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STABLE FAMILIES OF ANALYTIC SETS

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Abstract. We give a different proof of the well-known fact that any uncountable family of analytic subsets of a Polish space with the point-finite intersection property must contain a subfamily whose union is not analytic. Our approach is based on the Kunen–Martin theorem.

1. Introduction and notations. It is well known that any uncountable family of analytic subsets of a Polish space with the point-finite intersection property must contain a subfamily whose union is not analytic (see [1], [2], [3] and [5]). In [1], this (and in fact a much stronger) result is proved but the proof heavily depends on the Axiom of Choice. In [2], [3] and [5], the proofs are effective but the arguments are more complicated. In this note we give a short proof by using the Kunen–Martin theorem.

NOTATIONS. In what follows X and Y will be uncountable Polish spaces. By $\mathcal{N}$ we denote the Baire space. If $A \subseteq X \times Y$ and $U \subseteq Y$ is an arbitrary open set, we put

$$A(U) = \mathrm{proj}_X\{A \cap (X \times U)\}.$$

All the other notations we use are standard (for more information we refer to [4]).

2. Stable families of analytic sets. Departing from standard terminology, we make the following definition.

DEFINITION 1. A family $\mathcal{F} = (A_i)_{i\in I}$ of analytic subsets of X will be called *stable* if for every $J \subseteq I$ the set $\bigcup_{i\in J} A_i$ is an analytic subset of X.

Clearly any subfamily of a stable family is stable. Furthermore any countable family of analytic sets is stable. There exist however uncountable stable families of analytic sets.

EXAMPLE 1. Let $A \subset X$ be an analytic non-Borel set. By a classical result of Sierpiński (see [4, p. 201]), there exists a transfinite sequence $(B_\xi)_{\xi<\omega_1}$ of Borel sets such that $A = \bigcup_{\xi<\omega_1} B_\xi$. Clearly we may assume that

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the sequence $(B_\xi)_{\xi<\omega_1}$ is increasing. As A is not Borel, there exists $\Lambda \subseteq \omega_1$ uncountable such that $B_\xi \subsetneqq B_\zeta$ for every $\xi, \zeta \in \Lambda$ with $\xi < \zeta$. Then the family $\mathcal{F} = (B_\xi)_{\xi\in\Lambda}$ is an uncountable stable family of mutually different analytic sets (note that the members of $\mathcal{F}$ are actually Borel sets).

Definition 2. A family $\mathcal{F} = (A_i)_{i\in I}$ of subsets of X is said to have the *point-finite intersection property* (abbreviated as p.f.i.p.) if for every $x \in X$, the set $I_x = \{i \in I : x \in A_i\}$ is finite.

As before, any subfamily of a family with the point-finite intersection property has the point-finite intersection property. We will show that stable families of analytic sets with the p.f.i.p. are necessarily countable. First a couple of lemmas. The one that follows is elementary.

Lemma 3. *Let X and Y be Polish spaces. If $A \in \mathbf{\Pi}^1_1(X)$ and $U \subseteq Y$ is open, then $A \times U \in \mathbf{\Pi}^1_1(X \times Y)$.*

Lemma 4. *Let X and Y be Polish spaces. Assume that $A \subseteq X \times Y$ has closed sections (i.e. for every $x \in X$, the set $A_x = \{y \in Y : (x, y) \in A\}$ is closed) and moreover for every $U \subseteq Y$ open the set $A(U)$ is analytic. Then A is also analytic.*

Proof. Let $\mathcal{B} = (V_n)_n$ be a countable base for Y. Observe that $(x, y) \notin A$ if and only if there exists a basic open subset V_n of Y such that $x \notin A(V_n)$ and $y \in V_n$. It follows that

$$(X \times Y) \setminus A = \bigcup_n (X \setminus A(V_n)) \times V_n$$

and so, by Lemma 3, A is analytic. ∎

We have the following stability result.

Lemma 5. *Let $\mathcal{F} = (A_i)_{i\in I}$ be a stable family of analytic subsets of X with the point-finite intersection property. Then for every Polish space Y and every family $(B_i)_{i\in I}$ of analytic subsets of Y, the set*

$$A = \bigcup_{i\in I} (A_i \times B_i)$$

is an analytic subset of $X \times Y$.

Proof. Let $\mathcal{F} = (A_i)_{i\in I}$ and $(B_i)_{i\in I}$ as above. As every B_i is analytic, there exists $C_i \subseteq Y \times \mathcal{N}$ closed such that $B_i = \mathrm{proj}_Y C_i$. Define $\widetilde{A} \subseteq X \times Y \times \mathcal{N}$ by

$$\widetilde{A} = \bigcup_{i\in I} (A_i \times C_i).$$

Clearly $A = \mathrm{proj}_{X\times Y} \widetilde{A}$. Note that for every $x \in X$ the section

$$\widetilde{A}_x = \{(y, z) \in Y \times \mathcal{N} : (x, y, z) \in \widetilde{A}\}$$

is exactly the set $\bigcup_{i\in I_x} C_i$. As the family $\mathcal{F}$ has the point-finite intersection property, for every $x \in X$ the section $\widetilde{A}_x$ of $\widetilde{A}$ is closed. Observe that for every $U \subseteq Y \times \mathcal{N}$ open, we have

$$\begin{aligned}\widetilde{A}(U) &= \mathrm{proj}_X\{\widetilde{A} \cap (X \times U)\}\\ &= \{x \in X : \exists i \in I_x \text{ such that } C_i \cap U \neq \emptyset\}\\ &= \bigcup\{A_i : C_i \cap U \neq \emptyset\}.\end{aligned}$$

It follows directly from the stability of the family that $\widetilde{A}(U)$ is analytic. By Lemma 4, $\widetilde{A}$ is an analytic subset of $X \times Y \times \mathcal{N}$. Hence so is A. ■

Let $\prec$ be a strict well-founded binary relation on X. By recursion, we define the rank function $\varrho_\prec : X \to \mathrm{Ord}$ as follows. We set $\varrho_\prec(x) = 0$ if x is minimal, otherwise we set $\varrho_\prec(x) = \sup\{\varrho_\prec(y) + 1 : y \prec x\}$. Finally we define the rank of $\prec$ to be $\varrho(\prec) = \sup\{\varrho_\prec(x) + 1 : x \in X\}$. We will need the following boundedness principle of analytic well-founded relations due to Kunen and Martin (see [4] or [6]).

THEOREM 6. *Let $\prec$ be a strict well-founded relation and assume that $\prec$ is analytic* (*as a subset of* $X \times X$). *Then $\varrho(\prec)$ is countable.*

LEMMA 7. *Let $\mathcal{F} = (A_i)_{i\in I}$ be a stable family of mutually disjoint analytic subsets of X. Then $\mathcal{F}$ is countable.*

Proof. Assume that $\mathcal{F}$ is not countable. Pick an uncountable subfamily $\mathcal{F}'$ of $\mathcal{F}$ with $|\mathcal{F}'| = \aleph_1$ and let $\mathcal{F}' = (A_\xi)_{\xi<\omega_1}$ be a well-ordering of $\mathcal{F}'$. Clearly $\mathcal{F}'$ remains stable. As the sets A_ξ are pairwise disjoint let $\phi : \bigcup_{\xi<\omega_1} A_\xi \to \mathrm{Ord}$, where $\phi(x)$ is the unique ξ such that $x \in A_\xi$. Define the binary relation $\prec$ by

$$x \prec y \;\Leftrightarrow\; \phi(x) < \phi(y).$$

Clearly $\prec$ is well-founded and strict. Moreover note that $\prec$, as a subset of $X \times X$, is the set

$$\bigcup_{\xi<\omega_1} (A_\xi \times B_\xi),$$

where $B_\xi = \bigcup_{\zeta>\xi} A_\zeta$. From the stability of $\mathcal{F}'$, we see that the sets B_ξ are analytic subsets of X for every $\xi < \omega_1$. As $\mathcal{F}'$ is stable and has the p.f.i.p., by Lemma 5 we deduce that $\prec$ is a $\mathbf{\Sigma}^1_1$ relation. By Theorem 6, $\varrho(\prec)$ must be countable and we derive a contradiction. ■

Finally we have the following.

THEOREM 8. *Let $\mathcal{F}$ be a stable family of analytic sets with the point-finite intersection property. Then $\mathcal{F}$ is countable.*

Proof. Assume not. Let $\mathcal{F}'$ be as in Lemma 7. Let Y be an arbitrary uncountable Polish space and let $(y_\xi)_{\xi<\omega_1}$ be a transfinite sequence of distinct members of Y. For every $\xi < \omega_1$, set $L_\xi = A_\xi \times \{y_\xi\}$. Clearly every L_ξ is an analytic subset of $X \times Y$ and moreover $L_\xi \cap L_\zeta = \emptyset$ if $\xi \neq \zeta$. As the family (and every subfamily of) $\mathcal{F}'$ is stable and has the p.f.i.p., by Lemma 5, for every $G \subseteq \omega_1$ the set

$$\bigcup_{\xi\in G}(A_\xi \times \{y_\xi\}) = \bigcup_{\xi\in G} L_\xi$$

is an analytic subset of $X \times Y$. It follows that the family $\mathcal{L} = (L_\xi)_{\xi<\omega_1}$ is a stable family of mutually disjoint analytic subsets of $X \times Y$. By Lemma 7, the family $\mathcal{L}$ must be countable and again we derive a contradiction. ∎

A corollary of Theorem 8 is the following (see [7]).

COROLLARY 9. *Let X be a Polish space, Y a metrizable space and $A \in \mathbf{\Sigma}^1_1(X)$. Let also $f : A \to Y$ be a Borel measurable function. Then $f(A)$ is separable.*

Proof. Assume not. Let $C \subseteq f(A)$ be an uncountable closed discrete set with $|C| > \aleph_0$. For every $y \in C$, put $A_y = f^{-1}(\{y\})$. Then $\mathcal{F} = (A_y)_{y\in C}$ is a stable family of mutually disjoint analytic subsets of X. By Theorem 8, $\mathcal{F}$ must be countable and we derive a contradiction. ∎

REMARK 1. Say that a family $\mathcal{F} = (A_i)_{i\in I}$ has the point-countable intersection property if for every $x \in X$ the set $I_x = \{i \in I : x \in A_i\}$ is countable. One can easily see that Theorem 8 is not valid for stable families with the point-countable intersection property. For instance let $(A_\xi)_{\xi<\omega_1}$ be a strictly decreasing transfinite sequence of analytic sets with $\bigcap_{\xi<\omega_1} A_\xi = \emptyset$. As the sequence is decreasing, the family $\mathcal{F} = (A_\xi)_{\xi<\omega_1}$ is stable. Moreover note that for every $x \in X$ there exists $\xi < \omega_1$ such that $x \notin A_\zeta$ for every $\zeta > \xi$ (for if not there would exist an $x \in X$ such that $x \in A_\xi$ for every $\xi < \omega_1$, that is, $x \in \bigcap_{\xi<\omega_1} A_\xi$). Hence the family $\mathcal{F}$ is an uncountable stable family of analytic sets with the point-countable intersection property.

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