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THE QUASI-HEREDITARY ALGEBRA ASSOCIATED TO THE RADICAL BIMODULE OVER A HEREDITARY ALGEBRA

BY

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Dedicated to Raymundo Bautista on the occasion of his sixtieth birthday

Abstract. Let Γ be a finite-dimensional hereditary basic algebra. We consider the radical $\operatorname{rad}\Gamma$ as a Γ-bimodule. It is known that there exists a quasi-hereditary algebra $\mathcal{A}$ such that the category of matrices over $\operatorname{rad}\Gamma$ is equivalent to the category of Δ-filtered $\mathcal{A}$-modules $\mathcal{F}(\mathcal{A},\Delta)$. In this note we determine the quasi-hereditary algebra $\mathcal{A}$ and prove certain properties of its module category.

1. Introduction. Recently it was observed in [BH2, Theorem 1.1] that matrices over upper triangular bimodules are closely related to Δ-filtered modules over quasi-hereditary algebras. To be more precise let Γ be a finite-dimensional directed algebra over some field k and let B be an upper triangular bimodule over Γ. In particular, the radical of a directed finite-dimensional algebra and all its subbimodules are upper triangular. For any bimodule B one can define the category of matrices $\operatorname{mat} B$ over B. Then there exists a quasi-hereditary algebra $\mathcal{A}$ such that the category $\operatorname{mat} B$ and the category $\mathcal{F}(\mathcal{A},\Delta)$ of Δ-filtered $\mathcal{A}$-modules are equivalent as exact categories. If we assume $\mathcal{A}$ to be basic, then it is unique up to isomorphism. Note that $\operatorname{mat} B$ has a natural exact structure and $\mathcal{F}(\mathcal{A},\Delta)$ is an exact category as an extension closed full subcategory of a module category (for more details we refer to Section 2). Quasi-hereditary algebras are very well understood and the category of Δ-filtered modules has many nice properties (see e.g. [DR] and [R2]). The results in this note are motivated by these results and our previous work on actions of algebraic groups, where we use particular cases of the results in Theorem 1.1.

In this note we consider a finite-dimensional hereditary basic algebra Γ over an algebraically closed field k. We assume for simplicity that we have a fixed isomorphism $\Gamma \simeq kQ$ with the path algebra of the unique quiver Q. Thus Q is a finite quiver without oriented cycles. The aim of this note is to consider the radical bimodule $B = \operatorname{rad}\Gamma$. In this situation we can describe the corresponding quasi-hereditary algebra $\mathcal{A}$ explicitly as follows. Let $\overline{Q}$ be

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the double of Q. The vertices $\overline{Q}_0$ of $\overline{Q}$ are just the vertices Q_0 of Q. The set of arrows $\overline{Q}_1$ of $\overline{Q}$ consists of the arrows Q_1 of Q and the formal opposite arrows Q_1^* of Q. For any arrow α in Q we denote its opposite by α^*. Thus the starting vertex $s(\alpha^*)$ of α^* equals the terminal vertex $t(\alpha)$ of α and $t(\alpha^*) = s(\alpha)$. Further, for any arrow α in Q we consider an element r_α in the path algebra of the double quiver $k\overline{Q}$; note that the algebra $k\overline{Q}$ is not finite-dimensional if Q_1 is non-empty. We define

$$r_\alpha = \alpha^*\alpha - \sum_{\beta\in Q_1\,:\,t(\beta)=s(\alpha)} \beta\beta^*.$$

Then we define the algebra $\mathcal{A}(Q)$ as the quotient of $k\overline{Q}$ by the ideal generated by all elements r_α, where α runs through the arrows in Q, and all elements $\alpha^*\beta$ for all pairs (α,β) of arrows in Q with $\alpha \neq \beta$ and $t(\alpha) = t(\beta)$. Note that Γ and Γ^{op} are both subalgebras and quotient algebras of $\mathcal{A}(Q)$. In particular, we can consider any Γ-module and any Γ^{op}-module as an $\mathcal{A}$-module, and we can restrict an $\mathcal{A}(Q)$-module to Γ and Γ^{op}.

Theorem 1.1. (1) *The category of Δ-filtered $\mathcal{A}(Q)$-modules* (*see* 2.3 *below*) *is equivalent to the category of matrices over the radical bimodule* rad Γ *over Γ.*

(2) *For an $\mathcal{A}(Q)$-module M the following conditions are equivalent*:

(a) *M admits a Δ-filtration*,
(b) *M has projective dimension at most one*, *and*
(c) *M is projective as a Γ-module.*

We prove the theorem in Section 4. For the directed quiver of type $\mathbb{A}$ part (1) was already used in [HR] and part (2) was already proven in [BHRR, Lemma 1]. For unexplained terminology we refer to Section 2.

We note that the algebra $\mathcal{A}(Q)$ already appears in [D] in a different context. In Section 2 we start with basic notation on matrices and quasi-hereditary algebras and recall some results of [D] we use in this note. In the last section we mention several generalizations and further consequences. For unexplained terminology on matrices over bimodules we refer to [BH2] whereas our basic reference on quasi-hereditary algebras is [DR]. For further basic results on quivers and matrices over bimodules we mention [R1] and [GR].

Note that in the definition of a quasi-hereditary algebra $\mathcal{A}$ there appears a certain order on the vertices of the quiver of $\mathcal{A}$. In our previous work [HR], [BH1] and [BH2] we considered the opposite order to the traditional one in [DR] and [BHRR], since it is better adapted to actions of algebraic groups. Here we also use the first convention and follow closely the approach in [BH2].

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2. Basic notation

2.1. *Quivers and finite-dimensional algebras.* Let Q be a finite quiver and Γ the path algebra of Q. The radical $\operatorname{rad}\Gamma$ of Γ is generated by the arrows Q_1 of Q. For any arrow α we denote by $s(\alpha)$ its starting vertex and by $t(\alpha)$ its terminal vertex. We denote the idempotent corresponding to the trivial path i, for i a vertex in Q, by e_i. Further, let $\mathcal{A}$ be a finite-dimensional basic algebra. Then $\mathcal{A} \simeq kQ/I$ for some admissible ideal I and some finite quiver Q (here Q can have cycles). We denote by $\operatorname{mod}\mathcal{A}$ the category of finitely generated left $\mathcal{A}$-modules, by $\{P_{\mathcal{A}}(i)\}_{i\in Q_0}$ a set of representatives of the indecomposable projective modules in $\operatorname{mod}\mathcal{A}$ and by $\{S(i)\}_{i\in Q_0}$ a set of representatives of the simple left $\mathcal{A}$-modules. We denote the category of projective $\mathcal{A}$-modules by $\operatorname{proj}\mathcal{A}$. This category is equivalent to its full subcategory consisting of objects of the form $\bigoplus_{i\in Q_0} P_{\mathcal{A}}(i)\otimes M_i$ for finite-dimensional vector spaces M_i (tensor products are always taken over k). Let B be a bimodule over $\mathcal{A}$. Such a bimodule admits a decomposition $B = \bigoplus e_iBe_j$ as a vector space. It is convenient to consider this bimodule B as a biadditive bifunctor

$$(\operatorname{proj}\mathcal{A})^{\mathrm{op}} \times \operatorname{proj}\mathcal{A} \to \operatorname{mod} k$$

which is contravariant in the first argument and covariant in the second argument. For projective modules

$$P = \bigoplus_{i\in Q_0} P_{\mathcal{A}}(i)\otimes M_i \quad \text{and} \quad Q = \bigoplus_{i\in Q_0} P_{\mathcal{A}}(i)\otimes N_i$$

one can define the functor B directly as

$$B(P,Q) := \bigoplus_{i,j\in Q_0} M_i^* \otimes e_iBe_j \otimes N_j.$$

For more details we also refer to [BH3, Section 3].

2.2. *Matrices over bimodules.* Let Γ be a finite-dimensional directed algebra and B be a Γ-bimodule (in this part Γ is not necessarily hereditary). The category of matrices over B consists of objects (P,f), where P is a projective Γ-module and f is an element of $B(P,P)$, where we consider B as a functor over the category of projective Γ-modules $\operatorname{proj}\Gamma$. A morphism between two objects (P,f) and (P',f') is an element g in $\operatorname{Hom}(P,P')$ satisfying $f'g = gf$. In [BH3, Section 3] we gave a description of an equivalent category in terms of vector spaces and linear maps. This is used in Section 4, where we finally prove Theorem 1. We restrict to our main interest: Γ is the path algebra of the quiver Q and B is the radical of Γ. We denote the vector space with basis the set of paths starting in i and ending in j by $W(i,j)$. Then the category of matrices over B is equivalent to the following category: the objects are tuples $(M_i, \phi)_{i\in Q_0}$, where M_i is a finite-dimensional vector space for each $i \in Q_0$ and $\phi = (\phi_{i,j})_{i,j\in Q_0}$ with $\phi_{i,j} \in M_i^* \otimes W(j,i) \otimes M_j$ for $i < j$. So we can consider $\phi_{i,j}$ as a map

$$\phi_{i,j} : M_i \to M_j \otimes W(j,i).$$

2.3. *Quasi-hereditary algebras.* Let $\mathcal{A}$ be a finite-dimensional basic algebra and assume we have chosen a total order on the vertices Q_0 of the quiver of $\mathcal{A}$. One can define modules $\Delta(i)$ as follows:

$$\Delta(i) := P_{\mathcal{A}}(i)/\mathrm{Im}\Big(\bigoplus_{j<i} \mathrm{Hom}_{\mathcal{A}}(P_{\mathcal{A}}(j), P_{\mathcal{A}}(i)) \otimes P_{\mathcal{A}}(j) \to P_{\mathcal{A}}(i)\Big).$$

By definition, an $\mathcal{A}$-module M admits a Δ-*filtration* if there exists a sequence of submodules

$$M = M_t \supset M_{t-1} \supset \ldots \supset M_1 \supset M_0 = \{0\}$$

with M_i/M_{i-1} isomorphic to some module $\Delta(j)$ for j in Q_0. The algebra $\mathcal{A}$ together with the chosen order on Q_0 is called *quasi-hereditary* if each projective module $P_{\mathcal{A}}(i)$ admits a Δ-filtration and the endomorphism ring of each $\Delta(i)$ is isomorphic to k.

The following lemma was shown in [D] (where the opposite orientation is used).

LEMMA 2.1. *The algebra $\mathcal{A}(Q)$ is quasi-hereditary for any order of Q_0 satisfying $s(\alpha) < t(\alpha)$ for each arrow α in Q. Moreover, any two such orders define equivalent quasi-hereditary algebras, that is, the categories of Δ-filtered modules coincide.*

Moreover, there exists a basis of $\mathcal{A}(Q)$ consisting of paths uv^*, where u is a path in Q, v^* is a path in Q^{op}, and the starting point $s(u)$ of u coincides with the terminal point $t(v^*)$ of v^* (we write paths from right to left). It is obvious from the relations that these elements generate the algebra $\mathcal{A}(Q)$. Moreover, from the dimension formula for $\mathcal{A}(Q)$ in [D, Section 1, Corollary (3)], we conclude that these elements already form a basis of $\mathcal{A}(Q)$. In particular, for any vertex i we have an exact sequence

$$0 \to \bigoplus_{\alpha\in Q_1 \,:\, t(\alpha)=i} P_{\mathcal{A}(Q)}(s(\alpha)) \to P_{\mathcal{A}(Q)}(i) \to \Delta(i) \to 0,$$

where the component with index α of the first map is just multiplication by α^*, viewed as an element of $\mathcal{A}$. Consequently, $\Delta(i)$ is projective precisely when i is a source in Q.

LEMMA 2.2. *Consider a map*

$$\delta : \bigoplus_{i\in Q_0} P_{\mathcal{A}(Q)}(i)^{a_i} \to \bigoplus_{j\in Q_0} P_{\mathcal{A}(Q)}(j)^{b_j},$$

where each component $P_{\mathcal{A}(Q)}(i) \to P_{\mathcal{A}(Q)}(j)$ of δ is given by multiplication with an element in the radical of Γ and not all exponents a_i are zero. Then δ is not *injective.*

Proof. Let i_0 be a maximal element in Q_0 (with respect to the chosen order) with $a_{i_0} \neq 0$. We restrict δ to some direct summand $P_{\mathcal{A}(Q)}(i_0)$. It is sufficient to show that this restriction is not injective. Consider a basis element wu of $P_{\mathcal{A}(Q)}(i_0)$ as above of maximal length. Such an element will be annihilated by right multiplication with any element in the radical of Γ, which shows that δ is not injective. ■

The following properties of the algebra $\mathcal{A}(Q)$ can be easily checked. The first one follows from the lemma above and part (2) since the global dimension of Γ is one. Part (2) follows from the results in [D]. Finally (3) follows from the definition.

LEMMA 2.3. (1) *The global dimension of $\mathcal{A}(Q)$ is at most two.*
(2) *The module $\Delta(i)$ is isomorphic to $P_\Gamma(i)$.*
(3) *For a source i in Q the module $\Delta(i)$ is isomorphic to $P_{\mathcal{A}(Q)}(i)$.*

2.4. *Koszul algebras.* A finite-dimensional hereditary algebra is graded by the length of paths. If we consider a homogeneous ideal J in the path algebra of a quiver Q then the factor algebra kQ/J is a graded algebra. Note that $\mathcal{A}(Q)$ is graded, since all relations are quadratic. Such a graded finite-dimensional algebra is called a *Koszul algebra* if each simple $\mathcal{A}$-module $S(i)$ admits a minimal projective resolution

$$\ldots \to P^j(i) \to \ldots \to P^1(i) \to P^0(i) \to S(i) \to 0,$$

where each component of each differential is given by multiplication with an element of degree one (such a resolution is also called linear, cf. e.g. [GMRS]).

3. Some properties of $\mathcal{F}(\mathcal{A},\Delta)$. In this section we prove some basic results. These results are used for the proof of our principal result in Section 3, and they might be of independent interest. From now on the quiver Q is fixed, Γ is the path algebra of Q and $\mathcal{A}$ equals $\mathcal{A}(Q)$.

LEMMA 3.1. *A Γ-module M has projective dimension* 2 *over $\mathcal{A}$ precisely when it is not a projective Γ-module.*

Proof. Consider a minimal projective resolution of M as a Γ-module

$$0 \to \bigoplus_{i\in I} \Delta(i) \xrightarrow{f} \bigoplus_{j\in J} \Delta(j) \to M \to 0$$

and assume I is non-empty, that is, M is not projective. Note that f is given by a matrix consisting of elements in the radical of Γ. Consequently, f induces a homomorphism of the projective $\mathcal{A}$-modules $\bar{f} : \bigoplus_{i\in I} P_{\mathcal{A}}(i) \xrightarrow{f} \bigoplus_{j\in J} P_{\mathcal{A}}(j)$ that is not injective (Lemma 2.2). On the other hand, a minimal projective resolution of M over $\mathcal{A}$ starts with

$$\bigoplus_{i\in I} P_{\mathcal{A}}(i) \oplus P^0 \xrightarrow{(\bar{f},*)} \bigoplus_{j\in J} P_{\mathcal{A}}(j) \to M,$$

for some homomorphism $* : P^0 \to \bigoplus_{j\in J} P_{\mathcal{A}}(j)$. Consequently, $(\bar{f},*)$ is not injective. Thus M is of projective dimension 2. Conversely, if M is a projective Γ-module, then it is a direct sum of copies of the modules $\Delta(i)$. Thus, it is of projective dimension at most one, since the modules $\Delta(i)$ admit a short projective resolution (see Section 2).

Lemma 3.2. *Let M be an $\mathcal{A}$-module of projective dimension at most one.*

(1) *Let U be a submodule of M. Then U has projective dimension at most one.*

(2) *The natural map* $\mathrm{Hom}_{\mathcal{A}}(\Delta(i), M) \otimes \Delta(i) \to M$ *is injective.*

(3) *Let i be a source in Q and let U be the submodule* $\mathrm{Hom}_{\mathcal{A}}(\Delta(i), M) \otimes \Delta(i)$ *of M according to* (2). *Then U is a projective $\mathcal{A}$-module, the quotient M/U is of projective dimension at most one and* $\mathrm{Hom}(\Delta(i), M/U) = \mathrm{Hom}(P_{\mathcal{A}}(i), M/U) = 0$. *Moreover, the module M/U is of projective dimension at most one over $\mathcal{A}/\mathcal{A}e_i\mathcal{A}$.*

Proof. If M is of projective dimension at most one, then $\mathrm{Ext}^2_{\mathcal{A}}(M,S) = 0$ for all $\mathcal{A}$-modules S. We consider the exact sequence

$$0 \to U \to M \to M/U \to 0$$

and obtain an exact sequence

$$\mathrm{Ext}^1_{\mathcal{A}}(U,S) \to \mathrm{Ext}^2_{\mathcal{A}}(M/U,S) \to \mathrm{Ext}^2_{\mathcal{A}}(M,S) = 0 \to \mathrm{Ext}^2_{\mathcal{A}}(U,S) \to 0,$$

since the global dimension of $\mathcal{A}$ is two. This shows (1).

We consider the image U of $\mathrm{Hom}_{\mathcal{A}}(\Delta(i), M) \otimes \Delta(i)$ in M. Thus U is a Γ-module and of projective dimension one over $\mathcal{A}$ by (1). Consequently, U is a projective Γ-module by Lemma 3.1 and U is isomorphic to a direct sum of copies of $\Delta(i)$. Since $\mathrm{End}_{\mathcal{A}}(\Delta(i)) = k$ assertion (2) follows.

Note that $P_{\mathcal{A}}(i) = \Delta(i)$ for any source i in Q. Thus $\mathrm{Ext}^1_{\mathcal{A}}(U,S) = 0$ for any $\mathcal{A}$-module S. Consequently, $\mathrm{Ext}^2_{\mathcal{A}}(M/U,S) = 0$ for any $\mathcal{A}$-module S and M/U is of projective dimension at most one. We apply $\mathrm{Hom}(\Delta(i), -)$ to the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\Delta(i), M) \otimes \Delta(i) \to M \to M/U \to 0$$

and obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\Delta(i), M) \otimes k \to \mathrm{Hom}_{\mathcal{A}}(\Delta(i), M) \to \mathrm{Hom}_{\mathcal{A}}(\Delta(i), M/U) = 0.$$

Thus, M/U is an $\mathcal{A}/\mathcal{A}e_i\mathcal{A}$-module. To proceed, we take a minimal projective resolution

$$0 \to P^1 \to P^0 \to M/U \to 0$$

of M/U as an $\mathcal{A}$-module. Since $\mathrm{Hom}_{\mathcal{A}}(\Delta(i), M/U) = 0$ the projective module P^0 does not contain $\Delta(i) = P_{\mathcal{A}}(i)$ as a direct summand. If we apply $\mathrm{Hom}(P_{\mathcal{A}}(i), -)$ to the projective resolution above we obtain

$$\mathrm{Hom}_{\mathcal{A}}(P_{\mathcal{A}}(i), P^1) = \mathrm{Hom}_{\mathcal{A}}(P_{\mathcal{A}}(i), P^0).$$

Any projective $\mathcal{A}/\mathcal{A}e_i\mathcal{A}$-module is a quotient of a projective $\mathcal{A}$-module; if we take direct sums we get projective modules

$$R^l = P^l/\mathrm{Im}(\mathrm{Hom}(P_{\mathcal{A}}(i), P^l) \otimes P_{\mathcal{A}}(i) \to P^l)$$

for $l = 0, 1$. Consequently,

$$0 \to R^1 \to R^0 \to M/U \to 0$$

is a projective resolution of M/U as an $\mathcal{A}/\mathcal{A}e_i\mathcal{A}$-module. ■

4. Proof of Theorem 1.1. We start with part (1). First we give an idea of the proof: one may consider a Δ-filtered module M as a projective Γ-module together with linear maps $M(\alpha^*)$ corresponding to the opposite arrows α^* for α in Q. Thus, each Δ-filtered module is isomorphic as Γ-module to $\bigoplus_i V_i \otimes P_\Gamma(i)$. It is easy to see from the defining relations of $\mathcal{A}$ that the module M as an $\mathcal{A}$-module is already determined by the restriction of the maps $M(\alpha^*)$ to $V_{t(\alpha)}$ and each choice of these restrictions defines an $\mathcal{A}$-module structure on M. So we can define a category $\mathcal{C}$ equivalent to the category of Δ-filtered $\mathcal{A}$ modules. Moreover, we can consider the restricted maps as elements in a bimodule over the path algebra of Γ. It is easy to see that this bimodule is the radical bimodule of Γ. Thus we finally get the desired equivalence. The details of the proof can be found below.

First we introduce the category $\mathcal{C}$. This category is defined analogously to the category of flags introduced in [HR] and [BH1] (see also [BH3, Section 6]). First we fix the quiver Q and denote by t the number of vertices in Q. Further we identify Q_0 with the natural numbers $\{1, \ldots, t\}$ so that the order is preserved. An object in $\mathcal{C}$ of dimension vector $d = (d_1, \ldots, d_t)$ consists of vector spaces V_i of dimension d_i and certain linear maps ϕ_α for any α in Q. We consider linear maps

$$\phi_\alpha : V_{t(\alpha)} \to \bigoplus_{i \le s(\alpha)} V_i \otimes W(i, s(\alpha))$$

for α an arrow in Q. An object in $\mathcal{C}$ is a tuple $(V,\phi) = (V_i, \phi_\alpha)_{i\in Q_0,\,\alpha\in Q_1}$. A morphism $f : (V,\phi) \to (U,\psi)$ in $\mathcal{C}$ consists of linear maps

$$f_i : V_i \to \bigoplus_j U_j \otimes W(j,i) \quad \text{for any } i = 1,\dots,t,$$

such that f commutes with the maps ϕ_α and ψ_α in a natural way as follows: We extend the linear maps ϕ_α to linear maps

$$\varPhi_\gamma : \bigoplus_{j\le t(\gamma)} V_j \otimes W(j,t(\gamma)) \to \bigoplus_{i\le s(\gamma)} V_i \otimes W(i,s(\gamma))$$

defining $\varPhi_\gamma(v\otimes w) := \sum_{\alpha\,:\,t(\alpha)=i} \phi_\alpha(v)\otimes u\alpha$ for any path $w = \gamma u$ and any vector v in V_i. For a path w not of this form we define $\varPhi_\gamma(v\otimes w) := 0$. Analogously we extend the maps ψ to maps $\varPsi$. Then the maps $\varPhi = (\varPhi_\gamma)$, $\varPsi = (\varPsi_\gamma)$ and $f = (f_i)$ have to commute as follows:

$$\begin{tikzcd}
\bigoplus_j V_j \otimes W(j,t(\gamma)) \arrow[r, "\varPhi_\gamma"] \arrow[d, "\bigoplus_j f_j\otimes \mathrm{id}"] & \bigoplus_i V_i \otimes W(i,s(\gamma)) \arrow[d, "\bigoplus_i f_i\otimes \mathrm{id}"] \\
\bigoplus_{l,j} U_l \otimes W(l,j) \otimes W(j,t(\gamma)) \arrow[d, "\bigoplus \mathrm{id}_U\otimes h"] & \bigoplus_{k,i} U_k \otimes W(k,i) \otimes W(i,s(\gamma)) \arrow[d, "\bigoplus \mathrm{id}_U\otimes h'"] \\
\bigoplus_l U_l \otimes W(l,t(\gamma)) \arrow[r, "\varPsi_\gamma"] & \bigoplus_k U_k \otimes W(k,s(\gamma)),
\end{tikzcd}$$

where the linear maps h and h' are the natural composition maps for paths. Note that we define morphisms in $\mathcal{C}$ so that they coincide with morphisms for $\varDelta$-filtered modules in $\mathcal{A}$-mod, if we identify the objects in both categories (Lemma 4.2). To be more precise we describe the category $\mathcal{C}$ also in a different way. We can identify an $\mathcal{A}$-module M together with a fixed isomorphism $M \simeq \bigoplus V_i \otimes P(i)$ (as $\varGamma$-modules) with an object in $\mathcal{C}$, since it determines linear maps ϕ_α in an obvious way. Moreover, such a module M admits a $\varDelta$-filtration. Consequently, the category of $\varDelta$-filtered $\mathcal{A}$-modules is a subcategory of $\mathcal{C}$. We show in Lemma 4.2 that both categories are also equivalent.

There exists a natural exact structure in $\mathcal{C}$: a sequence of objects and morphisms in $\mathcal{C}$ is exact if the corresponding maps on the vector spaces V_i are exact for each vertex i in Q. This is equivalent to saying that the corresponding sequence of $\mathcal{A}$-modules is exact. A simple object in $\mathcal{C}$ is one with $\dim V_i = 1$ for one vertex i and $V_j = 0$ for the remaining ones. We denote such a simple object by $P_{\mathcal{C}}(i)$.

We define a functor $F : \mathcal{C} \to \operatorname{mat}\operatorname{rad}\varGamma$. Let (V,ϕ) be an object in $\mathcal{C}$. Note that $\operatorname{rad}\varGamma$ has a basis consisting of the non-trivial paths in Q. Thus $\operatorname{rad}\varGamma = \bigoplus_{i<j} W(i,j)$. For any projective object $\bigoplus_{i=1}^t P_\varGamma(i)\otimes V_i$ in $\varGamma$-mod

an element b in $\operatorname{rad}\Gamma(\bigoplus_{i=1}^{t} P_\Gamma(i)\otimes V_i, \bigoplus_{i=1}^{t} P_\Gamma(i)\otimes V_i)$ consists of a set of elements $b_{i,j}$ in $\operatorname{Hom}(V_i, V_j)\otimes W(j,i)$.

We define the matrix $F((V,\phi)) = (\bigoplus_i P_\Gamma(i)\otimes V_i, b)$, where b is an element corresponding to ϕ as follows. Remember that $W(j,i)$ is naturally isomorphic to the direct sum of the vector spaces $W(j,s(\alpha))$, where the sum is taken over all α with $t(\alpha)=i$. Thus we can decompose $b_i = \bigoplus_j b_{i,j}$ into elements $(b_i)_\alpha$ in $\bigoplus_j \operatorname{Hom}(V_i,V_j)\otimes W(j,s(\alpha))$, since $b_i = \bigoplus_{t(\alpha)=i}(b_i)_\alpha$. Then we define $(b_i)_\alpha := \phi_\alpha$. The map F defined on objects becomes a functor in a natural way, and it is fully faithful by definition. Moreover, F is obviously an exact functor. Thus we have already proven the following lemma.

Lemma 4.1. *The functor $F:\mathcal{C}\to \operatorname{mat}\operatorname{rad}\Gamma$ is an equivalence of exact categories. The image of the simple object $P_{\mathcal{C}}(i)$ is the simple object $(P_\Gamma(i),0)$ in $\operatorname{mat}\operatorname{rad}\Gamma$.*

We also define a functor $G:\mathcal{C}\to \operatorname{mod}\mathcal{A}$. Let (V,ϕ) be an object in $\mathcal{C}$. First we define a projective Γ-module M just by $M := \bigoplus_{i\in Q_0} P_\Gamma(i)\otimes V(i)$. We can define an $\mathcal{A}$-module structure on M by specifying linear maps

$$M(\beta):\bigoplus V(i)\otimes W(i,s(\beta))\to \bigoplus V(j)\otimes W(j,t(\beta)) \quad \text{for } \beta \text{ in } Q_1^*$$

satisfying the desired relations. Then all these maps are uniquely determined by the component

$$\begin{aligned}\psi_\beta := M(\beta)|_{V(s(\beta))} : V(s(\beta)) = V(s(\beta))\otimes W(s(\beta),s(\beta))&\\ \to \bigoplus_j V(j)\otimes W(j,t(\beta))&\end{aligned}$$

for i in Q_0, and any choice of linear maps ψ_β defines an $\mathcal{A}$-module structure on M. Here we use the relations in $\mathcal{A}$: since $M(\alpha)$ is injective the maps $M(\beta^*)$ determine the map $M(\alpha^*)$ in the relation r_α, and conversely, any choice of the maps ψ defines a representation M. Given a matrix (V,ϕ) we define $\psi_{\alpha^*} := \phi_\alpha$. This way G becomes a fully faithful functor. Since any $\mathcal{A}$-module M is uniquely determined by the linear maps $M(\beta^*)$ and the restriction $M|_\Gamma$, and M is Δ-filtered precisely when $M|_\Gamma$ is projective, the functor G is also dense. Moreover G is exact by definition.

Lemma 4.2. *The functor $G:\mathcal{C}\to\operatorname{mod}\mathcal{A}$ induces an exact equivalence between $\mathcal{C}$ and $\mathcal{F}(\mathcal{A},\Delta)$. Moreover, the image of the simple object $P_{\mathcal{C}}(i)$ is the standard module $\Delta(i)$.*

Proof of Theorem 1.1. (1) Using the two lemmata above we obtain an equivalence between $\operatorname{mat}\operatorname{rad}\Gamma$ and $\mathcal{F}(\mathcal{A},\Delta)$ (the equivalence satisfies the conditions of [BH2, Theorem 1.1]).

(2) We prove (a) is equivalent to (b). Since the projective dimension of any module $\Delta(i)$ is at most one it is sufficient to show that any module

with projective dimension at most one admits a Δ-filtration. First we note that part (3) of Lemma 3.2 shows the claim via induction on the number of vertices of Q: if M/U admits a Δ-filtration then so does M. Since M/U is an $\mathcal{A}/\mathcal{A}e_i\mathcal{A}$-module of projective dimension at most one and $\mathcal{A}/\mathcal{A}e_i\mathcal{A}$ is isomorphic to the quasi-hereditary algebra $\mathcal{A}(Q\setminus\{i\})$ corresponding to the radical of the path algebra $k(Q\setminus\{i\})$, where $Q\setminus\{i\}$ is the quiver obtained from Q by deleting the source i and all arrows starting in i, we can apply our induction hypothesis to $\mathcal{A}(Q\setminus\{i\})$.

Finally, since the global dimension of $\mathcal{A}$ is 2, the equivalence between (a) and (c) follows from Lemma 3.1.

5. Generalizations. Finally, in this section we mention several generalizations and further properties of the quasi-hereditary algebra $\mathcal{A}$.

PROPOSITION 5.1. *The algebra $\mathcal{A}$ is a Koszul algebra.*

Proof. The simple module $S_{\mathcal{A}}(i)$ admits a short resolution with projective Γ-modules

$$0 \to \bigoplus_{\alpha\in Q_1\,:\,s(\alpha)=i} P_\Gamma(t(\alpha)) \to P_\Gamma(i) \to S_{\mathcal{A}}(i) \to 0.$$

Moreover, the module $P_\Gamma(i)$ is isomorphic to $\Delta(i)$ and the latter admits a short projective resolution as an $\mathcal{A}$-module

$$0 \to \bigoplus_{\alpha\in Q_1\,:\,t(\alpha)=i} P_{\mathcal{A}(Q)}(s(\alpha)) \to P_{\mathcal{A}(Q)}(i) \to \Delta(i) \to 0.$$

Combining both resolutions we get a double complex. The total complex of this double complex is a projective resolution

$$\begin{aligned} 0 \to \bigoplus_{\alpha\in Q_1\,:\,s(\alpha)=i}\ \bigoplus_{\beta\in Q_1\,:\,t(\beta)=t(\alpha)} P_{\mathcal{A}(Q)}(s(\beta)) \to &\bigoplus_{\alpha\in Q_1\,:\,t(\alpha)=i} P_{\mathcal{A}(Q)}(s(\alpha)) \\ &\oplus \bigoplus_{\alpha\in Q_1\,:\,s(\alpha)=i} P_{\mathcal{A}(Q)}(t(\alpha)) \to P_{\mathcal{A}(Q)}(i) \to S_{\mathcal{A}}(i) \to 0. \end{aligned}$$

This resolution is linear and, consequently, $\mathcal{A}$ is Koszul. ∎

A first generalization concerns the powers of the radical bimodule. Let $B^l := \operatorname{rad}^l \Gamma$ be the lth power of the radical of Γ. Then one can describe in a similar way the corresponding quasi-hereditary algebra $\mathcal{A}^l$ so that $\operatorname{mat} B^l$ and $\mathcal{F}(\mathcal{A}^l,\Delta)$ are equivalent categories. Instead of the quiver $\overline{Q}$ we consider the quiver $\overline{Q}^l$. Its vertices are the same as for Q. It consists of arrows α for each arrow α in Q, and for any path $u=\alpha_1\ldots\alpha_l$ of length l in Q of an arrow $\beta(u)$ with $s(\beta(u))=t(u)$ and $t(\beta(u))=s(u)$. Obviously, $k\overline{Q}^l$ is a subalgebra of $k\overline{Q}$. Let R be the ideal in $k\overline{Q}$ so that $\mathcal{A}=k\overline{Q}/R$. Then we obtain

$$\mathcal{A}^l = k\overline{Q}^l/(R\cap k\overline{Q}^l).$$

The algebra $\mathcal{A}^l$ is also Koszul and it has properties similar to $\mathcal{A}$, but a precise description is much more technical. Finally we note that even more general subbimodules of $\operatorname{rad} \Gamma$ can be handled if one combines our results with those in [BH3].

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