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# FULL MATRIX ALGEBRAS WITH STRUCTURE SYSTEMS

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**Abstract.** We study associative, basic $n\times n$ $\mathbb{A}$-full matrix algebras over a field, whose multiplications are determined by structure systems $\mathbb{A}$, that is, $n$-tuples of $n\times n$ matrices with certain properties.

**Introduction.** Let $D$ be a discrete valuation ring with a unique maximal ideal $\pi D$. It is standard to reduce homological properties of $D$-orders $\Lambda$ to those of factor algebras $\Lambda/\pi\Lambda$. For example, Gorenstein $D$-orders can be reduced to quasi-Frobenius $D/\pi D$-algebras. (See e.g. [7] and [9].) As another example, we recall a theorem of Jategaonkar. It is proved in [4] that there are only finitely many tiled $D$-orders in $\mathbb{M}_n(D)$ having finite global dimension for a fixed integer $n$ ($\geq 2$). The key idea of its proof comes from a fact concerning the structure of factor algebras $\Lambda/\pi\Lambda$ of tiled $D$-orders $\Lambda$, and [5, III Theorem 9]. We note that further information can be found in [3].

In this paper we introduce $\mathbb{A}$-full matrix algebras over a field to provide a framework for such factor algebras $\Lambda/\pi\Lambda$ of tiled $D$-orders $\Lambda$, and as an application of the framework, we study Frobenius $\mathbb{A}$-full matrix algebras. We also give an example to show that the class of $\mathbb{A}$-full matrix algebras is strictly larger than that of factor algebras of tiled $D$-orders.

In Section 1, we define an $n\times n$ $\mathbb{A}$-full matrix algebra $A$ determined by an $n$-tuple $\mathbb{A}$ of $n\times n$ matrices which we call a *structure system*, and we examine the Gabriel quiver of an $\mathbb{A}$-full matrix algebra $A$. We note that the notion of structure systems is a modification of structure constants of finite-dimensional algebras. (See e.g. [2].) In the study of tiled $D$-orders, irreducible $\Lambda$-lattices play an important role. For an irreducible $\Lambda$-lattice $L$, the $\Lambda/\pi\Lambda$-module $L/\pi L$ has dimension type $(1,\ldots,1)$. In Section 2, we define representation matrices of right $A$-modules of dimension type $(1,\ldots,1)$, and we show that for each indecomposable projective right $A$-module and each indecomposable injective right $A$-module, their representation matrices consist of a part of a structure system. In Section 3, we notice a relationship between tiled orders and $\mathbb{A}$-full matrix algebras. The factor algebras of tiled $D$-orders form a large class of $\mathbb{A}$-full matrix algebras. However, we

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give an example of a structure system which does not have corresponding tiled $D$-orders. In Section 4, we show that for an arbitrary permutation $\sigma$ such that $\sigma(i) \neq i$ for all $i$, there exists a Frobenius $\mathbb{A}$-full matrix algebra with Nakayama permutation $\sigma$, which is determined by a special structure system. We give a procedure to find other structure systems of Frobenius $\mathbb{A}$-full matrix algebras for a given permutation.

**1. Structure systems for full matrix algebras.** Let $K$ be a field and $n$ an integer with $n \geq 2$. Let $\mathbb{A} = (A_1, \ldots, A_n)$ be an $n$-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(k = 1, \ldots, n)$, which satisfies the following conditions:

(A1) $\quad a_{ij}^{(k)} a_{il}^{(j)} = a_{il}^{(k)} a_{kl}^{(j)} \quad$ for all $1 \leq i, j, k, l \leq n$.

(A2) $\quad a_{kj}^{(k)} = a_{ik}^{(k)} = 1 \quad$ for all $1 \leq i, j, k \leq n$.

(A3) $\quad a_{ii}^{(k)} = 0 \quad$ whenever $i \neq k,\ 1 \leq i, k \leq n$.

Let $A = \bigoplus_{1 \leq i,j \leq n} K u_{ij}$ be a $K$-vector space with basis $\{u_{ij} \mid 1 \leq i, j \leq n\}$. Then we define multiplication in $A$ by

$$u_{ik} u_{lj} := \begin{cases} a_{ij}^{(k)} u_{ij} & \text{if } k = l, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 1.1. *$A$ is an associative basic $K$-algebra and $u_{11}, \ldots, u_{nn}$ are orthogonal primitive idempotents with $1 = u_{11} + \ldots + u_{nn}$.*

*Proof.* For all $1 \leq i, j, k, l \leq n$, we have

$$(u_{ik} u_{kj}) u_{jl} = a_{ij}^{(k)} u_{ij} u_{jl} = a_{ij}^{(k)} a_{il}^{(j)} u_{il}$$

and

$$u_{ik} (u_{kj} u_{jl}) = u_{ik} (a_{kl}^{(j)} u_{kl}) = a_{il}^{(k)} a_{kl}^{(j)} u_{il}.$$

Hence the multiplication is associative if and only if (A1) holds.

It follows from (A2) that $u_{ii} u_{ij} = u_{ij} u_{jj} = u_{ij}$ and $u_{ii} A u_{ii} \cong K$ for all $1 \leq i, j \leq n$. Hence $u_{11}, \ldots, u_{nn}$ are orthogonal primitive idempotents with $1 = u_{11} + \ldots + u_{nn}$.

It follows from (A3) that $u_{ik} u_{ki} = 0$ whenever $i \neq k$, so that $u_{ii} A$ is not isomorphic to $u_{kk} A$. Hence $A$ is basic. This completes the proof. ∎

DEFINITION. An $n$-tuple $\mathbb{A} = (A_1, \ldots, A_n)$ of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ is said to be a *structure system* for a $K$-algebra $A = \bigoplus_{1 \leq i,j \leq n} K u_{ij}$ provided that (A1)–(A3) hold. In this case, we call $A$ an $\mathbb{A}$-*full matrix $K$-algebra*.

Next, we examine the Gabriel quiver $\mathcal{Q}(A)$ of an $\mathbb{A}$-full matrix algebra $A$ (see [1]). Since $A$ is basic, the set of vertices is $\mathcal{Q}(A)_0 = \{1, \ldots, n\}$.

PROPOSITION 1.2. (1) *The Jacobson radical of $A$ is* $J=\bigoplus\{Ku_{ij} \mid i \neq j, 1 \leq i,j \leq n\}$.

(2) *For any* $i,j \in \mathcal{Q}(A)_0$ *with* $i \neq j$, *there exists an arrow* $j \to i \in \mathcal{Q}(A)_1$ *if and only if* $a_{ij}^{(k)} = 0$ *for any* $k \neq i,j$.

(3) $\mathcal{Q}(A)$ *has no loops*, *and there exists at most one arrow from* $j$ *to* $i$ *in* $\mathcal{Q}(A)$, *for any* $i \neq j$.

*Proof.* (1) We can show that $J$ is a two-sided ideal of $A$, $J^n = 0$ and that $A/J \cong Ku_{11} \oplus \ldots \oplus Ku_{nn}$ is semisimple. Hence $J$ is the Jacobson radical of $A$.

(2) Note that $J^2 = \bigoplus\{Ku_{ij} \mid i \neq j, \text{ and } u_{ik}u_{kj} \neq 0 \text{ for some } k \neq i,j\}$. As $j \to i \in \mathcal{Q}(A)_1$ if and only if $u_{ii}(J/J^2)u_{jj} \neq 0$, (2) follows from the multiplication $u_{ik}u_{kj} = a_{ij}^{(k)}u_{ij}$.

(3) Note that the number of arrows from $j$ to $i$ is $\dim_K u_{ii}(J/J^2)u_{jj} \leq 1$, and that $u_{ii}Ju_{ii} = 0$. Hence (3) holds. ∎

Let $B$ be a finite-dimensional basic $K$-algebra, and let $e_1,\ldots,e_n$ be orthogonal primitive idempotents of $B$ with $1 = e_1 + \ldots + e_n$. For a right $B$-module $M$, let $m_i$ be the length of $Me_i$ as $e_iBe_i$-module. Then $(m_1,\ldots,m_n)$ is called the *dimension type* of $M$, denoted by $\underline{\dim}\, M$. The *Cartan matrix* $C_B$ of $B$ is the $n \times n$ matrix whose $i$th row is $\underline{\dim}\, e_iB$. It is well known that if $\mathrm{gl.dim}\, B < \infty$ then $\det C_B = \pm 1$. For $\mathbb{A}$-full matrix algebras $A$, we have the following proposition, whose proof is straightforward.

PROPOSITION 1.3. *Every entry of the Cartan matrix of an* $\mathbb{A}$-*full matrix algebra* $A$ *is* 1. *Therefore* $\mathrm{gl.dim}\, A = \infty$.

**2. Representation matrices of projectives and injectives.** Let $A$ be an $\mathbb{A}$-full matrix $K$-algebra with structure system $\mathbb{A} = (A_1,\ldots,A_n)$, where $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(k = 1,\ldots,n)$. In this section we study representation matrices of right $A$-modules of dimension type $(1,\ldots,1)$ to distinguish indecomposable projective $A$-modules and indecomposable injective $A$-modules.

PROPOSITION 2.1. *Let $M$ be a $K$-vector space with basis* $\{v_i \mid 1 \leq i \leq n\}$. *Then a right $A$-module structure of $M$ with* $\underline{\dim}\, M = (1,\ldots,1)$ *is determined by a matrix* $S = (s_{ij}) \in \mathbb{M}_n(K)$ *satisfying the condtion*

$$(*) \qquad s_{ik}s_{kj} = a_{ij}^{(k)}s_{ij} \quad \text{and} \quad s_{ii} = 1, \qquad \text{for all } 1 \leq i,j,k \leq n.$$

*In this case* $S = (s_{ij})$ *is determined by* $v_iu_{ij} = s_{ij}v_j$ *for all* $1 \leq i,j \leq n$.

*Proof.* It is well known that a right $A$-module structure of $M$ is determined by a $K$-algebra homomorphism $\varphi : A \to \mathbb{M}_n(K)$ as follows. For any $a \in A$, let $\varphi(a) = (a_{ij}) \in \mathbb{M}_n(K)$. Then $M$ becomes a right $A$-module if we set

$$v_i a := \sum_{j=1}^{n} a_{ij} v_j \quad \text{for all } 1 \le i \le n.$$

Conversely, if $M$ is a right $A$-module then the above equation defines a $K$-algebra homomorphism $\varphi : A \to \mathbb{M}_n(K)$, $a \mapsto (a_{ij})$.

Let $\varphi : A \to \mathbb{M}_n(K)$ be a $K$-algebra homomorphism. Put $\varphi(u_{ij}) = M_{ij} \in \mathbb{M}_n(K)$. Since $\underline{\dim}\, M = (1, \dots, 1)$, $\varphi(u_{ii}) \ne 0$ for all $1 \le i \le n$. Hence by Proposition 1.1, $M_{11}, \dots, M_{nn}$ are orthogonal primitive idempotents in $\mathbb{M}_n(K)$ with $E_n = M_{11} + \ldots + M_{nn}$, where $E_n$ is the identity matrix. Hence for some invertible matrix $P \in \mathbb{M}_n(K)$, $P^{-1} M_{ii} P = E_{ii}$ for all $1 \le i \le n$, where $E_{ii}$ is the usual matrix unit. (See e.g. [6, §3.7, Proposition 3].) Hence we may assume that $M_{ii} = E_{ii}$ for all $1 \le i \le n$. Since $M_{ij} = M_{ii} M_{ij} = M_{ij} M_{jj}$, excepting the $(i, j)$-entry of $M_{ij}$, all other entries are 0. We let $s_{ij}$ be the $(i, j)$-entry of $M_{ij}$, and let $S$ be the matrix $(s_{ij}) \in \mathbb{M}_n(K)$. Since $M_{ik} M_{kj} = \varphi(u_{ik} u_{kj}) = a_{ij}^{(k)} M_{ij}$, the condition $(*)$ holds.

Conversely, for a given matrix $S = (s_{ij}) \in \mathbb{M}_n(K)$ with $(*)$, we can define a $K$-algebra homomorphism $\varphi : A \to \mathbb{M}_n(K)$ by

$$\varphi\Big(\sum_{i,j} a_{ij} u_{ij}\Big) := (a_{ij} s_{ij})$$

for all elements $\sum_{i,j} a_{ij} u_{ij} \in A$. Since $\varphi(u_{ii}) = E_{ii}$, $\underline{\dim}\, M = (1, \dots, 1)$ and $v_i u_{ij} = s_{ij} v_j$. This completes the proof. ∎

We call the above $S = (s_{ij})$ a *representation matrix* of a right $A$-module $M$. Note that

$$v_i u_{jk} = \begin{cases} s_{ik} v_k & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 2.2. (1) *For each indecomposable projective right $A$-module $u_{ii} A$, its representation matrix is given by $(a_{ij}^{(k)})_{k,j}$, i.e., an $n \times n$ matrix whose $(k, j)$-entry is $a_{ij}^{(k)}$.*

(2) *Let $M$ be a right $A$-module with $\underline{\dim}\, M = (1, \dots, 1)$ and representation matrix $S = (s_{ij})$. Then $M$ is isomorphic to $u_{ll} A$ if and only if $s_{lk} = 1$ for all $1 \le k \le n$.*

*Proof.* (1) Note that $u_{ii} A$ is a $K$-vector space with basis $\{u_{ik} \mid 1 \le k \le n\}$ and $\dim u_{ii} A = (1, \dots, 1)$. Since $u_{ik} u_{kj} = a_{ij}^{(k)} u_{ij}$, the $(k, j)$-entry of the representation matrix of $u_{ii} A$ is $a_{ij}^{(k)}$.

(2) The "only if" part follows from (1) and (A2). Assume that $s_{lk} = 1$ for all $1 \le k \le n$. Then $s_{ij} = s_{li} s_{ij} = a_{lj}^{(i)} s_{lj} = a_{lj}^{(i)}$. This completes the proof. ∎

We denote the duality functor $\mathrm{Hom}_K(\ ,K)$ by $(\ )^*$. As a dual of Proposition 2.2, we have the following.

PROPOSITION 2.3. (1) *For each indecomposable injective right $A$-module $(Au_{jj})^*$, its representation matrix is given by $(a_{ij}^{(k)})_{i,k}$, i.e., an $n \times n$ matrix whose $(i,k)$-entry is $a_{ij}^{(k)}$.*

(2) *Let $M$ be a right $A$-module with $\underline{\dim}\, M = (1,\ldots,1)$ and representation matrix $S = (s_{ij})$. Then $M$ is isomorphic to $(Au_{kk})^*$ if and only if $s_{lk} = 1$ for all $1 \le l \le n$.*

Let $M$ be a right $A$-module with $\underline{\dim}\, M = (1,\ldots,1)$ and representation matrix $S = (s_{ij})$. Then we draw a *diagram* of $M$ as follows. (See [3].) The diagram has vertices $1,\ldots,n$ corresponding to composition factors of $M$. There is an arrow $j \to i$ if $s_{ij} \neq 0$ and $s_{ik}s_{kj} = 0$ for any $k \neq i,j$. Note that $a$ is in the top if $s_{ia} = 0$ for all $i \neq a$, and that $b$ is in the socle if $s_{bj} = 0$ for all $j \neq b$.

The following example illustrates the above observations.

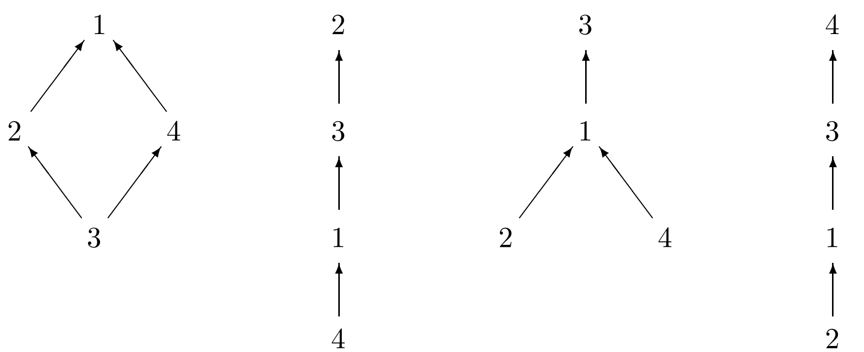
EXAMPLE 2.4. Let

$$\mathbb{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & \; & 0 & 1 & 1 & 0 & \; & 0 & 0 & 1 & 0 & \; & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & & 1 & 1 & 1 & 1 & & 1 & 0 & 1 & 1 & & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & & 0 & 1 & 0 & 0 & & 1 & 1 & 1 & 1 & & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & & 0 & 1 & 0 & 0 & & 1 & 1 & 1 & 0 & & 1 & 1 & 1 & 1 \end{pmatrix}$$

be a structure system of an $\mathbb{A}$-full matrix algebra $A$. Then representation matrices of $u_{11}A,\ldots,u_{44}A$ are given by

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

with the following diagrams, respectively:



**3. $\mathbb{A}$-full matrix algebras and tiled orders.** In this section we study a certain relationship between $\mathbb{A}$-full matrix algebras and tiled orders. We begin with the following simple example.

EXAMPLE 3.1. When $n = 2$, there exists a unique structure system, namely

$$\mathbb{A} = (A_1, A_2) = \begin{pmatrix} 1 & 1 & & 0 & 1 \\ 1 & 0 & & 1 & 1 \end{pmatrix}.$$

Now we recall the definition of tiled orders (see [4], [8], [10], [11]) . Let $D$ be a commutative discrete valuation domain with a unique maximal ideal $\pi D$. Let $n \geq 2$ be an integer. Let $\{\lambda_{ij} \mid 1 \leq i, j \leq n\}$ be the set of non-negative integers satisfying

$$\lambda_{ik} + \lambda_{kj} \geq \lambda_{ij}, \qquad \lambda_{ii} = 0, \qquad \lambda_{ij} + \lambda_{ji} > 0 \quad \text{if } i \neq j$$

for all $1 \leq i, j, k \leq n$. Then $\Lambda = (\pi^{\lambda_{ij}} D)$ is a $D$-subalgebra of $\mathbb{M}_n(D)$. We call $\Lambda$ an $n \times n$ *tiled $D$-order*. The following example provides us a prototype of $\mathbb{A}$-full matrix algebras.

EXAMPLE 3.2. Let $\Lambda$ be an $n \times n$ tiled $D$-order and $A = \Lambda/\pi\Lambda$ the factor ring of $\Lambda$. For each matrix unit $e_{ij} \in \mathbb{M}_n(D)$, let $u_{ij} \in A$ be the image of $\pi^{\lambda_{ij}} e_{ij} \in \Lambda$ via the canonical epimorphism $\Lambda \to A$. Let $K = D/\pi D$ be the residue field. Then $A$ is a $K$-algebra with basis $\{u_{ij} \mid 1 \leq i, j \leq n\}$. For each $k = 1, \ldots, n$, define $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ by

$$a_{ij}^{(k)} := \begin{cases} 1 & \text{if } \lambda_{ik} + \lambda_{kj} = \lambda_{ij}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{A} = (A_1, \ldots, A_n)$ is the structure system for the $K$-algebra $A$.

In what follows, we assume that every entry of structure systems of $\mathbb{A}$-full matrix algebras is 0 or 1.

When $n \leq 3$, for every structure system one can find a corresponding tiled $D$-order as in Example 3.2. The following example shows that, for $n = 4$, there exists a structure system which has no corresponding tiled $D$-orders.

EXAMPLE 3.3. Consider the following structure system:

$$\mathbb{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & & 0 & 1 & 1 & 0 & & 0 & 0 & 1 & 0 & & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & & 1 & 1 & 1 & 1 & & 0 & 0 & 1 & 1 & & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 1 & 0 & 0 & & 1 & 1 & 1 & 1 & & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & & 0 & 1 & 0 & 0 & & 0 & 1 & 1 & 0 & & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Suppose, to the contrary, that there exists a $4 \times 4$ tiled $D$-order $\Lambda = (\pi^{\lambda_{ij}} D)$ corresponding to $\mathbb{A}$. By [4, Lemma 1.1], we may assume that $\lambda_{1j} = 0$ for

$1 \leq j \leq 4$. Since $\lambda_{24} = \lambda_{21} + \lambda_{14}$, $\lambda_{13} = \lambda_{12} + \lambda_{23}$ and $\lambda_{24} = \lambda_{23} + \lambda_{34}$, we have $\lambda_{21} = \lambda_{24} = \lambda_{34}$. Since $\lambda_{13} = \lambda_{14} + \lambda_{43}$, $\lambda_{42} = \lambda_{43} + \lambda_{32}$ and $\lambda_{42} = \lambda_{41} + \lambda_{12}$, we have $\lambda_{32} = \lambda_{42} = \lambda_{41}$. Hence $\lambda_{31} < \lambda_{32} + \lambda_{21} = \lambda_{34} + \lambda_{41} = \lambda_{31}$, a contradiction.

**4. Frobenius $\mathbb{A}$-full matrix algebras.** In this section we study Frobenius $\mathbb{A}$-full matrix algebras. We begin by recalling the following well known fact. (See e.g. [2].)

PROPOSITION 4.1. *Let $B$ be a finite-dimensional basic $K$-algebra, and let $e_1, \ldots, e_n$ be orthogonal primitive idempotents of $B$ with $1 = e_1 + \ldots + e_n$. Then $B$ is Frobenius if and only if the socle of each $e_iB$ is simple and* $\mathrm{soc}(e_iB) \not\cong \mathrm{soc}(e_jB)$ *whenever $i \neq j$ $(1 \leq i, j \leq n)$. In this case, there is a permutation $\sigma$ of $\{1, \ldots, n\}$ (called a Nakayama permutation) such that* $\mathrm{soc}(e_iB) \cong \mathrm{top}(e_{\sigma(i)}B)$.

LEMMA 4.2. *Let $A$ be an $n \times n$ $\mathbb{A}$-full matrix algebra with structure system $\mathbb{A} = (A_1, \ldots, A_n)$ where $A_k = (a_{ij}^{(k)})$ $(1 \leq k \leq n)$. Then the following are equivalent.*

(1) *$A$ is a Frobenius algebra with Nakayama permutation $\sigma$.*

(2) *There exists a permutation $\sigma$ of $\{1, \ldots, n\}$ such that $\sigma(i) \neq i$ for all $1 \leq i \leq n$, and $a_{ij}^{(k)} = 1$ if $i = k$, $j = k$, or if $j = \sigma(i)$, for all $1 \leq i, j, k \leq n$.*

*Proof.* (1)⇒(2): Since $\dim u_{ii}A = (1, \ldots, 1)$, $\sigma(i) \neq i$ for all $1 \leq i \leq n$. Since $\mathrm{soc}(u_{ii}A) \cong \mathrm{top}(u_{\sigma(i)\sigma(i)}A)$, it follows from Propositions 2.2 and 2.3 that $a_{ij}^{(k)} = 1$ if $i = k$, $j = k$ or if $j = \sigma(i)$, for all $1 \leq i, k, j \leq n$.

(2)⇒(1): This follows from Propositions 2.2, 2.3 and 4.1. ∎

As an immediate application of Lemma 4.2, we have the following.

COROLLARY 4.3. *When $n = 2$, there is a unique structure system of a Frobenius $\mathbb{A}$-full matrix algebra.*

*Proof.* The structure system of Example 3.1 defines a Frobenius $\mathbb{A}$-full matrix algebra with Nakayama permutation $\sigma = (1\ 2)$. ∎

THEOREM 4.4. *Let $\sigma \in S_n$ be an arbitrary permutation such that $\sigma(i) \neq i$ for all $1 \leq i \leq n$. Then there exists a Frobenius $n \times n$ $\mathbb{A}$-full matrix algebra with Nakayama permutation $\sigma$.*

*Proof.* For all $1 \leq i, k, j \leq n$, we put

$$a_{ij}^{(k)} := \begin{cases} 1 & \text{if } i = k \text{ or } j = k \text{ or } j = \sigma(i), \\ 0 & \text{otherwise.} \end{cases}$$

Then by Lemma 4.2, it is sufficient to show that (A1)–(A3) hold. It is clear that (A2) holds. Since $\sigma(i) \neq i$ for all $1 \leq i \leq n$, (A3) holds. In order to show (A1), that is, $a_{ij}^{(k)} a_{il}^{(j)} = a_{il}^{(k)} a_{kl}^{(j)}$ for all $1 \leq i, k, j, l \leq n$, we need to check the following.

(1) If $a_{ij}^{(k)} = 0$ then $a_{il}^{(k)} = 0$ or $a_{kl}^{(j)} = 0$.
(2) If $a_{il}^{(j)} = 0$ then $a_{il}^{(k)} = 0$ or $a_{kl}^{(j)} = 0$.
(3) If $a_{il}^{(k)} = 0$ then $a_{ij}^{(k)} = 0$ or $a_{il}^{(j)} = 0$.
(4) If $a_{kl}^{(j)} = 0$ then $a_{ij}^{(k)} = 0$ or $a_{il}^{(j)} = 0$.

Suppose that $a_{ij}^{(k)} = 0$ and $a_{il}^{(k)} \neq 0$. Then we obtain $i \neq k$, $j \neq k$, $j \neq \sigma(i)$ and also $l = k$ or $l = \sigma(i)$. We need to show that $k \neq j$, $l \neq j$, $l \neq \sigma(k)$. In the case of $l = k$, we have $l \neq j$ because $j \neq k$, and since $\sigma(k) \neq k$, it follows that $l \neq \sigma(k)$. In the case of $l = \sigma(i)$, we have $l \neq j$ because $j \neq \sigma(i)$, and since $i \neq k$, it follows that $l = \sigma(i) \neq \sigma(k)$. Therefore we have $a_{kl}^{(j)} = 0$, so that (1) has been checked. We can check (2), (3) and (4) in a similar way. This completes the proof. ∎

It is obvious that the structure system given in the proof of Theorem 4.4 is not unique for Frobenius $\mathbb{A}$-full matrix algebras with a given Nakayama permutation. In order to find other structure systems, we use the following lemma.

LEMMA 4.5. *Let $\mathbb{A} = (A_1, \ldots, A_n) = (a_{ij}^{(k)})$ be a structure system whose $\mathbb{A}$-full matrix algebra is Frobenius with Nakayama permutation $\sigma$. Then the following statements hold.*

(1) *For distinct $1 \leq i, k, j \leq n$, $a_{ij}^{(k)} = 0$ whenever $j = \sigma(k)$ or $k = \sigma(i)$.*
(2) *Consider the set*

$$X := \{(i, k, j) \mid 1 \leq i, k, j \leq n \textit{ are distinct}, \ j \neq \sigma(i),\ j \neq \sigma(k),\ k \neq \sigma(i)\}.$$

*Then for any $(i, k, j) \in X$, $a_{ij}^{(k)} = a_{k\sigma(i)}^{(j)}$, and the correspondence $(i, k, j) \mapsto (k, j, \sigma(i))$ defines a bijection $\varphi : X \to X$.*

*Proof.* (1) For $(i, k, i, \sigma(k))$, $a_{i\sigma(k)}^{(k)} a_{k\sigma(k)}^{(i)} = a_{ii}^{(k)} a_{i\sigma(k)}^{(i)} = 0$ if $i \neq k$. Since $a_{k\sigma(k)}^{(i)} = 1$ by Lemma 4.2, we have $a_{ij}^{(k)} = 0$ if $j = \sigma(k)$.

For $(i, j, \sigma(i), j)$, $a_{i\sigma(i)}^{(j)} a_{ij}^{(\sigma(i))} = a_{ij}^{(j)} a_{jj}^{(\sigma(i))} = 0$ if $\sigma(i) = k (\neq j)$. Hence $a_{ij}^{(k)} = 0$ if $k = \sigma(i)$.

(2) For $(i, k, j, \sigma(i))$, since $a_{ij}^{(k)} a_{i\sigma(i)}^{(j)} = a_{i\sigma(i)}^{(k)} a_{k\sigma(i)}^{(j)}$, we have $a_{ij}^{(k)} = a_{k\sigma(i)}^{(j)}$. If $(i, k, j) \in X$ then we can verify that $(k, j, \sigma(i)) \in X$. Since $\sigma$ is a permutation, $\varphi : (i, k, j) \mapsto (k, j, \sigma(i))$ defines a bijection from $X$ to $X$. ∎

REMARK 4.6. When $n = 3$, the Nakayama permutation is cyclic and hence the set $X$ is empty, so that there is a unique structure system $\mathbb{A}$ whose $\mathbb{A}$-full matrix algebra is Frobenius.

In the following example, by applying the bijection $\varphi : X \to X$ of Lemma 4.5, we obtain structure systems of Frobenius $\mathbb{A}$-full matrix algebras in the case of $n = 4, 5$.

EXAMPLE 4.7. (1) Let $n = 4$ and $\sigma = (1\ 2\ 3\ 4)$. First observe that the set $X$ of Lemma 4.5 has the form $X = \{(1,4,3),(2,1,4),(3,2,1),(4,3,2)\}$. Next note that $X$ itself is a unique $\varphi$-orbit, i.e.,

$$(1,4,3) \mapsto (4,3,2) \mapsto (3,2,1) \mapsto (2,1,4)\ (\mapsto (1,4,3)).$$

If we put $a = a_{ij}^{(k)}$ for all $(i,k,j) \in X$, then Lemma 4.5(1) yields the following two structure systems:

$$\mathbb{A} = \begin{pmatrix} 1&1&1&1 & 0&1&0&0 & 0&1&1&0 & 0&1&a&1 \\ 1&0&1&a & 1&1&1&1 & 0&0&1&0 & 0&0&1&1 \\ 1&0&0&1 & a&1&0&1 & 1&1&1&1 & 0&0&0&1 \\ 1&0&0&0 & 1&1&0&0 & 1&a&1&0 & 1&1&1&1 \end{pmatrix},$$

where $a = 0$ or $1$.

(2) $n = 4$ and $\sigma = (1\ 2)(3\ 4)$: Observe that the set $X$ is empty. Hence the structure system is unique.

(3) $n = 5$ and $\sigma = (1\ 2\ 3\ 4\ 5)$: Observe that the set $X$ has two $\varphi$-orbits, i.e.,

$$X_1 = \{\varphi^t((2,1,4)) \mid 0 \leq t \leq 14\}, \qquad X_2 = \{\varphi^t((4,1,3)) \mid 0 \leq t \leq 4\}.$$

Put $a = a_{ij}^{(k)}$ for all $(i,k,j) \in X_1$ and $b = a_{ij}^{(k)}$ for all $(i,k,j) \in X_2$. Since $(2,1,4) \in X_1$ and $(2,4,1) \in X_2$, we have

$$ab = a_{24}^{(1)} a_{21}^{(4)} = a_{21}^{(1)} a_{11}^{(4)} = 0.$$

Hence we obtain three structure systems depending on $(a,b) = (0,0), (1,0)$, or $(0,1)$.

(4) $n = 5$ and $\sigma = (1\ 2)(3\ 4\ 5)$: Observe that the set $X$ is a $\varphi$-orbit $\{\varphi^t((3,1,5)) \mid 0 \leq t \leq 17\}$. Put $a = a_{ij}^{(k)}$ for all $(i,k,j) \in X$. Since $(1,3,5) = \varphi^{13}((3,1,5)) \in X$, we have $a^2 = a_{15}^{(3)} a_{35}^{(1)} = a_{11}^{(3)} a_{15}^{(1)} = 0$. Hence $a = 0$. Therefore the structure system is unique.

We note that there are corresponding Gorenstein tiled orders in each case, which can be found in [9, Examples].

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