lambda}} \widetilde{G}_p$. We have $\widetilde{G}'_p \subset \widetilde{C}_p$ and $\widetilde{c}$ is a central element of order p. Let

$$\widetilde{b}_i = \widetilde{a}_i^{p^{r_i - 1}},$$

where p^{r_i} is the order of $a_i C_p$, $1 \le i \le m$. Denote by $\widetilde{T}$ the subgroup of $\widetilde{G}_p$ generated by $\widetilde{c}, \widetilde{b}_1, \dots, \widetilde{b}_m$. The exponent of $\widetilde{T}$ is p. From Lemma 1.2 and the result of case I, we conclude that $F^{\widetilde{\lambda}} \widetilde{T} / F^{\widetilde{\lambda}} \widetilde{T} \cdot \operatorname{rad} F^{\widetilde{\lambda}} \widetilde{C}_p$ is a field. Then so is $F^{\widetilde{\lambda}} \widetilde{G}_p / F^{\widetilde{\lambda}} \widetilde{G}_p \cdot \operatorname{rad} F^{\widetilde{\lambda}} \widetilde{C}_p$, and hence also $F^\lambda G_p / F^\lambda G_p \cdot \operatorname{rad} F^\lambda C_p$.

Sufficiency. If $F^\lambda G_p / F^\lambda G_p \cdot \operatorname{rad} F^\lambda C_p$ is a field, then $F^\lambda G_p$ is uniserial, and hence by Theorem 1.1 the algebra $F^\lambda G$ is of finite representation type. ■

Remark 1.1. If $p = 2$, then the necessity in Lemma 1.7 does not hold. Indeed, let F be a field of characteristic 2 with $i_F \ne 0$, and $G_2 = \langle a, b \rangle$ the dihedral group of order 8. Assume that $F^\lambda G_2$ is given by the defining relations

$$u_a^4 = u_e, \quad u_b^2 = \gamma u_e, \quad u_b^{-1} u_a u_b = u_a^3,$$

where $\gamma \in F^*$ and $\gamma \notin F^2$. In this case, $\operatorname{rad} F^\lambda G_2 = F^\lambda G_2(u_a - u_e)$. The algebra $F^\lambda G_2$ is uniserial, and hence of finite representation type. At the same time, $C_2 = G_2' = \langle a^2 \rangle$, $G_2/C_2 = \langle abC_2 \rangle \times \langle bC_2 \rangle$, $C_2 \not\subset \langle ab \rangle$, $C_2 \not\subset \langle b \rangle$ and $F^\lambda G_2 / F^\lambda G_2 \cdot \operatorname{rad} FC_2$ is not a field.

THEOREM 1.2. *Let G be a finite group, $p \neq 2$, $\overline{G}_p = G_p/C_p$, $\overline{g} = gC_p$ for every $g \in G_p$, $\lambda \in Z^2(G, F^*)$ and $\overline{\lambda}_{\bar{a},\bar{b}} = \lambda_{a,b}$ for any $a, b \in G_p$. The algebra $F^\lambda G$ is of finite representation type if and only if C_p is cyclic and one of the following conditions is satisfied*:

(1) *$F^{\overline{\lambda}} \overline{G}_p$ is a field*;

(2) *there is a decomposition $\overline{G}_p = \langle \overline{a}_1 \rangle \times \overline{D}$ with $\overline{D} = \langle \overline{a}_2 \rangle \times \ldots \times \langle \overline{a}_s \rangle$ such that $F^{\overline{\lambda}} \overline{D}$ is a field, and if $C_p \neq \{e\}$ then $C_p \subset \langle a_1 \rangle$ and $C_p \not\subset \langle a_j \rangle$ for all $j \in \{2, \ldots, s\}$.*

Proof. Suppose $F^\lambda G_p$ is of finite representation type. From Lemmas 1.2, 1.4 and 1.5 we deduce that C_p is a cyclic group. Let $C_p = \langle c \rangle$. Assume that G_p is not cyclic. In view of Proposition 1.3 we also suppose $c \neq e$. Suppose $\overline{G}_p = \langle \overline{a}_1 \rangle \times \ldots \times \langle \overline{a}_s \rangle$ is a group of type $(p^{m_1}, \ldots, p^{m_s})$. If

$$a_i^{p^{m_i}} = c^{pt_i}$$

for all $i \in \{1, \ldots, s\}$, then by Lemma 1.7, $F^{\overline{\lambda}} \overline{G}_p$ is a field. Suppose

$$a_1^{p^{m_1}} = c^{k_1}, \quad a_2^{p^{m_2}} = c^{k_2},$$

where $(k_1, p) = 1$, $(k_2, p) = 1$ and $m_1 \geq m_2$. There exists an integer l such that $lk_1 + k_2 \equiv 0 \pmod{p}$. Let $\widetilde{G}_p = G_p / \langle c^p \rangle$ and $\widetilde{g} = g\langle c^p \rangle$ for any $g \in G_p$. From the equality

$$(\widetilde{a}_1^{lp^{m_1 - m_2}} \cdot \widetilde{a}_2)^{p^{m_2}} = \widetilde{a}_1^{lp^{m_1}} \cdot \widetilde{a}_2^{p^{m_2}} = \widetilde{c}^{lk_1 + k_2} = \widetilde{e}$$

it follows that

$$(a_1^{lp^{m_1 - m_2}} \cdot a_2)^{p^{m_2}} = c^{pt},$$

so we may assume that

$$C_p = \langle a_1^{p^{m_1}} \rangle \quad \text{and} \quad a_j^{p^{m_j}} = c^{pt_j} \tag{1.5}$$

for all $j \in \{2, \ldots, s\}$. Let $\overline{D} = \langle \overline{a}_2 \rangle \times \ldots \times \langle \overline{a}_s \rangle$ and D be the subgroup of G_p generated by $c, a_2, \ldots, a_s$. By Lemma 1.2 the algebra $F^\lambda D$ is of finite representation type. In view of Lemma 1.7, $F^{\overline{\lambda}} \overline{D}$ is a field. This proves the necessity.

Let us prove the sufficiency. Keep the notation used in the proof of the necessity, and suppose that conditions (1.5) are satisfied. Assume also that

$F^{\bar\lambda}\overline{D}$ is a field and $F^{\bar\lambda}\overline{G}_p$ is not. Let $\{u_g : g \in G_p\}$ be a natural F-basis of $F^\lambda G_p$ and

$$u_{a_1}^{p^{m_1}} = \gamma_1 u_c, \quad u_{a_j}^{p^{m_j}} = \gamma_j u_c^{pt_j}, \quad 2 \le j \le s, \tag{1.6}$$

where $\gamma_i \in F^*$, $1 \le i \le s$. Let $c \ne e$, $U = F^\lambda G_p(u_c - u_e)$, and $V = F^\lambda G_p(u_c^p - u_e)$. We have

$$u_c u_g \equiv u_g u_c \pmod V, \quad u_a^p u_g \equiv u_g u_a^p \pmod V \tag{1.7}$$

for all $a, g \in G_p$. We suppose that $F^{\bar\lambda}\overline{G}_p = F^\lambda G_p/U$ and a natural F-basis of $F^{\bar\lambda}\overline{G}_p$ is formed by elements $u_{\bar g}$, where $u_{\bar g} := u_g + U$. Let K be the F-subalgebra of $F^\lambda G_p/U$ generated by $u_{a_j}^p + U$, $2 \le j \le s$, and L the F-subalgebra of $F^\lambda G_p/V$ generated by $u_{a_j}^p + V$, $2 \le j \le s$. By (1.7), L is commutative. In view of (1.6) the correspondence

$$u_{a_j}^p + U \mapsto u_{a_j}^p + V, \quad 2 \le j \le s,$$

extends to an F-homomorphism f of the field K onto L. Hence f is an isomorphism and L is a field.

Let p^d be the nilpotency index of the radical of the algebra $F^\lambda G_p/U$. Evidently $d \le m_1$. There exists an element

$$x = \sum_{i_2,\ldots,i_s} \alpha_{i_2,\ldots,i_s} u_{a_2}^{i_2} \ldots u_{a_s}^{i_s},$$

where $\alpha_{i_2,\ldots,i_s} \in F$, $0 \le i_j < p^{m_j}$, such that

$$x^{p^d} \equiv \gamma_1^{-1} u_e \pmod U.$$

Applying the isomorphism f, we obtain

$$\sum_{i_2,\ldots,i_s} \alpha_{i_2,\ldots,i_s}^{p^d} u_{a_2}^{i_2 p^d} \ldots u_{a_s}^{i_s p^d} \equiv \gamma_1^{-1} u_e \pmod V. \tag{1.8}$$

Let

$$w = x u_{a_1}^{p^{m_1-d}} - u_e.$$

Then $(F^\lambda G_p w + U)/U$ is the radical of the algebra $F^\lambda G_p/U$. By Lemma 1.6,

$$\begin{aligned} w^p &\equiv x^p u_{a_1}^{p^{m_1-d+1}} - u_e + (u_c - u_e)^2 z' \pmod V, \\ x^p &\equiv \sum_{i_2,\ldots,i_s} \alpha_{i_2,\ldots,i_s}^p u_{a_2}^{pi_2} \ldots u_{a_s}^{pi_s} + (u_c - u_e)^2 z'' \pmod V, \end{aligned} \tag{1.9}$$

where $z', z'' \in F^\lambda G_p$. It follows from (1.6), (1.8) and (1.9) that

$$w^{p^d} \equiv u_c - u_e + (u_c - u_e)^{2p^{d-1}} z \pmod V, \quad z \in F^\lambda G_p,$$

and hence

$$w^{p^d} = (u_c - u_e) y,$$

where y is an invertible element of $F^\lambda G_p$. We proved that $F^\lambda G_p w$ is the radical of the algebra $F^\lambda G_p$. Therefore, $F^\lambda G_p$ is uniserial. By Theorem 1.1 the algebra $F^\lambda G$ is of finite representation type. ∎

Corollary. *Let G be a finite group. If the algebra $F^\lambda G$ is of finite representation type for some $\lambda \in Z^2(G, F^*)$, then C_p is a cyclic group and the number of invariants of the group G_p/C_p does not exceed $i_F + 1$.*

Remark 1.2. Theorem 1.2 is true for $p = 2$ as well if we suppose that $G_2' \neq C_2$ in the case when G_2' is not the identity subgroup and C_2 is a cyclic group.

Theorem 1.3. *Suppose $G = G_p \times B$, $\lambda \in Z^2(G, F^*)$, and $F^\lambda G_p$ is a uniserial algebra. Then every indecomposable $F^\lambda G$-module can be uniquely represented, up to isomorphism, in the form $V\#W$, where V is an indecomposable $F^\lambda G_p$-module and W is a simple $F^\lambda B$-module. Moreover, the outer tensor product of any indecomposable $F^\lambda G_p$-module and any simple $F^\lambda B$-module is an indecomposable $F^\lambda G$-module.*

The proof of Theorem 1.3 is analogous to the one of Theorem 3.1 in [1], where the case of G_p abelian is investigated.

Lemma 1.8. *Suppose $p \neq 2$, $p \,|\, |G'|$ and C_p is cyclic. Assume that G contains $G_p \rtimes B$, where $[G_p, B] \neq \{e\}$. Then $G_p = C_p \rtimes H$, where H is an abelian subgroup and $[B, H] = \{e\}$.*

Proof. By hypothesis, $C_p = \langle c \rangle$, $|c| = p^n$ and $n \geq 1$. Let $T = G_p \rtimes B$. The subgroup C_p is normal in T. Let $b \in B$ and φ_b be the automorphism of C_p such that $\varphi_b(c) = bcb^{-1}$. The mapping $\varphi : b \mapsto \varphi_b$ is a homomorphism of the group B into $\operatorname{Aut} C_p$. Since $\operatorname{Aut} C_p$ is a cyclic group it follows that $\varphi(B)$ is cyclic. Let K be the kernel of φ. If $B/K = \langle gK \rangle$, then

$$(g^t k) c (g^t k)^{-1} = g^t c g^{-t}, \qquad k x k^{-1} = x$$

for all $k \in K$ and $x \in G_p$.

Let $gcg^{-1} = c^i$. Then $i \not\equiv 1 \pmod{p}$. Let $h \in G_p$ and $ghg^{-1} = hc^l$. Then $g(hc^s)g^{-1} = hc^{l+si}$. We choose s in such a way that $l + si \equiv s \pmod{p^n}$. If $gc^j g^{-1} = c^j$, then $j \equiv 0 \pmod{p^n}$. From this and the equality $h = hc^s c^{-s}$ it follows that $G_p = C_p \rtimes H$, where $H = \{h \in G_p : ghg^{-1} = h\}$. ∎

Remark 1.3. Suppose $p = 2$, $G = G_2 \rtimes B$ and $[G, G_2]$ is a cyclic group. Then $G = G_2 \times B$.

Theorem 1.4. *Suppose $p \neq 2$, $G = G_p \rtimes B$, $[G, G_p] = \langle c \rangle$, $|c| = p^n$ $(n > 0)$ and $[B, G_p] \neq \{e\}$. Then:*

(1) *$G_p = \langle c \rangle \rtimes H$, where H is abelian and $[B, H] = \{e\}$.*

(2) *Let $\lambda \in Z^2(G, F^*)$. The algebra $F^\lambda G$ is of finite representation type if and only if $F^\lambda H$ is a field.*

(3) *Suppose that $F^\lambda H$ is a field. Let $e_1, \ldots, e_d$ be a complete system of primitive pairwise orthogonal idempotents of the semisimple algebra $F^\lambda B$, and $V_{ij} = F^\lambda G(u_c - u_e)^i e_j$, where $i \in \{0, 1, \ldots, p^n - 1\}$, $j \in \{1, \ldots, d\}$. Then every left ideal V_{ij} of the algebra $F^\lambda G$ is indecomposable as a left $F^\lambda G$-module and any indecomposable $F^\lambda G$-module is isomorphic to one of these ideals. The ideals $V_{i_1 j_1}$ and $V_{i_2 j_2}$ are isomorphic if and only if $i_1 = i_2$ and the ideals $F^\lambda B e_{j_1}$, $F^\lambda B e_{j_2}$ of the algebra $F^\lambda B$ are isomorphic as $F^\lambda B$-modules.*

Proof. The first statement is a particular case of Lemma 1.8. The second statement follows from Lemma 1.7.

Suppose $F^\lambda H$ is a field. Then $\operatorname{rad} F^\lambda G = F^\lambda G(u_c - u_e)$. From the Morita Theorem (see [10, p. 507]) we conclude that $F^\lambda G$ is a serial algebra. In view of [2, Theorem 2], $e_1, \ldots, e_d$ is a complete system of primitive pairwise orthogonal idempotents of the semisimple algebra $A = F^\lambda H \otimes_F F^\lambda B$. By the Deuring–Noether Theorem ([8, p. 200]), we also have

$$Ae_r \cong Ae_s \ \Leftrightarrow\ F^\lambda B e_r \cong F^\lambda B e_s.$$

In view of [9, Theorem 6.8, p. 124], $e_1, \ldots, e_d$ is a complete system of primitive pairwise orthogonal idempotents of $F^\lambda G$. Furthermore, for $1 \le r, s \le d$ we have

$$F^\lambda G e_r \cong F^\lambda G e_s \ \Leftrightarrow\ Ae_r \cong Ae_s.$$

Applying Lemma 1.1 and [10, Lemma 62.28, p. 508], we finish the proof. ■

COROLLARY. *Keep the notation of Theorem* 1.4 *and suppose that $F^\lambda H$ is a field. Then every simple $F^\lambda G$-module is isomorphic to one of the ideals $V_{p^n-1,j}$; moreover, any ideal $V_{p^n-1,j}$, $1 \le j \le d$, is minimal.*

2. Projective representation types of finite groups. A group G is said to be of *finite* (resp. *infinite*) *PFR-type* if the number of indecomposable projective F-representations of the group G with a cocycle λ is finite (resp. infinite) for any $\lambda \in Z^2(G, F^*)$. Other groups are said to be of *mixed PFR-type.*

Let Γ and Γ' be equivalent projective matrix F-representations of G with a cocycle λ. Then there exists an invertible matrix C over F and a mapping $\alpha : G \to F^*$ such that $C^{-1}\Gamma(g)C = \alpha_g \Gamma'(g)$ for all $g \in G$. In this case,

$$\lambda_{a,b} = \frac{\alpha_a \alpha_b}{\alpha_{ab}}\, \lambda_{a,b}$$

for all $a, b \in G$. Hence, α is a linear F-character of the group G. But the number of linear F-characters of G is finite. Therefore, the number of pairwise inequivalent indecomposable projective F-representations of G with a cocycle λ is finite if and only if the algebra $F^\lambda G$ is of finite representation type. This allows one to define the type of projective F-representations of G as in the Introduction.

Applying Lemma 1.3 we may establish some connection between *PFR*-type of a group G and *PFR*-type of a Sylow p-subgroup G_p of G. If G_p is of finite (resp. infinite) *PFR*-type, then so is G. Suppose G_p is of mixed *PFR*-type. In view of Corollary 3 to Proposition 1.3, G_p is not cyclic. By Lemma 1.4 the group algebra FG is of infinite representation type. It follows that G is not of finite *PFR*-type. If G is of finite *PFR*-type, then by Lemma 1.4, G_p is cyclic, and hence, in view of Corollary 3 to Proposition 1.3, G_p is of finite *PFR*-type. If G is of infinite *PFR*-type, then G_p is not of finite *PFR*-type. If G is of mixed *PFR*-type, then G_p is also of mixed *PFR*-type.

Let G be a finite group and $p \,|\, |G'|$. The group G/G' can be written as a direct product of its Sylow q-subgroups G_qG'/G', where G_q is a Sylow q-subgroup of G and q is a prime divisor of $|G : G'|$. Denote by C_p a Sylow p-subgroup of G'. We shall assume that $C_p \subset G_p$ and $C_p \neq G_p$. Then $G'_p \subset C_p$, and hence $C_p \lhd G_p$. The group G_p/C_p is isomorphic to the Sylow p-subgroup G_pG'/G' of G/G'. Let $\varphi : G \to G/G'$ be the canonical homomorphism, $\psi : G/G' \to G_pG'/G'$ a projector and $\chi : G_pG'/G' \to G_p/C_p$ the isomorphism defined by $\chi(aG') = aC_p$ for any $a \in G_p$. Then

$$f = \chi\psi\varphi \tag{2.1}$$

is a homomorphism of G onto G_p/C_p. The restriction of f to G_p is the canonical homomorphism of G_p onto G_p/C_p.

LEMMA 2.1. *Let* $H = G_p/C_p$, $f : G \to H$ *be the epimorphism* (2.1), $\mu \in Z^2(H, F^*)$ *and* $\lambda_{a,b} = \mu_{f(a),f(b)}$ *for any* $a, b \in G$. *Then* $\lambda \in Z^2(G, F^*)$ *and* $\lambda_{x,y} = \lambda_{y,x} = 1$ *for all* $x \in G_p$, $y \in C_p$. *If* $V = F^\lambda G_p \cdot \operatorname{rad} FC_p$, *then* V *is an ideal of the algebra* $F^\lambda G_p$ *and* $F^\lambda G_p/V \cong F^\mu H$.

Proof. Direct calculation. ∎

THEOREM 2.1. *Suppose* $i_F \neq 0$, G *is a finite group*, $p \,|\, |G'|$ *and* G_p/C_p *is a direct product of* s *cyclic* p*-subgroups for* $C_p \neq G_p$. *Then*:

(1) *If* C_p *is not cyclic or* $s \geq i_F + 2$, *then* G *is of infinite PFR-type.*

(2) *If* G_p *is cyclic*, *then* G *is of finite PFR-type.*

(3) *If* C_p *is a cyclic group and* G_p *is not a cyclic group and* $1 \leq s \leq i_F$, *then* G *is of mixed PFR-type.*

(4) *Suppose* $C_p = \langle c \rangle$, $G_p/C_p = \langle a_1C_p \rangle \times \ldots \times \langle a_sC_p \rangle$ *and* $s = i_F + 1$. *If* $c \in \langle a_r \rangle$ *for some* $r \in \{1, \ldots, s\}$, *then* G *is of mixed PFR-type. If* $c \notin \langle a_j \rangle$ *for every* $j \in \{1, \ldots, s\}$ *and* $C_2 \neq G'_2$ *for* $p = 2$ *then* G *is of infinite PFR-type.*

Proof. The assertion for $p \neq 2$ follows from Theorem 1.2 and Lemmas 1.5, 2.1. Now we turn to the case when p is an arbitrary prime. Statements (1)–(3) follow from Lemmas 1.2–1.5, 2.1 and Corollary 3 to Proposition 1.3.

We prove (4). Let

$$c = a_1^{p^{m_1}}, \quad H = G_p/C_p, \quad \overline{H} = G_p/\langle a_1\rangle.$$

Then

$$\overline{H} \cong H/(\langle a_1\rangle/C_p) \cong \langle a_2C_p\rangle \times \ldots \times \langle a_sC_p\rangle.$$

There is a cocycle $\overline{\mu} \in Z^2(\overline{H}, F^*)$ such that $F^{\overline{\mu}}\overline{H}$ is a field. Let $\varphi : G_p \to \overline{H}$ be the canonical homomorphism. Put $\mu_{x,y} = \overline{\mu}_{\varphi(x),\varphi(y)}$ for any $x, y \in G_p$. Then $\mu \in Z^2(G_p, F^*)$. Let $\{u_x : x \in G_p\}$ be a natural F-basis of the algebra $F^\mu G_p$. We have

$$u_{a_1}^{p^{m_1}} = u_c, \quad u_c^{|c|} = u_e,$$

$\operatorname{rad} F^\mu G_p = F^\mu G_p(u_{a_1} - u_e)$ and $F^\mu G_p/\operatorname{rad} F^\mu G_p \cong F^{\overline{\mu}}\overline{H}$. Let $\pi : G_p \to G_p/C_p$ be the canonical homomorphism. If $\pi(x) = \pi(x')$ then $\varphi(x) = \varphi(x')$. It follows that the formula $\nu_{\pi(x),\pi(y)} = \overline{\mu}_{\varphi(x),\varphi(y)}$, where $x, y \in G_p$, gives a cocycle $\nu \in Z^2(H, F^*)$. In view of Lemma 2.1 there is a cocycle $\lambda \in Z^2(G, F^*)$ such that $\lambda_{a,b} = \nu_{f(a),f(b)}$ for all $a, b \in G$, where f is the epimorphism (2.1). If $a, b \in G_p$ then $\lambda_{a,b} = \nu_{\pi(a),\pi(b)} = \mu_{a,b}$. It follows that $F^\lambda G_p \cong F^\mu G_p$, and hence $F^\lambda G_p$ is a uniserial algebra. Applying Theorem 1.1 we conclude that $F^\lambda G$ is of finite representation type. But G_p is not cyclic. Therefore, by Lemma 1.4 the group algebra FG is of infinite representation type. Thus, the group G is of mixed *PFR*-type.

Let $|a_jC_p| = p^{m_j}$ and

$$a_j^{p^{m_j}} = c^{pt_j}$$

for every $j \in \{1, \ldots, s\}$. If $p \neq 2$ then by Lemma 1.7, G is of infinite *PFR*-type. Suppose $p = 2$, $G_2' \neq C_2$, $H = \langle c^2\rangle$ and $\lambda \in Z^2(G, F^*)$. Then $G_2' \subset H$ and $G_2/H = \langle cH\rangle \times \langle a_1H\rangle \times \ldots \times \langle a_sH\rangle$. In view of Lemma 1.5, $F^\lambda H$ is a group algebra and the set $V = F^\lambda G_2 \cdot \operatorname{rad} F^\lambda H$ is a two-sided ideal of the algebra $F^\lambda G_2$. The quotient algebra $F^\lambda G_2/V$ is a commutative twisted group algebra of the group G_2/H and the field F. From Corollary 3 to Proposition 1.3 we conclude that $F^\lambda G/V$ is of infinite representation type. From this and Lemma 1.3 it follows that G is of infinite *PFR*-type. ∎

Corollary 1. *Suppose $i_F = \infty$. If C_p is a non-cyclic group then G is of infinite PFR-type. If C_p is cyclic and G_p is not cyclic then G is of mixed PFR-type. If G_p is a cyclic group then G is of finite PFR-type.*

Corollary 2. *Suppose $i_F \neq 0$, $p \neq 2$, $G = G_p \rtimes B$, $[G, G_p] = \langle c\rangle$ and $[B, G_p] \neq \{e\}$. Suppose $G_p/\langle c\rangle$ is a direct product of s cyclic subgroups for $G_p \neq \langle c\rangle$. If $1 \leq s \leq i_F$ then G is of mixed PFR-type. If $s \geq i_F + 1$ then G is of infinite PFR-type. For $G_p = \langle c\rangle$ the group G is of finite PFR-type.*

Proof. Apply Theorems 1.4 and 2.1. ■

THEOREM 2.2. *Suppose $i_F \neq 0$, G is a finite group and $p \mid |G'|$. Assume that G_p is abelian and C_p is cyclic. Let s be the number of invariants of G_p. If $s = 1$ then G is of finite PFR-type. If $1 < s \leq i_F + 1$ then G is of mixed PFR-type. If $s \geq i_F + 2$ then G is of infinite PFR-type.*

Proof. From Lemma 1.3 and Corollary 3 to Proposition 1.3 we conclude that if $s = 1$ then G is of finite *PFR*-type, and if $s \geq i_F + 2$ then G is of infinite *PFR*-type. Let $1 < s \leq i_F + 1$ and $C_p = \langle c \rangle$. We have $G_p/C_p = \langle a_1 C_p \rangle \times \ldots \times \langle a_t C_p \rangle$, $t \leq s$. If $t \leq i_F$ then by Lemmas 1.3 and 2.1, G is of mixed *PFR*-type. Suppose that $t = i_F + 1$. If $c \notin \langle a_i \rangle$ for all $i \in \{1, \ldots, t\}$ then $G_p/H = \langle cH \rangle \times \langle a_1 H \rangle \times \ldots \times \langle a_t H \rangle$, where $H = \langle c^p \rangle$. This contradiction shows that $c \in \langle a_r \rangle$ for some $r \in \{1, \ldots, t\}$. In this case, G is also of mixed *PFR*-type, by Lemmas 1.3 and 2.1, Corollary 3 to Proposition 1.3 and Theorem 2.1. ■

PROPOSITION 2.1. *Suppose $i_F = 0$. If G_p is not cyclic then G is of infinite PFR-type. If G_p is cyclic then G is of finite PFR-type.*

Proof. The algebra $F^\lambda G_p$ is the group algebra FG_p for every $\lambda \in Z^2(G, F^*)$ (see [26, p. 43]). It remains to apply Lemmas 1.3 and 1.4. ■

We remark that Proposition 2.1 was, in fact, formulated in [7].

Two groups are said to be *PFR-isotypic* if they are of the same *PFR*-type. From the above results, we will derive necessary and sufficient conditions for G and G_p to be *PFR*-isotypic. In view of Lemmas 1.3, 1.5 and 2.1 groups G and G_p are *PFR*-isotypic if $C_p = G'_p$.

PROPOSITION 2.2. *Let G be a finite group with $p \mid |G'|$ and G_p an abelian group, and s the number of invariants of G_p. If C_p is cyclic then G and G_p are PFR-isotypic. If C_p is not cyclic then G and G_p are PFR-isotypic if and only if $s \geq i_F + 2$.*

Proof. If C_p is cyclic we apply Theorem 2.2. If C_p is not cyclic we apply the Corollary of Theorem 1.2 and Theorem 2.2. ■

PROPOSITION 2.3. *Suppose $i_F \neq 0$, G is a finite group, $p \mid |G'|$, and s is the number of invariants of G_p/G'_p. Assume that G_p is non-abelian and if G'_p is cyclic then $s \neq i_F+1$ for $p = 2$. The groups G and G_p are PFR-isotypic if and only if one of the following conditions holds*:

(1) *$s \geq i_F + 2$ or G'_p is non-cyclic*;
(2) *$s \leq i_F + 1$and C_p is cyclic*;
(3) *$s = i_F + 1$, G'_p is cyclic, C_p is non-cyclic and $G_p/G'_p = \langle b_1 G'_p \rangle \times \ldots \times \langle b_s G'_p \rangle$, where $G'_p \not\subset \langle b_j \rangle$ for every $j \in \{1, \ldots, s\}$.*

Proof. Apply Theorem 2.1. If condition (1) holds, then G_p is of infinite *PFR*-type. If condition (2) holds and $G'_p \neq C_p$, then by the same arguments as in the proof of Theorem 2.2 we can establish that G is of mixed *PFR*-type. Suppose that conditions (1) and (2) do not hold. Then $s \leq i_F + 1$, G'_p is cyclic and C_p is non-cyclic. In this case, G is of infinite *PFR*-type. The subgroup G_p is of infinite *PFR*-type if and only if $s = i_F + 1$ and $G_p/G'_p = \langle b_1 G'_p\rangle \times \ldots \times \langle b_s G'_p\rangle$, where $G'_p \not\subset \langle b_j\rangle$ for every $j \in \{1, \ldots, s\}$. ∎

COROLLARY. *Suppose* $i_F = \infty$, G *is a finite group and* $p \mid |G'|$. *The groups* G *and* G_p *are PFR-isotypic if and only if* C_p *is cyclic or* G'_p *is not cyclic.*

REFERENCES

[1] L. F. Barannyk, *Modular projective representations of direct products of finite groups*, Publ. Math. Debrecen 64 (2004) (in press).
[2] L. F. Barannyk and K. Sobolewska, *On modular projective representations of finite nilpotent groups*, Colloq. Math. 87 (2001), 181–193.
[3] R. Bautista, *On algebras of strongly unbounded representation type*, Comment. Math. Helv. 60 (1985), 392–399.
[4] R. Bautista, P. Gabriel, A.V. Roĭter and L. Salmerón, *Representation-finite algebras and multiplicative bases*, Invent. Math. 81 (1985), 217–285.
[5] S. D. Berman, *Representations of finite groups over an arbitrary field and over rings of integers*, Izv. Akad. Nauk SSSR Ser. Mat. 30 (1966), 69–132 (in Russian); English transl.: Amer. Math. Soc. Transl. (2) 64 (1967), 147–215.
[6] K. Bongartz, *Indecomposables are standard*, Comment. Math. Helv. 60 (1985), 400–410.
[7] S. B. Conlon, *Twisted group algebras and their representations*, J. Austral. Math. Soc. 4 (1964), 152–173.
[8] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York, 1962 (2nd ed., 1966).
[9] —, —, *Methods of Representation Theory with Applications to Finite Groups and Orders*, Vol. 1, Wiley, New York, 1981.
[10] —, —, *Methods of Representation Theory with Applications to Finite Groups and Orders*, Vol. 2, Wiley, New York, 1987.
[11] E. C. Dade, *Blocks with cyclic defect groups*, Ann. of Math. 84 (1966), 20–48.
[12] V. Dlab and C. M. Ringel, *Decomposition of modules over right uniserial rings*, Math. Z. 129 (1972), 207–230.
[13] —, —, *On algebras of finite representation type*, J. Algebra 33 (1975), 306–394.
[14] P. Dowbor and D. Simson, *Quasi-Artin species and rings of finite representation type*, J. Algebra 63 (1980), 435–443.
[15] Yu. A. Drozd and V. V. Kirichenko, *Finite Dimensional Algebras*, Springer, Berlin, 1994.
[16] W. Feit, *The Representation Theory of Finite Groups*, North-Holland, Amsterdam, 1982.
[17] J. A. Green, *On the indecomposable representations of a finite group*, Math. Z. 70 (1959), 430–445.
[18] P. M. Gudivok, *On modular representations of finite groups*, Dokl. Uzhgorod. Univ. Ser. Fiz.-Mat. 4 (1961), 86–87 (in Russian).

[19] P. M. Gudivok, *On boundedness of degrees of indecomposable modular representations of finite groups over principal ideal rings*, Dopovīdī Akad. Nauk USSR Ser. A 1971, 683–685 (in Ukrainian).

[20] W. H. Gustafson, *Group rings of finite representation type*, Math. Scand. 34 (1974), 58–60.

[21] D. G. Higman, *Indecomposable representations at characteristic p*, Duke Math. J. 21 (1954), 377–381.

[22] G. J. Janusz, *Indecomposable representations of groups with a cyclic Sylow subgroup*, Trans. Amer. Math. Soc. 125 (1966), 288–295.

[23] —, *Indecomposable modules for finite groups*, Ann. of Math. (2) 89 (1969), 209–241.

[24] —, *Faithful representations of p-groups at characteristic p, I*, J. Algebra 15 (1970), 335–351.

[25] —, *Faithful representations of p-groups at characteristic p, II*, ibid. 22 (1972), 137–160.

[26] G. Karpilovsky, *Group Representations*, Vol. 2, North-Holland Math. Stud. 177, North-Holland, 1993.

[27] F. Kasch, M. Kneser und H. Kupisch, *Unzerlegbare modulare Darstellungen endlicher Gruppen mit zyklischer p-Sylow-Gruppe*, Arch. Math. (Basel) 8 (1957), 320–321.

[28] H. Kupisch, *Projektive Moduln endlicher Gruppen mit zyklischer p-Sylow-Gruppe*, J. Algebra 10 (1968), 1–7.

[29] —, *Unzerlegbare Moduln endlicher Gruppen mit zyklischer p-Sylow-Gruppe*, Math. Z. 108 (1969), 77–104.

[30] H. Meltzer and A. Skowroński, *Group algebras of finite representation type*, Math. Z. 182 (1983), 129–148.

[31] —, —, *Correction to "Group algebras of finite representation type"*, ibid. 187 (1984), 563–569.

[32] L. A. Nazarova und A. V. Roĭter, *Kategorielle Matrizen-Probleme und die Brauer–Thrall-Vermutung*, Mitt. Math. Sem. Giessen 115 (1975), 1–153.

[33] W. F. Reynolds, *Twisted group algebras over arbitrary fields*, Illinois J. Math. 15 (1971), 91–103.

[34] A. V. Roĭter, *Unboundedness of the dimensions of the indecomposable representations of an algebra which has infinitely many indecomposable representations*, Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968), 1275–1282 (in Russian).

[35] A. Skowroński, *The representation type of group algebras*, in: CISM Courses and Lectures 287, Springer, Wien, 1984, 517–531.

[36] —, *Group algebras of polynomial growth*, Manuscripta Math. 59 (1987), 499–516.

[37] K. Sobolewska, *On the number of indecomposable representations with a given degree of a twisted group algebra over a field of characteristic p*, Słupskie Prace Mat.-Fiz. 2 (2002), 81–89.

Institute of Mathematics
Pedagogical Academy
Arciszewskiego 22b
76-200 Słupsk, Poland
E-mail: barannykleo@poczta.onet.pl
kamiles@poczta.onet.pl

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