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ON SPLIT-BY-NILPOTENT EXTENSIONS

BY

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Dedicated to Raymundo Bautista and Roberto Martínez-Villa on the occasion of their 60th birthdays

Abstract. Let A and R be two artin algebras such that R is a split extension of A by a nilpotent ideal. We prove that if R is quasi-tilted, or tame and tilted, then so is A. Moreover, generalizations of these properties, such as laura and shod, are also inherited. We also study the relationship between the tilting R-modules and the tilting A-modules.

Introduction. It is frequent in the representation theory of artin algebras to consider problems of the following type: Let A and R be artin algebras over a commutative artin ring k, and assume that the category $\operatorname{mod} A$ of finitely generated right A-modules is embedded in the category $\operatorname{mod} R$ of finitely generated right R-modules; then which properties of $\operatorname{mod} R$ are inherited by $\operatorname{mod} A$? In this paper, we study this problem in the following context: we let R and A be such that there exists a split surjective morphism $R \to A$ whose kernel Q is a nilpotent ideal of R. We say then that R is a split-by-nilpotent extension of A by Q; see [5, 6, 11, 15, 16]. We start by considering some of the classes of algebras that have been extensively studied in recent years in the representation theory of artin algebras, namely the quasi-tilted algebras [14], the shod algebras [9, 18], the weakly shod algebras [10], the left and right glued algebras [1], and finally, the laura algebras [3, 19, 22]. Our first main theorem says that, if R belongs to one of these classes, then so does A.

Theorem A. *Let R be a split-by-nilpotent extension of A by Q. Then*:

(a) *If R is laura, then so is A.*
(b) *If R is left (or right) glued, then so is A.*
(c) *If R is weakly shod, then so is A.*
(d) *If R is shod, then so is A.*
(e) *If R is quasi-tilted, then so is A.*

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We conjecture that, if R is a tilted algebra, then so is A. We prove here that this conjecture is true in the case when R is a tame algebra (see 2.6 below). In order to investigate the general case, we start with a given tilting R-module U, and we study under which conditions $U \otimes_R A$ is a tilting A-module. Such a tilting R-module is called restrictable. We show that this is indeed the case whenever $\mathrm{Tor}_1^R(U, A) = 0$ (see 3.2 below). This sufficient condition was obtained independently by Fuller [12] and Miyashita [17]. We recall that a tilting A-module T is extendable if $T \otimes_A R$ is a tilting R-module, and an R-module is induced if it is of the form $M \otimes_A R$ for some A-module M. The extendable tilting A-modules have been characterized in [5]. This leads us to our second main result.

THEOREM B. *The functors* $-\otimes_R A$ *and* $-\otimes_A R$ *induce mutually inverse bijections between the class of the induced tilting* R*-modules* U *such that* $\mathrm{Tor}_1^R(U, A) = 0$ *and the class of extendable tilting* A*-modules.*

We conclude the paper by giving conditions which are equivalent to the condition $\mathrm{Tor}_1^R(U, A) = 0$, and with some remarks and examples.

This paper consists of three sections. Section 1 is devoted to some basic facts about split-by-nilpotent extensions, Section 2 to our Theorem A, and Section 3 to our Theorem B.

1. Basic facts on split-by-nilpotent extensions. Throughout this paper, all algebras are artin algebras over a commutative artinian ring k. Unless otherwise specified, the modules are finitely generated right modules. We use freely, and without further reference, properties of the module categories and the almost split sequences as can be found, for instance, in [7, 20]. Let A and R be two artin algebras.

DEFINITION 1.1. We say that R is a *split extension* of A by the two-sided nilpotent ideal Q, or briefly a *split-by-nilpotent extension*, if there exists a surjective algebra morphism $\pi : R \to A$ whose kernel Q is a nilpotent ideal. This means that there exists a short exact sequence of A-A-bimodules

$$0 \longrightarrow Q \xrightarrow{\iota} R \underset{\sigma}{\overset{\pi}{\rightleftarrows}} A \longrightarrow 0$$

where ι denotes the inclusion and σ is an algebra map such that $\pi\sigma = 1_A$. In particular, A is a k-subalgebra of R. Note that since Q is nilpotent, Q is contained in $\operatorname{rad} R$ so that $\operatorname{rad} A = (\operatorname{rad} R)/Q$.

Let R and A be as above. We have the usual change of rings functors $-\otimes_A R : \operatorname{mod} A \to \operatorname{mod} R$, $-\otimes_R A : \operatorname{mod} R \to \operatorname{mod} A$, $\mathrm{Hom}_A(R_A, -) : \operatorname{mod} A \to \operatorname{mod} R$, and $\mathrm{Hom}_R(A_R, -) : \operatorname{mod} R \to \operatorname{mod} A$. The image of the functor $-\otimes_A R$ in $\operatorname{mod} R$ (or of the functor $\mathrm{Hom}_A(R_A, -)$ in $\operatorname{mod} R$) is called the subcategory of *induced* (or *coinduced*, respectively) *modules.*

We have the obvious natural isomorphisms $-\otimes_A R_R \otimes_R A_A \cong 1_{\operatorname{mod} A}$ and $\operatorname{Hom}_R(A_R, \operatorname{Hom}_A(R_A, -)) \cong 1_{\operatorname{mod} A}$. Moreover, an indecomposable R-module X is projective (or injective) if and only if there exists an indecomposable projective A-module P such that $X \cong P \otimes_A R$ (or an indecomposable injective A-module I such that $X \cong \operatorname{Hom}_A(R, I)$, respectively).

LEMMA 1.2. *If A is a connected algebra, then so is R.*

Proof. Since, for any two indecomposable projective A-modules P and P', the fact that $\operatorname{Hom}_A(P, P') \neq 0$ implies that $\operatorname{Hom}_R(P \otimes_A R, P' \otimes_A R) \neq 0$, the statement follows from the connectedness of A and from the fact that every indecomposable projective R-module is induced from an indecomposable projective A-module. ∎

The converse is not true as we shall see in the Example after 1.3.

We now explain how to construct examples of split extensions of algebras given by quivers and relations. We first give a necessary condition: let R and A be as above, and assume that R is given by a quiver with relations. We prove that A is obtained from R by factoring an ideal Q generated by arrows in the quiver of R. Conversely, starting from any given set of arrows in the quiver of R, and factoring out the ideal Q they generate, it is easy to check whether the induced surjection $R \to R/Q$ is a retraction or not. This technique, which we illustrate in an example below and later after our main theorem (A) in 2.5, has been used essentially in the proofs of the main results of [4] and [8].

Assume that $R = k\Gamma/I$ is a presentation of R as a quiver with relations. We say that a set S of generators of Q is *minimal* if, for each $\varrho + I$ in S, we have:

(a) If ϱ is a path in Γ, then for each proper subpath ϱ' of ϱ, $\varrho' + I$ does not belong to Q.

(b) If $\varrho = \sum_{1 \leq i \leq m} \lambda_i w_i$ with $m \geq 2$, the λ_i nonzero scalars and the w_i paths in Γ of positive length, all having the same source and the same target, then for each nonempty proper subset $J \subset \{1, \ldots, m\}$, we have that $\sum_{j \in J} \lambda_j w_j + I$ is not in Q.

PROPOSITION 1.3. *Let $R = k\Gamma/I$ be a split extension of A by Q. Then Q is generated by classes of arrows of Γ.*

Proof. We will construct a minimal set of generators of Q of the desired type. Let $\{\varrho_1, \ldots, \varrho_s\}$ be the preimages modulo I of any finite set of generators of Q. Notice that the set $\{e_a \varrho_i e_b : a, b \in \Gamma_0, 1 \leq i \leq s\}$ is a set of linear combinations of paths having the same source and the same target in Γ. Further, since $Q \subseteq \operatorname{rad} R$, all the paths involved in these linear combinations have length at least 1. Let $\sigma = \sum_{1 \leq j \leq m} \lambda_j w_j$ belong to this set, with $m \geq 2$, and assume that σ does not satisfy condition (b) in the definition of minimality. Then there exists a nonempty proper subset $J \subset \{1, \ldots, m\}$ such that, if $\sigma' = \sum_{j \in J} \lambda_j w_j$, then $\sigma' + I \in Q$. Since $\sigma = \sigma' + (\sigma - \sigma')$, we may replace σ by σ' in the above set of generators. Since the sum defining σ is finite, applying this procedure finitely many times yields another finite set $\{\sigma_1, \ldots, \sigma_n\}$ where all linear combinations of at least two paths satisfy condition (b). Furthermore, the set $\{\sigma_1 + I, \ldots, \sigma_n + I\}$ generates Q.

Assume that σ_i is a path and that it does not satisfy condition (a). Then there exist paths w_1, w_2 and σ_i' such that $\sigma_i' + I$ is in Q, and $\sigma_i = w_1 \sigma_i' w_2$. This procedure yields after at most finitely many steps the required minimal set of generators for Q. Let thus $\{\varrho_1, \ldots, \varrho_t\}$ be the preimages modulo I of such a minimal set. We now show that each $\varrho = \varrho_i$ is an arrow. Assume first that $\varrho = \sum_{1 \le j \le m} \lambda_j w_j$ with $m \ge 2$. By minimality, $w_j \notin Q$ for each j, thus $\lambda_j w_j + I$ is identified with a nonzero element of $A = R/Q$. So $\sum \lambda_j w_j + I$ belongs to A and it is nonzero in A since it is nonzero in R. On the other hand, $\varrho + Q = \sum_{1 \le j \le m} \lambda_j w_j + Q$ is zero in $A = R/Q$ since $\varrho + I \in Q$. This is a contradiction if $m \ge 2$, so we have established that each ϱ is a path. Assume now that ϱ is of length $l \ge 2$, thus $\varrho = \alpha_1 \ldots \alpha_l$, where the α_j are arrows. By minimality, $\alpha_j \notin Q$ for each j. Hence, for each j, $\alpha_j + I$ can be identified with a nonzero element of A. So $(\alpha_1 + I) \ldots (\alpha_l + I) = \alpha_1 \ldots \alpha_l + I \in A$ and is nonzero in A since it is nonzero in R. On the other hand, $\varrho + Q$ is zero in A so ϱ must be an arrow. ∎

Example. The converse of 1.2 is usually false; almost any algebra over an algebraically closed field is a counterexample. For instance, if we let $R = k\Gamma/I$, and Q be the ideal generated by an arbitrary set of arrows in Γ, then R need not be a split extension of R/Q by Q. More specifically, let $R = k\Gamma/I$, where Γ is the quiver

$$\begin{tikzcd}
 & \bullet \arrow[dr, "\beta"] & \\
\bullet \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & \bullet \\
 & \bullet \arrow[ur, "\delta"'] &
\end{tikzcd}$$

and I is the ideal generated by $\alpha\beta - \gamma\delta$. Let $Q_1 = \langle \alpha + I, \delta + I \rangle$ and $A_1 = R/Q_1$. Then it is easily seen that R is a split-by-nilpotent extension of A_1 by Q_1. However, if we let $Q_2 = \langle \alpha + I \rangle$ and $A_2 = R/Q_2$, then R is not a split-by-nilpotent extension of A_2 by an ideal, because A_2 is not a subalgebra of R. Indeed, in this case, $(\gamma + I)(\delta + I)$ is zero in A_2 but not in R.

Lemma 1.4. *Let R be a split extension of A by Q and let e be an idempotent of A. Then eRe is a split extension of eAe by eQe.*

Proof. Clearly, eQe is an ideal of eRe and it is nilpotent since $eQe \subseteq Q$. The map $\pi' : eRe \to eAe$ defined by $\pi'(e(a,q)e) = eae$ is a surjective algebra map having the map $eae \mapsto e(a,0)e$ as a section. Moreover, $\operatorname{Ker}\pi'$ contains eQe. Since $eRe = eAe \oplus eQe$ as k-modules, counting lengths yields the result. ∎

2. Inherited properties in split-by-nilpotent extensions. In this section, R denotes a split extension of A by the nilpotent ideal Q. It follows from [5, 2.2] that if R is hereditary, then so is A. This section is devoted to proving analogous results for other classes of algebras. Before proving our first results of this section, we recall that, by [5, 1.1], for any R-module X, the canonical epimorphism of R-modules $p_X : X \to X/XQ \cong X \otimes_R A$ is minimal.

LEMMA 2.1. *If $f : P_A \to M_A$ is a projective cover in* $\operatorname{mod} A$, *then the composition $p_{M\otimes R}(f \otimes R) : P \otimes_A R \to M$ is a projective cover in* $\operatorname{mod} R$.

Proof. Indeed, it is shown in [5, 1.3] that the induced homomorphism $f \otimes R : P \otimes_A R \to M \otimes_A R$ is a projective cover in $\operatorname{mod} R$. On the other hand, the above observation applied to $X = M \otimes_A R$ yields that the morphism $p_{M\otimes R} : M \otimes_A R \to M \cong M \otimes_A R \otimes_R A$ is a minimal morphism, and, since the composition of two minimal epimorphisms is a minimal epimorphism, our result follows directly from [7, I.4.1]. ∎

LEMMA 2.2. *Let M be an A-module. If* $\operatorname{pd} M_R \leq 1$, *then* $\operatorname{pd}(M \otimes_A R)_R \leq 1$.

Proof. (Compare [6, 1.1].) Let $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$ be a minimal projective presentation of M as an A-module. By [5, 1.3], we have an induced minimal projective presentation of $M \otimes_A R$ over R:

$$P_1 \otimes_A R \xrightarrow{f_1\otimes R} P_0 \otimes_A R \xrightarrow{f_0\otimes R} M \otimes_A R \longrightarrow 0.$$

Let now $0 \longrightarrow \widetilde{P}_1 \xrightarrow{\widetilde{f}_1} \widetilde{P}_0 \xrightarrow{\widetilde{f}_0} M \longrightarrow 0$ denote a minimal projective resolution of M over R. By 2.1, $\widetilde{P}_0 \cong P_0 \otimes_A R$ and we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & P_1 \otimes_A R \arrow[r, "f_1\otimes R"] \arrow[d] & P_0 \otimes_A R \arrow[r, "f_0\otimes R"] \arrow[d, equal] & M \otimes_A R \arrow[r] \arrow[d, "p_M"] & 0 \\
0 \arrow[r] & \widetilde{P}_1 \arrow[r] & P_0 \otimes_A R \arrow[r, "p_M(f_0\otimes R)"] & M \arrow[r] & 0
\end{tikzcd}$$

where $p_M : x \otimes (a,q) \mapsto xa$ (for $x \in M$, $a \in A$, $q \in Q$). In order to determine $\widetilde{P}_1$, we consider the bottom exact sequence as a sequence of A-modules. As A-modules, we have $P_0 \otimes_A R \cong P_0 \oplus (P_0 \otimes_A Q)$ and $M \otimes_A R \cong M \oplus (M \otimes_A Q)$.

As A-linear maps, $p_M = [1\ 0]$ and

$$f_0 \otimes R = \begin{bmatrix} f_0 & 0 \\ 0 & f_0 \otimes Q \end{bmatrix},$$

so that $p_M(f_0 \otimes R) = [f_0\ 0]$. We deduce an isomorphism of A-modules $\widetilde{P}_1 \cong \mathrm{Ker}[f_0\ 0] \cong \Omega^1_A M \oplus (P_0 \otimes_A Q)$. Let P be the projective cover of $P_0 \otimes_A Q$ in $\mathrm{mod}\, A$. We have a projective cover morphism in $\mathrm{mod}\, R$, denoted by $p : P \otimes_A R \to P_0 \otimes_A Q$. Since P_0 is projective and ${}_AQ_R$ is a subbimodule of ${}_AR_R$, it follows that $P_0 \otimes_A Q$ is an R-submodule of $P_0 \otimes_A R$. Letting $\widetilde{f}$ denote the composition of the inclusion with p, we get a commutative diagram with exact rows in $\mathrm{mod}\, R$:

$$\begin{tikzcd}
P_1 \otimes_A R \arrow[r, "f_1 \otimes R"] \arrow[d, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & P_0 \otimes_A R \arrow[r, "f_0 \otimes R"] \arrow[d, equal] & M \otimes_A R \arrow[r] \arrow[d, "p_M"] & 0 \\
(P_1 \oplus P) \otimes_A R \arrow[r, "{[f_1 \otimes R\ \widetilde{f}]}"] & P_0 \otimes_A R \arrow[r, "p_M(f_0 \otimes R)"] & M \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is a (usually not minimal) projective presentation of M_R. We claim that there exists a summand P' of P such that we have a commutative diagram with exact rows where the bottom row is a minimal projective presentation of M_R:

$$\begin{tikzcd}
 & P_1 \otimes_A R \arrow[r, "f_1 \otimes R"] \arrow[d, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & P_0 \otimes_A R \arrow[r, "f_0 \otimes R"] \arrow[d, equal] & M \otimes_A R \arrow[r] \arrow[d, "p_M"] & 0 \\
0 \arrow[r] & (P_1 \oplus P') \otimes_A R \arrow[r, "{[f_1 \otimes R\ \widetilde{f}']}"] & P_0 \otimes_A R \arrow[r, "p_M(f_0 \otimes R)"] & M \arrow[r] & 0
\end{tikzcd}$$

where $\widetilde{f}'$ denotes the restriction of $\widetilde{f}$ to $P' \otimes_A R$. In order to prove the claim, let P'' be a summand of $P_1 \oplus P$ such that

$$0 \longrightarrow P'' \otimes_A R \longrightarrow P_0 \otimes_A R \xrightarrow{p_M(f_0 \otimes R)} M \longrightarrow 0$$

is a minimal projective resolution of M over R. Tensoring this resolution with ${}_RA$, and using the fact that $M_A \cong M \otimes_R A$ (because M is annihilated by Q), we obtain a commutative diagram with exact rows in $\mathrm{mod}\, A$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^R(M, A) \arrow[r] & P'' \arrow[r, "f''"] & P_0 \arrow[r, "f_0"] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & 0 \\
 & & P_1 \arrow[r, "f_1"] & P_0 \arrow[r, "f_0"] & M \arrow[r] & 0
\end{tikzcd}$$

Since P_1 is a projective cover of $\Omega^1_A M$, there exists an epimorphism $P'' \to P_1$ induced from f''. Hence there is a decomposition $P'' \cong P_1 \oplus P'$ and the claim follows. Therefore,

$$f_1 \otimes R = [f_1 \otimes R\ \widetilde{f'}] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is the composition of two monomorphisms, so it is also a monomorphism and we have $\operatorname{pd} M \otimes_A R \leq 1$. ■

COROLLARY 2.3. *Let M be an A-module.*

(a) *If* $\operatorname{pd} M_R \leq 1$, *then* $\operatorname{pd} M_A \leq 1$.
(b) *If* $\operatorname{id} M_R \leq 1$, *then* $\operatorname{id} M_A \leq 1$.

Proof. (a) The previous lemma implies that $\operatorname{pd}(M \otimes_A R)_R \leq 1$, and by [5, 2.2], we get $\operatorname{pd} M_A \leq 1$.

(b) Assume that $\operatorname{id} M_R \leq 1$. Then $\operatorname{pd}_R(DM) \leq 1$. Observe that, as R-modules, ${}_R(DM)$ and ${}_A(DM)$ are isomorphic because M is annihilated by Q, hence the projective dimension in $\operatorname{mod} R^{\mathrm{op}}$ of ${}_A(DM)$ is at most one. By the first part of the corollary, $\operatorname{pd}_A(DM) \leq 1$ as an A-module, hence $\operatorname{id} M_A \leq 1$. ■

Let C be an algebra and let $\operatorname{ind} C$ denote a full subcategory of $\operatorname{mod} C$ consisting of a complete set of representatives of the isomorphism classes of indecomposable C-modules. Following [14], we let $\mathcal{L}_C$ denote the full subcategory of $\operatorname{ind} C$ consisting of those indecomposable C-modules U such that, if there exists an indecomposable C-module V and a sequence of nonzero C-morphisms

$$V = V_0 \longrightarrow V_1 \longrightarrow \ldots \longrightarrow V_n = U$$

with all the V_i indecomposable, then $\operatorname{pd} V_C \leq 1$. The subcategory $\mathcal{R}_C$ is defined dually.

LEMMA 2.4. *Let M be an indecomposable A-module.*

(a) *If $M \otimes_A R$ belongs to $\mathcal{L}_R$, then M belongs to $\mathcal{L}_A$.*
(b) *If $M \otimes_A R$ belongs to $\mathcal{R}_R$, then M belongs to $\mathcal{R}_A$.*
(c) *If* $\operatorname{Hom}_A(R, M)$ *belongs to $\mathcal{R}_R$, then M belongs to $\mathcal{R}_A$.*
(d) *If* $\operatorname{Hom}_A(R, M)$ *belongs to $\mathcal{L}_R$, then M belongs to $\mathcal{L}_A$.*

Proof. (a) Assume that we have a sequence of nonzero morphisms between indecomposable A-modules:

$$L = L_0 \longrightarrow L_1 \longrightarrow \ldots \longrightarrow L_n = M.$$

For each i, the R-module $L_i \otimes_A R$ is indecomposable and the induced R-homomorphism $f_i \otimes R : L_{i-1} \otimes_A R \to L_i \otimes_A R$ is nonzero. Thus we have an induced sequence of nonzero morphisms between indecomposable R-modules:

$$L \otimes_A R = L_0 \otimes_A R \longrightarrow L_1 \otimes_A R \longrightarrow \ldots \longrightarrow L_n \otimes_A R = M \otimes_A R$$

and, since $M \otimes_A R \in \mathcal{L}_R$, we have $\operatorname{pd}(L \otimes_A R)_R \leq 1$. By [5, 2.2], we infer that $\operatorname{pd} L_A \leq 1$.

(c) The proof is similar to that of (a).

(b) We have the following sequence of isomorphisms of k-modules:

$$\begin{aligned}\operatorname{Hom}_R(M \otimes_A R, \operatorname{Hom}_A({}_R R_A, M)) &\cong \operatorname{Hom}_A(M \otimes_A R \otimes_R R, M)\\ &\cong \operatorname{Hom}_A(M \otimes_A R_A, M)\\ &\cong \operatorname{Hom}_A(M \otimes_A (A \oplus Q), M)\\ &\cong \operatorname{Hom}_A(M, M) \oplus \operatorname{Hom}_A(M \otimes_A Q, M).\end{aligned}$$

Since $\operatorname{Hom}_A(M, M) \neq 0$, there exists a nonzero homomorphism of R-modules from $M \otimes_A R$ to $\operatorname{Hom}_A(R, M)$. Since $M \otimes_A R$ is in $\mathcal{R}_R$, we see that $\operatorname{Hom}_A(R, M) \in \mathcal{R}_R$. By (c), $M \in \mathcal{R}_A$.

(d) The proof is similar to that of (b). ■

We recall the following definitions. An artin algebra C is called a *laura* algebra if $\mathcal{L}_C \cup \mathcal{R}_C$ is cofinite in $\operatorname{ind} C$ (see [2, 22]). An artin algebra C is called *left* or *right glued* if the class of all U in $\operatorname{ind} C$ such that $\operatorname{id} U \leq 1$ (or $\operatorname{pd} U \leq 1$, respectively) is cofinite in $\operatorname{ind} C$ (see [1]). It is called *weakly shod* if the length of any path of nonzero morphisms between indecomposable modules from an injective module to a projective module is bounded (see [10]). It is *shod* if for each indecomposable C-module U, we have $\operatorname{pd} U \leq 1$ or $\operatorname{id} U \leq 1$ (see [9]). Finally, C is *quasi-tilted* if it is shod and $\operatorname{gldim} C \leq 2$ (see [14]). We are now able to prove the main result of this section.

THEOREM 2.5. (a) *If R is laura, then so is A.*

(b) *If R is left or right glued, then so is A.*

(c) *If R is weakly shod, then so is A.*

(d) *If R is shod, then so is A.*

(e) *If R is quasi-tilted, then so is A.*

Proof. (a) We first observe that if M is an indecomposable A-module and $M \notin \mathcal{L}_A \cup \mathcal{R}_A$ then, by 2.4, the R-module $M \otimes_A R \notin \mathcal{L}_R \cup \mathcal{R}_R$. Since R is a laura algebra, $\mathcal{L}_R \cup \mathcal{R}_R$ is cofinite in $\operatorname{ind} R$, hence $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\operatorname{ind} A$.

(b) The proof is similar since an algebra C is left glued (or right glued) if and only if $\mathcal{R}_C$ (or $\mathcal{L}_C$, respectively) is cofinite in $\operatorname{ind} C$ (see [2, 2.2]).

(c) It is proved in [3, 1.4] that an algebra C is weakly shod if and only if the length of any path of nonzero morphisms between indecomposable C-modules from a module $U \notin \mathcal{L}_C$ to a module $V \notin \mathcal{R}_C$ is bounded. Let $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} M_n$ be such a path in $\operatorname{ind} A$ with $M_0 \notin \mathcal{L}_A$ and $M_n \notin \mathcal{R}_A$. Then

$$M_0 \otimes_A R \xrightarrow{f_1 \otimes R} M_1 \otimes_A R \xrightarrow{f_2 \otimes R} \ldots \xrightarrow{f_n \otimes R} M_n \otimes_A R$$

is a path of nonzero morphisms in $\operatorname{ind} R$. Moreover, by 2.4, $M_0 \otimes_A R \notin \mathcal{L}_R$ and $M_n \otimes_A R \notin \mathcal{R}_R$. Since R is weakly shod, n is bounded.

(d) Let M be an indecomposable A-module. Since R is shod, $\operatorname{pd} M_R \leq 1$ or $\operatorname{id} M_R \leq 1$. The result follows now from 2.3.

(e) By [14, II.1.14] it suffices to show that if P is any indecomposable projective A-module, then $P \in \mathcal{L}_A$. Since $P \otimes_A R$ is an indecomposable projective R-module and R is quasi-tilted, we have $P \otimes_A R \in \mathcal{L}_R$. The result follows now from 2.4. ∎

Examples. Since, as observed in [6], one-point extensions are special cases of split-by-nilpotent extensions, it follows from [3, 3.4] that, if A is a tubular algebra, and R is a laura algebra, then R must be quasi-tilted. The following examples show that any of the remaining cases may occur. Here, in order to verify these examples, we apply systematically the technique outlined in 1.3. Let R be given by the quiver

$$\begin{tikzcd}
 & & 3 \arrow[dl, "\gamma_1"'] & & \\
1 & 2 \arrow[l, bend right, "\delta_1"'] \arrow[l, "\delta_2"] \arrow[l, bend left, "\delta_3"] & & 5 \arrow[ul, "\beta_1"'] \arrow[dl, "\beta_2"] & 6 \arrow[l, shift right, "\alpha_1"'] \arrow[l, shift left, "\alpha_2"] \\
 & & 4 \arrow[ul, "\gamma_2"] & &
\end{tikzcd}$$

where $\alpha_i\beta_j = 0$, $\gamma_i\delta_j = 0$ for all i, j, and $\beta_1\gamma_1 = 0$. Then R is a laura algebra that is not weakly shod.

(1) Let Q_1 be the ideal of R generated by δ_3; then R is a split extension of $A_1 = R/Q_1$ by Q_1, and A_1 is laura but not weakly shod.

(2) Let Q_2 be the ideal of R generated by α_1, α_2; then R is a split extension of $A_2 = R/Q_2$ by Q_2, and A_2 is right glued but not weakly shod.

(3) Let Q_3 be the ideal of R generated by β_2, γ_2; then R is a split extension of $A_3 = R/Q_3$ by Q_3, and A_3 is weakly shod but not shod.

(4) Let Q_4 be the ideal of R generated by β_1, γ_1; then R is a split extension of $A_4 = R/Q_4$ by Q_4, and A_4 is shod but not quasi-tilted.

(5) Let Q_5 be the ideal of R generated by $\alpha_1, \alpha_2, \beta_1, \beta_2$; then R is a split extension of $A_5 = R/Q_5$ by Q_5, and A_5 is quasi-tilted, and even tilted.

We conjecture that if R is a tilted algebra, then so is A. We have the following lemma.

Lemma 2.6. *Assume that A is connected. If R is a tilted algebra having a projective* (*or injective*) *indecomposable module in a connecting component of its Auslander–Reiten quiver*, *then A is tilted.*

Proof. Since A is connected, so is R by 1.2. Moreover, by 2.5, A is quasi-tilted. By [14, II 3.4], the hypothesis implies that up to duality there exists an indecomposable projective A-module P such that $P \otimes_A R \in \mathcal{R}_R$. By 2.4, $P_A \in \mathcal{R}_A$. Another application of [14, II.3.4]) establishes the statement. ∎

THEOREM 2.7. *Let R be a tame tilted algebra. Then so is A.*

Proof. By [13, III.6.5], and by 1.2 and 1.4, we may assume that A and R are both connected. Since R is tame, so is A. Therefore, there exists a projective or an injective indecomposable module in a connected component of the Auslander–Reiten quiver of R (see [21]). By 2.6, A is tilted. ∎

3. Restrictable and extendable tilting modules. As in Section 2, we assume that R is a split extension of A by the nilpotent ideal Q. Motivated by our conjecture just before 2.6, we now study the relationship between the tilting A-modules and the tilting R-modules. We recall from [5] that given an A-module T, the induced module $T \otimes_A R$ is a (partial) tilting R-module if and only if T_A is a (partial) tilting A-module and that we also have $\mathrm{Hom}_A(T \otimes_A Q, \tau_A T) = 0 = \mathrm{Hom}_A(D({}_AQ), \tau_A T)$. Such (partial) tilting modules are then called *extendable*. We now consider the opposite problem, namely: given a (partial) tilting R-module U, under which conditions is $U \otimes_R A$ a (partial) tilting A-module? We first give a sufficient condition. This condition has been obtained independently by Fuller [12] and Miyashita [16] using different proofs.

LEMMA 3.1. *Let $f : \widetilde{P} \to X$ be a projective cover in* $\mathrm{mod}\, R$. *Then $f \otimes A : \widetilde{P} \otimes_R A \to X \otimes_R A$ is a projective cover in* $\mathrm{mod}\, A$.

Proof. Clearly, $f \otimes A$ is an epimorphism and $\widetilde{P} \otimes_R A$ is a projective A-module. Moreover,

$$\begin{aligned}\mathrm{top}(\widetilde{P} \otimes_R A) &\cong \mathrm{top}(\widetilde{P}/\widetilde{P}Q) \cong \frac{\widetilde{P}/\widetilde{P}Q}{(\widetilde{P}/\widetilde{P}Q)\cdot \mathrm{rad}\, A} \\ &\cong \frac{\widetilde{P}/\widetilde{P}Q}{(\widetilde{P}/\widetilde{P}Q)(\mathrm{rad}\, R/Q)} \cong \frac{\widetilde{P}/\widetilde{P}Q}{(\widetilde{P}\cdot \mathrm{rad}\, R)/\widetilde{P}Q} \\ &\cong \widetilde{P}/\widetilde{P}\cdot \mathrm{rad}\, R \cong X/X\cdot \mathrm{rad}\, R \cong \mathrm{top}(X \otimes_R A).\end{aligned}$$

This establishes the result. ∎

REMARK. If $\widetilde{P}_1 \xrightarrow{f_1} \widetilde{P}_0 \xrightarrow{f_0} X \longrightarrow 0$ is a minimal projective presentation in $\mathrm{mod}\, R$, it does not follow that

$$\widetilde{P}_1 \otimes_R A \xrightarrow{f_1\otimes A} \widetilde{P}_0 \otimes_R A \xrightarrow{f_0\otimes A} X \otimes_R A \longrightarrow 0$$

is a minimal projective presentation in $\mathrm{mod}\, A$. Let, for instance, R be given by the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

and the relations $\alpha\beta\alpha = 0$ and $\beta\alpha\beta = 0$, and let A be the hereditary subalgebra with quiver

$$1 \xrightarrow{\alpha} 2.$$

Then the simple R-module S_2 corresponding to the vertex 2 has a minimal projective presentation $e_1R \xrightarrow{f_1} e_2R \xrightarrow{f_0} S_2 \longrightarrow 0$, where the image of f_1 is the radical of e_2R. Applying $- \otimes_R A$ yields a projective presentation

$$e_1A \xrightarrow{f_1\otimes A} e_2A \xrightarrow{f_0\otimes A} S_2 \otimes_R A \longrightarrow 0.$$

However, $\mathrm{Hom}_A(e_1A, e_2A) = 0$. Hence $f_1 \otimes A = 0$ and $f_0 \otimes A$ is an isomorphism $S_2 \otimes_R A \cong e_2A$. It is easy to show that $e_1A \cong \mathrm{Tor}_1^R(S_2, A)$.

LEMMA 3.2. *Let U be an R-module such that* $\mathrm{pd}\, U_R \le 1$ *and* $\mathrm{Tor}_1^R(U, A) = 0$. *Then*:

(a) $\mathrm{pd}(U \otimes_R A)_A \le 1$.
(b) $\tau_A(U \otimes_R A) \cong \mathrm{Hom}_A(A, \tau_R U)$.

Proof. (a) Let $0 \longrightarrow \widetilde{P}_1 \xrightarrow{f_1} \widetilde{P}_0 \xrightarrow{f_0} U \longrightarrow 0$ be a minimal projective resolution of U over R. In view of 3.1, the vanishing of $\mathrm{Tor}_1^R(U, A)$ implies that

$$(*) \qquad 0 \longrightarrow \widetilde{P}_1 \otimes_R A \xrightarrow{f_1\otimes A} \widetilde{P}_0 \otimes_R A \xrightarrow{f_0\otimes A} U \otimes_R A \longrightarrow 0$$

is a minimal projective resolution of $U \otimes_R A$ in $\mathrm{mod}\, A$.

(b) Applying $\mathrm{Hom}_A(-, A)$ to the sequence $(*)$ above, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Hom}_A(\widetilde{P}_0 \otimes_R A, A) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_A(\widetilde{P}_1 \otimes_R A, A) \arrow[r] \arrow[d, "\cong"] & \mathrm{Tr}(U \otimes_R A)_A \arrow[r] & 0 \\
\mathrm{Hom}_R(\widetilde{P}_0, \mathrm{Hom}_A({}_RA, A)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(\widetilde{P}_1, \mathrm{Hom}_A({}_RA, A)) \arrow[d, "\cong"] & & \\
\mathrm{Hom}_R(\widetilde{P}_0, A_R) \arrow[r] & \mathrm{Hom}_R(\widetilde{P}_1, A_R) \arrow[r] & \mathrm{Ext}_R^1(U, A) \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is obtained by applying $\mathrm{Hom}_R(-, A)$ to the given minimal projective resolution of U. Hence we have an isomorphism of A-modules

$$\mathrm{Tr}(U \otimes_R A)_A \cong \mathrm{Ext}_R^1(U, A)$$

and therefore we also get

$$\tau_A(U \otimes_R A) \cong D\,\mathrm{Ext}_R^1(U, A) \cong \mathrm{Hom}_R({}_AA_R, \tau_R U)$$

because $\mathrm{pd}\, U_R \le 1$ see [20]. ∎

The following is a sufficient condition for obtaining (partial) tilting modules over A from (partial) tilting modules over R. It would be interesting to know whether this condition is also necessary. Note that this result is a special case of [17, Lemma 5.1 and Theorem 5.2]. We give however an independent proof for the benefit of the reader, and also because we believe that our approach is more suitable for the actual computation of examples.

THEOREM 3.3. *Let U_R be a (partial) tilting module such that $\operatorname{Tor}_1^R(U,A) = 0$. Then $U \otimes_R A$ is a (partial) tilting A-module.*

Proof. By 3.2, $\operatorname{pd} U \otimes_R A \le 1$. We prove that $\operatorname{Ext}_A^1(U \otimes_R A, U \otimes_R A) = 0$. We have the following sequence of isomorphisms of k-modules:

$$\begin{aligned} D\operatorname{Ext}_A^1(U \otimes_R A, U \otimes_R A) &\cong \operatorname{Hom}_A(U \otimes_R A, \tau_A(U \otimes_R A)) \\ &\cong \operatorname{Hom}_R(U, \operatorname{Hom}_A({}_RA, \tau_A(U \otimes_R A))) \\ &\cong \operatorname{Hom}_R(U, \operatorname{Hom}_A({}_RA, \operatorname{Hom}_R({}_AA_R, \tau_R U))) \\ &\cong \operatorname{Hom}_R(U, \operatorname{Hom}_R({}_RA \otimes_A A_R, \tau_R U)) \\ &\cong \operatorname{Hom}_R(U, \operatorname{Hom}_R({}_RA_R, \tau_R U)). \end{aligned}$$

Applying the functor $\operatorname{Hom}_R(-, \tau_R U)$ to the exact sequence of R-R-bimodules

$$0 \longrightarrow Q \longrightarrow R \longrightarrow A \longrightarrow 0$$

yields a monomorphism of R-modules

$$0 \longrightarrow \operatorname{Hom}_R({}_RA_R, \tau_R U) \longrightarrow \operatorname{Hom}_R(R, \tau_R U) \cong \tau_R U$$

and we obtain an injection

$$0 \longrightarrow \operatorname{Hom}_R(U, \operatorname{Hom}_R({}_RA_R, \tau_R U)) \longrightarrow \operatorname{Hom}_R(U, \tau_R U) \cong D\operatorname{Ext}_R^1(U,U).$$

Since $\operatorname{Ext}_R^1(U,U) = 0$, we get $\operatorname{Ext}_A^1(U \otimes_R A, U \otimes_R A) = 0$, and so $U \otimes_R A$ is a partial tilting A-module. Finally, let $0 \to R \to U' \to U'' \to 0$ be a short exact sequence in $\operatorname{mod} R$ with U' and U'' in $\operatorname{add} U$. Tensoring this sequence with ${}_RA$ yields the exact sequence $0 \to A \to U' \otimes_R A \to U'' \otimes_R A \to 0$ since $\operatorname{Tor}_1^R(U,A) = 0$. Also, $U' \otimes_R A$ and $U'' \otimes_R A$ are both in $\operatorname{add}(U \otimes_R A)$ and this completes the proof of the theorem. ∎

We call a (partial) tilting R-module U *restrictable* if $U \otimes_A R$ is a (partial) tilting A-module. We have just shown that if $\operatorname{Tor}_1^R(U,A) = 0$, then U is restrictable. We now prove the main result of this section.

THEOREM 3.4. *The functors $- \otimes_R A$ and $- \otimes_A R$ induce mutually inverse bijections between the class of the induced tilting R-modules U such that $\operatorname{Tor}_1^R(U,A) = 0$ and the class of the extendable tilting A-modules.*

Proof. Assume that T is an extendable tilting A-module. We show first that $\operatorname{Tor}_1^R(T \otimes_A R, A) = 0$. Let $0 \to P_1 \to P_0 \to T \to 0$ be a minimal projective resolution of T over A. Using [5, 1.3], we have a minimal projective resolution of $T \otimes_A R$ in $\operatorname{mod} R$:

$$0 \longrightarrow P_1 \otimes_A R \longrightarrow P_0 \otimes_A R \longrightarrow T \otimes_A R \longrightarrow 0.$$

Applying $- \otimes_R A$ to this resolution yields a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Tor}_1^R(T \otimes_A R, A) \arrow[r] & P_1 \otimes_A R \otimes_R A \arrow[r] \arrow[d, "\cong"] & P_0 \otimes_A R \otimes_R A \arrow[d, "\cong"] \\ & 0 \arrow[r] & P_1 \arrow[r] & P_0 \end{tikzcd}$$

hence $\operatorname{Tor}_1^R(T \otimes_A R, A) = 0$. Suppose now that $U_R = T \otimes_A R$ is an induced tilting R-module such that $\operatorname{Tor}_1^R(U, A) = 0$. By 3.3, $U \otimes_R A \cong T$ is a tilting A-module and it is clearly extendable. ∎

We now discuss the torsion pair corresponding to a restrictable tilting R-module U. We recall that if W is a tilting module over an algebra C, then W determines a torsion pair $(\mathcal{T}(W), \mathcal{F}(W))$, where $\mathcal{T}(W) = \{V_C : \operatorname{Ext}_C^1(W, V) = 0\}$ and $\mathcal{F}(W) = \{V_C : \operatorname{Hom}_C(W, V) = 0\}$.

PROPOSITION 3.5. *Let U be a restrictable tilting R-module and let M be an A-module. Then*:

(a) $M_A \in \mathcal{F}(U \otimes_R A)$ *if and only if* $M_R \in \mathcal{F}(U)$.
(b) $M_A \in \mathcal{T}(U \otimes_R A)$ *if and only if* $M_R \in \mathcal{T}(U)$.

Moreover, if $(\mathcal{T}(U), \mathcal{F}(U))$ is a splitting torsion pair, then so is the torsion pair $(\mathcal{T}(U \otimes_R A), \mathcal{F}(U \otimes_R A))$.

Proof. For (a) we have

$$\operatorname{Hom}_A(U \otimes_R A, M) \cong \operatorname{Hom}_R(U, \operatorname{Hom}_A({}_A A_R, M)) \cong \operatorname{Hom}_R(U, M_R).$$

For (b) we have

$$\begin{aligned} &\operatorname{Ext}_A^1(U \otimes_R A, M) \\ &\quad \cong D\operatorname{Hom}_A(M, \tau_A(U \otimes_R A)) \cong D\operatorname{Hom}_A(M, \operatorname{Hom}_R({}_R A, \tau_R U)) \\ &\quad \cong D\operatorname{Hom}_A(M \otimes_A A_R, \tau_R U)) \cong D\operatorname{Hom}_R(M, \tau_R U) \cong \operatorname{Ext}_R^1(U, M). \end{aligned}$$

The last statement follows immediately. ∎

In what follows, we study the condition $\operatorname{Tor}_1^R(U, A) = 0$. We start with the following lemma.

LEMMA 3.6. *Let U be an R-module of projective dimension less than or equal to one. Let $0 \to \widetilde{P}_1 \to \widetilde{P}_0 \to U \to 0$ be a minimal projective resolution of U_R. Then $\widetilde{P}_1 Q = \widetilde{P}_0 Q \cap \widetilde{P}_1$ if and only if the multiplication map $U \otimes_R Q \to UQ$ is an isomorphism.*

Proof. Since $\operatorname{pd} U_R \leq 1$, applying $U \otimes_R -$ to the sequence of R-R-bimodules $0 \to Q \to R \to A \to 0$ yields the exact sequence $0 \to \operatorname{Tor}_1^R(U,Q) \to \operatorname{Tor}_1^R(U,R)$ hence we obtain $\operatorname{Tor}_1^R(U,Q) = 0$. Applying $- \otimes_R Q$ to the given projective resolution of U, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{P}_1 \otimes_R Q & \longrightarrow & \widetilde{P}_0 \otimes_R Q & \longrightarrow & U \otimes_R Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \| & & \\
0 & \longrightarrow & \widetilde{P}_1 Q & \longrightarrow & \widetilde{P}_0 Q & \longrightarrow & U \otimes_R Q & \longrightarrow & 0
\end{array}$$

We also have the following exact sequence of R-modules:

$$0 \longrightarrow \widetilde{P}_0 Q \cap \widetilde{P}_1 \longrightarrow \widetilde{P}_0 Q \longrightarrow UQ \longrightarrow 0.$$

Thus we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{P}_1 Q & \longrightarrow & \widetilde{P}_0 Q & \longrightarrow & U \otimes_R Q & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & \widetilde{P}_1 \cap \widetilde{P}_1 Q & \longrightarrow & \widetilde{P}_0 Q & \longrightarrow & UQ & \longrightarrow & 0
\end{array}$$

where $p(u \otimes q) = uq$ for $u \in U$ and $q \in Q$. The lemma follows. ∎

LEMMA 3.7. *Let U be an R-module. The multiplication map $U \otimes_R Q \to UQ$ is an isomorphism of R-modules if and only if* $\operatorname{Tor}_1^R(U,A) = 0$.

Proof. Applying the functor $U \otimes_R -$ to the exact sequence of R-R-bimodules $0 \to Q \to R \to A \to 0$ yields a commutative diagram with exact rows in $\operatorname{mod} R$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \operatorname{Tor}_1^R(U,A) & \longrightarrow & U \otimes_R Q & \longrightarrow & U \otimes_R R & \longrightarrow & U \otimes_R A & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \\
 & & 0 & \longrightarrow & UQ & \longrightarrow & U & \longrightarrow & U/UQ & \longrightarrow & 0
\end{array}$$

where p is the multiplication map. Since p is surjective, the result follows. ∎

Combining the previous two lemmata we obtain the following corollary.

COROLLARY 3.8. *Let U be an R-module such that its projective dimension is at most one, and let $0 \to \widetilde{P}_1 \to \widetilde{P}_0 \to U \to 0$ be a minimal projective resolution of U_R. The following statements are equivalent*:

(a) $\operatorname{Tor}_1^R(U,A) = 0$.

(b) *The multiplication map $U \otimes_R Q \to UQ$ is an isomorphism of R-modules.*

(c) $\widetilde{P}_1 Q = \widetilde{P}_0 Q \cap \widetilde{P}_1$. ∎

REMARK. Assume that, in addition, U is a tilting R-module. Then the conditions of the corollary are equivalent to the condition that $D({}_RA)$ is generated by U_R. This follows from the well-known isomorphism $D\operatorname{Ext}^1_R(U, DA) \cong \operatorname{Tor}_1^R(U, A)$.

The next result holds for instance when R is hereditary and also in the case of one-point extensions.

COROLLARY 3.9. *Assume that Q is projective as a left R-module. Then every* (*partial*) *tilting R-module is restrictable.*

Proof. This follows from condition (b) of 3.8. ∎

EXAMPLES. (a) The following is an example of a restrictable tilting module that is not induced. Let R be the hereditary algebra with quiver

$$1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$$

and A be the hereditary subalgebra given by the quiver

$$1 \xleftarrow{\alpha} 2.$$

The APR-tilting module $U_R = \tau_R^{-1}(e_1R) \oplus e_2R$ is restrictable by 3.9. In order to show that U_R is not induced, it suffices to show that the indecomposable module $\tau_R^{-1}(e_1R)$ is not induced. Notice that the top of $\tau_R^{-1}(e_1R)$ is isomorphic to a sum of two copies of S_2, and its socle is isomorphic to a sum of three copies of S_1. Since there are only three isomorphism classes of indecomposable A-modules of which two are projective, it suffices to compute $S_2 \otimes_A R$. The projective resolution

$$0 \longrightarrow e_1A \longrightarrow e_2A \longrightarrow S_2 \longrightarrow 0$$

in $\operatorname{mod} A$ lifts to a projective resolution over the algebra R:

$$0 \longrightarrow e_1R \longrightarrow e_2R \longrightarrow S_2 \otimes_A R \longrightarrow 0$$

Hence $S_2 \otimes_A R$ is a two-dimensional uniserial R-module and is not isomorphic to $\tau_R^{-1}(e_1R)$. Finally, we compute $U \otimes_R A$. The projective resolution

$$0 \longrightarrow e_1R \longrightarrow (e_2R)^2 \longrightarrow \tau_R^{-1}(e_1R) \longrightarrow 0$$

yields a minimal projective resolution in $\operatorname{mod} A$:

$$0 \longrightarrow e_1A \longrightarrow (e_2A)^2 \longrightarrow \tau_R^{-1}(e_1R) \otimes_R A \longrightarrow 0.$$

Therefore $\tau_R^{-1}(e_1R) \otimes_R A \cong S_2 \oplus e_2A$. Since $e_2R \otimes_R A \cong e_2A$, we conclude that $U \otimes_R A \cong S_2 \oplus (e_2A)^2$.

(b) We now give an example of a restrictable induced tilting module over R. Let R be given by the quiver

$$\begin{tikzcd} 1 \arrow[rr, bend left, "\eta"] & 3 \arrow[l, "\beta"] \arrow[d, "\gamma"] & 4 \arrow[l, "\alpha"'] \\ & 2 & \end{tikzcd}$$

subject to the relations $\eta\alpha\beta\eta\alpha = 0$, $\alpha\gamma = 0$, and let A be the subalgebra given by the quiver

$$\begin{tikzcd} 1 & 3 \arrow[l, "\beta"'] \arrow[d, "\gamma"] & 4 \arrow[l, "\alpha"'] \\ & 2 & \end{tikzcd}$$

with $\alpha\gamma = 0$. It is easily verified that

$$U_R = e_2R \oplus e_4R \oplus \begin{pmatrix} 4 \\ 3 \end{pmatrix} \oplus \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

is a tilting R-module. Applying the functor $-\otimes_R A$ to the minimal projective resolutions

$$0 \longrightarrow e_1R \longrightarrow e_4R \longrightarrow \begin{pmatrix} 4 \\ 3 \end{pmatrix} \longrightarrow 0, \qquad 0 \longrightarrow e_1R \longrightarrow e_3R \longrightarrow \begin{pmatrix} 3 \\ 2 \end{pmatrix} \longrightarrow 0$$

we see at once that U_R is restrictable. The same calculation shows that

$$U \otimes_R A \cong e_2A \oplus e_4A \oplus \begin{pmatrix} 4 \\ 3 \end{pmatrix} \oplus \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

which is a tilting A-module. Since it is easily verified that $U \otimes_R A \otimes_A R \cong U$, we infer that U is induced.

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