ON v-POSITIVE TYPE TRANSFORMATIONS
IN INFINITE MEASURE

BY

TUDOR PĂDURARIU (Los Angeles, CA), CESAR E. SILVA (Williamstown, MA) and
EVANGELIE ZACHOS (Princeton, NJ)

Abstract. For each vector v we define the notion of a v-positive type for infinite-measure-preserving transformations, a refinement of positive type as introduced by Hajian and Kakutani. We prove that a positive type transformation need not be $(1,2)$-positive type. We study this notion in the context of Markov shifts and multiple recurrence, and give several examples.

1. Introduction. Let $(X, \mathcal{B}, \mu)$ be a standard Borel measure space with a sigma-finite nonatomic measure. Let $T : X \to X$ be a *measure-preserving* transformation (i.e., $T^{-1}(A)$ is measurable and $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$). We will assume that T is invertible and T^{-1} is measure-preserving. We concentrate on measure spaces with infinite measure. A transformation is *ergodic* if every invariant set is null or conull. *Wandering sets* A are those for which $\mu(A \cap T^n A) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. A transformation is *conservative* if it admits no wandering sets of positive measure. It follows that a transformation T is conservative ergodic if for any positive measure sets A and B there exists a positive integer n such that $\mu(A \cap T^n B) > 0$. If an invertible measure-preserving transformation is ergodic on a nonatomic space, then it is conservative (see e.g. [S08, Lemma 3.9.1]). All of our transformations will be assumed to be conservative ergodic.

To classify infinite-measure-preserving transformations, Hajian and Kakutani [HK64] introduced the notions of zero type and positive type. A transformation T is of *zero type* if for all sets A of finite measure,

$$\lim_{n \to \infty} \mu(A \cap T^{-n}A) = 0.$$

A transformation T is of *positive type* if for all sets A of positive finite measure,

$$\limsup_{n \to \infty} \mu(A \cap T^{-n}A) > 0.$$

If T is infinite-measure-preserving and ergodic, the Birkhoff ergodic theorem immediately implies that

$$\liminf_n \mu(A \cap T^{-n}A) = 0 \quad \text{for all finite measure } A.$$

The $\limsup$ definition for positive type thus captures the greatest possible variation in the limit behavior of these intersections. In [HK64], Hajian and Kakutani note that conservative ergodic infinite-measure-preserving transformations are either zero type or positive type. Additionally, it is known that such a transformation T is zero type if and only if all powers T^k are zero type (where the proof does *not* depend on ergodicity).

In this paper, we introduce a new concept inspired by the definition of positive type, called v-positive type, for each vector v in $\mathbb{Z}^d$. This notion provides a further classification of positive type transformations. We also define a concept related to v-positive type that we call v-multiplicative-positive type. In Section 2 we introduce the main definitions and prove that v-multiplicative-positive type is equivalent to the corresponding Cartesian product transformation being positive type. It is a direct consequence of the definitions that v-positive type implies v-multiplicative-positive type, but we prove that the converse does not hold. In fact, in Section 4 we prove that for every v such that $v_d/v_1 \geq d$, there is a transformation that is v-multiplicative-positive type but not v-positive type. We also construct v-positive type transformations for each v. These constructions are rank-one examples and use basic approximation results from Section 3.

In Section 5, we use results involving Markov shifts to discuss connections with multiple recurrence, specifically that v-positive type is a stronger condition than k-recurrence. Section 6 provides miscellaneous results about the behavior of $\mu(A \cap T^{n_i}A)$ for more general sequences $\{n_i\}$. These results are both interesting in their own right and explain why the definition of v-positive type is the most natural definition to consider. Section 7 discusses varying the vector v to compare v-positive type and w-positive type and proves that except for the obvious obstructing relationships between v and w, v-positive type and w-positive type are independent.

2. v-positive type and v-multiplicative-positive type. We begin with a vector, $v = (v_1, \ldots, v_d) \in (\mathbb{Z}^+)^d$. We study each vector essentially independently of all other vectors until Section 7. Section 6 provides some motivation for the following definitions.

DEFINITION 2.1. A transformation T is *v-positive type for a set A* if

$$\limsup_{n \to \infty} \mu(A \cap T^{v_1 n}A \cap \cdots \cap T^{v_d n}A) > 0.$$

We define T to be *v-positive type* if it is v-positive for all sets of positive finite measure.

DEFINITION 2.2. A transformation T is *v-multiplicative-positive type for a set A* if
$$\limsup_{n \to \infty} \mu(A \cap T^{v_1 n} A) \cdots \mu(A \cap T^{v_d n} A) > 0.$$
We similarly define T to be *v-multiplicative-positive type* if it is v-multiplicative-positive type for all sets of positive finite measure.

Of course, these definitions can only be satisfied if T is positive type. They offer, however, a way for further classifying into subcategories different positive type transformations. Note that we can easily confine ourselves to vectors in $(\mathbb{Z}^+)^d$, for given $(v_1, \ldots, v_d) \in \mathbb{Z}^d$ where v_1 is the negative number of largest absolute value, then $0 < \mu(A \cap T^{v_1 n} A \cap \cdots \cap T^{v_d n} A) = \mu(T^{-v_1 n} A \cap A \cap T^{(-v_1+v_2)n} A \cap \cdots \cap T^{(-v_1+v_d)n} A)$, so that we can investigate the v-positive type of T by evaluating the w-positive type of T where $w = (v_2 - v_1, v_3 - v_1, \ldots, v_d - v_1, -v_1) \in (\mathbb{Z}^+)^d$. It is typical to list these vectors with v_i distinct and $v_{i+1} > v_i$.

In general, of the two definitions, T v-multiplicative-positive type is easier to work with. The following theorem explains some of this ease, where the proof technique is adapted from Aaronson and Nakada [AN00]. We start with two lemmas before presenting the theorem.

LEMMA 2.3. *Suppose that T is conservative ergodic and v-multiplicative-positive type. Then, for every collection $A_1, \ldots, A_d$ of sets of positive finite measure,*
$$\limsup \mu(A_1 \cap T^{v_1 n} A_1) \cdots \mu(A_d \cap T^{v_d n} A_d) > 0.$$

Proof. As T is ergodic, we can find $c_2, \ldots, c_d$ such that
$$C = A_1 \cap T^{c_2} A_2 \cap \cdots \cap T^{c_d} A_d$$
has $\mu(C) > 0$. Then $\limsup \mu(C \cap T^{v_1 n} C) \cdots \mu(C \cap T^{v_d n} C) > 0$ but we note that $\mu(A_j \cap T^{v_j n} A_j) = \mu(T^{c_j} A_j \cap T^{v_j n + c_j} A_j) \geq \mu(C \cap T^{n v_j} C)$ so that we can deduce the desired result. ∎

As a corollary which is not used in the main proof of the following theorem, we have:

COROLLARY 2.4. *Suppose that T is conservative ergodic and is positive type. Then, for every finite collection of sets $A_1, \ldots, A_m$ of positive finite measure, we have*
$$\limsup \mu(A_1 \cap T^n A_1) \cdots \mu(A_m \cap T^n A_m) > 0.$$

The first part of the following lemma is standard.

LEMMA 2.5. *Let T be a conservative, ergodic and measure-preserving transformation and let $k \in \mathbb{Z}^+$. Then there exists a measurable set B and an integer $1 \leq \ell \leq k$ such that $B, TB, \ldots, T^{\ell-1} B$ is a partition of X and every positive measure invariant subset of T^k contains (mod μ) at least one of $B, TB, \ldots, T^{\ell-1} B$.*

Proof. First, consider all T^k-invariant sets of positive measure. Given such a set A, we can create a partition $\mathcal{B}$ with base set B from A in the following canonical method: We consider intersections of the form $A \cap T^{i_1} A \cap \cdots \cap T^{i_r} A$ where $i_j < k$ and the i_j are distinct. There are a finite number of such sets, and we pick $B = A \cap T^{i_1} A \cap \cdots \cap T^{i_r} A$ of positive measure with highest possible value of r and then, to break any ties, of minimal measure. (When all powers of T are conservative ergodic, for example, both A and B will be forced to be X, the entire space.) Now, as A is T^k-invariant, by definition B is as well, and so $B \cup \cdots \cup T^{k-1} B = X$ by the ergodicity of T. We then define $\ell = \ell_B$ to be the smallest positive number such that $B \cap T^\ell B = B$. Such an $\ell \leq k$ must exist by the T^k-invariance of B. (For example, note that if T^k is ergodic, then $\ell = 1$. In fact, one can construct examples B, T for any $\ell \,|\, k$.) Then by the maximality of r, if $1 \leq j < \ell$, then $\mu(B \cap T^j B) = 0$. In fact, if $1 \leq j < i < \ell$, then $\mu(T^i B \cap T^j B) = \mu(T^{i-j} B \cap B) = 0$ by the same argument. By our choice of ℓ, we have $B \cup \cdots \cup T^{\ell-1} B = X$. Thus, $\mathcal{B} = \{B, TB, \ldots, T^{\ell-1} B\}$ is a partition of X.

Now, we use a minimizing argument. Given any two partition bases, B and C, there is some choice of i such that $\mu(B \cap T^i C) > 0$. Let $\mathcal{S}$ be the set of partition bases. Note of course that every set in $\mathcal{S}$ has positive measure and $\mathcal{S}$ is nonempty. Decreasing chains in $\mathcal{S}$ are of the form $\cdots \subset B \subset C \subset D$ where the inclusions are proper with respect to the measure. Next, note that for each $B \in \mathcal{S}$, $B \cup T^1 B \cup \cdots \cup T^{\ell_B} B = X$. Now, if $B \subset C$, note that $\mu(X \setminus (B \cup T^1 B \cup \cdots \cup T^{\ell_C} B)) > 0$ as $\mu(C \setminus B) > 0$ and C is a partition base, so we can conclude that $\ell_B > \ell_C$. As $\ell_B \leq k$ for all $B \in \mathcal{S}$, we see that chains in $\mathcal{S}$ must be finite. In fact, as we have a bound k, there must be a maximum ℓ_B for $B \in \mathcal{S}$ and we pick a B with corresponding maximum ℓ_B, noting that it necessarily must be the base set of a minimal partition, i.e. a base not positive-measure containing the base set to any other partition.

Given a T^k-invariant set A of positive measure, chosen without loss of generality such that $\mu(B \cap A) > 0$, we wish to show that $B \subset A \bmod \mu$. If, not, i.e. if $\mu(B \setminus A) > 0$, then $C = A \cap B$ is T^k invariant as well and $C \subset B$ with positive measure inclusion and so we can find the corresponding partition base set $C' \subset C$ which has the property that $C' \subset B$ with strict μ-inclusion. This violates the minimality property of B, and so we can conclude that $B \subset A \bmod \mu$. ∎

THEOREM 2.6. *Let T be a conservative ergodic transformation on $(X, \mathcal{B}, \mu)$ and let $v = (v_1, \ldots, v_d)$ where each v_i is in $\mathbb{Z}^+$. Then $T^{v_1} \times \cdots \times T^{v_n}$ is positive type if and only if T is v-multiplicative-positive type.*

Proof. One direction is clear. If $T^{v_1} \times \cdots \times T^{v_n}$ is positive type, then in particular it is positive type for the set $A \times \cdots \times A$ for each $A \subset X$ of positive

finite measure, so T clearly is v-multiplicative-positive type with respect to each of these A.

For the other direction, starting from the assumption that T has v-multiplicative-positive type, we proceed as follows. Let $S := T^{v_1} \times \cdots \times T^{v_d}$ be a measure-preserving transformation on the space $(X^d, \mathcal{B}^d, \mu^d = \mu \times \cdots \times \mu)$. To show that S has positive type, define

$$\mathcal{W} := \{B \in \mathcal{B}^d : 0 < \mu^d(B) < \infty, \lim \mu^d(B \cap S^n B) = 0\}.$$

Then, by a standard argument, we can find a set Z which is a countable union of sets in $\mathcal{W}$ such that every set in $\mathcal{W}$ is contained up to a set of measure zero in Z. We observe that

$$\lim \mu^d(B \cap S^n C) = 0$$

for all finite measure sets $B, C \in \mathcal{W}$. This can be seen in a short number of steps: If $\limsup \mu^d(B \cap S^n C) > 0$, we could find a sequence $\{n_i\}$ satisfying $\mu^d(B \cap S^{n_i} C) > \epsilon$ for a fixed $\epsilon > 0$ and then as B and C are each of finite measure, we could find $N \in n_i$ and a subsequence $\{n_j'\}$ such that $\mu^d(B \cap S^N C \cap S^{n_j'} C) > \epsilon'$ for some fixed $\epsilon' > 0$. Of course, we then have $\limsup \mu^d(S^N C \cap S^{n_j'} C) \geq \epsilon'$, which is a contradiction of the defining property of $\mathcal{W}$.

This fact gives us that if $A, B \in \mathcal{W}$, then $A \cup B \subset \mathcal{W}$. If $A \subset Z$ has $\mu(A) < \infty$, then it can be approximated up to epsilon measure by sets in $\mathcal{W}$, which then quickly implies that $A \in \mathcal{W}$. So, the argument above shows that $\lim \mu^d(B \cap S^n C) = 0$ for $B, C \subset Z$ where $\mu(B), \mu(C) < \infty$.

Returning to the main proof, it suffices to show that if T is v-multiplicative-positive type, then $\mu(Z) = 0$. We will stop briefly to note that if T^{v_i} is ergodic for some $1 \leq i \leq d$, then the argument in [AN00, Proposition 2.2] can be used to finish our proof. However, the following proof is more general, working even when T^{v_i} is nonergodic for all i. To continue, assume for the sake of contradiction that Z has positive measure in X^d.

If we define $R_i := I \times \cdots \times T^{v_i} \times \cdots \times I$, then for any $A \in \mathcal{W}$,

$$\mu^d(R_i A \cap S R^i A) = \mu^d(A \cap S) = 0,$$

so that $R_i A \in \mathcal{W}$ and therefore $R_i Z = Z$ mod μ.

Then define $p_i : Z \to X$ and $\pi_i : Z \to X^{d-1}$ by $p_i(x_1, \ldots, x_d) = x_i$ and $\pi_i(x_1, \ldots, x_d) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$. We start an inductive process, beginning with $i = 1$. First note that for any measurable subset A, by our restrictions at the beginning of the paper, we can conclude that $p_i(A)$ is measurable. For $x \in \pi_1(Z)$, we define $V_x^1 := p_1(\pi_1^{-1}(x))$, a measurable set. V_x^1 is a T^{v_1}-invariant subset of X, and so if it is of positive measure, by Lemma 2.5 applied to $k = v_1$, it contains one $T^{j(x)} B_1$ of $B_1, T^1 B_1, \ldots, T^{\ell-1} B$. If $D_1 \subset \pi_1(Z)$ is the subset containing x such that

$\mu(V_x^1) > 0$, then as Z has positive measure, $\mu^{d-1}(D_1) > 0$. We then have a map $f : D_1 \to \{1, \ldots, \ell\}$. There must be some j such that $f^{-1}(j)$ has positive measure. In fact, define $C_1 = \{x \in D_1 : T^j B \subset V_x^1\}$, where $f^{-1}(j) \subset C_1$, then define $A_1 = T^j B_1$ and note that $Z_1 = A_1 \times C_1 \subset Z$ has positive measure. Furthermore, for $i \geq 2$, we see that $R_i(Z_1) = Z_1$, and so we can proceed inductively to find $Z_d \subset Z_{d-1} \subset \cdots \subset Z_1 \subset Z$ where Z_d has positive measure and in fact $A_1 \times \cdots \times A_d \subset Z_d \subset Z$. For each i, take a subset $A_i' \subset A_i$ of positive finite measure.

By Lemma 2.3 and the fact that T is v-multiplicative-positive type, we can conclude that $\limsup \mu^d(A_1' \times \cdots \times A_d' \cap S(A_1' \times \cdots \times A_d')) > 0$ so that $\prod_i A_i' \not\subset Z$. Thus, by contradiction, Z is trivial, and so S is positive type. $\blacksquare$

Therefore, v-multiplicative-positive type of T is just another name for positive type of $T^{v_1} \times \cdots \times T^{v_d}$. All results obtained for positive type transformations apply. For example, if $T^{v_1} \times \cdots \times T^{v_d}$ is ergodic, a transformation is either v-multiplicative-positive type or v-multiplicative-zero type. For this reason, the remainder of the paper will focus primarily on v-positive type, a property that is less well understood.

3. Rank-one constructions and approximation. Rank-one transformations are a much-studied type of transformation, easily described by defining the transformation iteratively on intervals; see e.g. [S08]. We begin with a review of the cutting and stacking method of constructing rank-one examples and then proceed to a number of lemmas which help in the transition from considering columns and levels to considering arbitrary positive measure sets, as far as it can be accomplished.

A *column*, denoted here by C, is an ordered set of $h > 0$ pairwise disjoint intervals in $\mathbb{R}$ of the same measure. (Cutting and stacking can be adapted to construct nonsingular examples, but we do not consider them here.) These intervals are called *levels*, and C is said to have *height* h. We consider these levels to be "stacked" so that the element $i + 1$ is directly above i, for all $0 \leq i \leq h-2$. For a level I in a column C, denote by $h(I) \in \{0, \ldots, h-1\}$ the height of I, its position in C. Let $w(C)$ denote the measure of each level in C. We define the *column map* T_C to map a point in the level i, $0 \leq i \leq h - 2$, to the point directly above in the level $i+1$. Thus, if we let J be the bottom level of the column C, then the ith level will be $T_C^i(J)$. The map T_C brings the bottom $h - 1$ levels to the top $h - 1$ levels. A cutting and stacking construction for a measure-preserving transformation $T : X \to X$ consists of a sequence of columns $C_n = \{J_n, \ldots, T^{h_n-1}J_n\}$ of height h_n such that:

(i) J_n is a disjoint union of elements from $\{J_{n+1}, \ldots, T^{h_{n+1}-1}J_{n+1}\}$.
(ii) C_{n+1} is obtained from C_n by cutting C_n into $r_n \geq 2$ subcolumns C_n^i of equal measure, putting a number of spacers (new levels of the

same measure as any of the levels in the r_n subcolumns) above each subcolumn, mapping the top level of each subcolumn to the spacer above it, and stacking left under right. In this way, C_{n+1} consists of r_n copies of C_n, possibly separated by spacers, which are also included in C_n.

(iii) $\bigcup_n C_n$ is a generating subalgebra of the Borel sets $\mathcal{B}$. In our case, this usually requires that there be enough spacers for $\bigcup_n C_n$ to have infinite measure.

(iv) The pointwise limit of T_{C_n} as n increases is T.

To discuss these cutting and stacking constructions, we introduce the idea of **descendants**. Given a level J ($J \in C_n$) and any column C_m where $m \geq n$, we define the *m-descendants* of J to be the collection of levels in C_m whose disjoint union is J. We denote this set by $D(J, m)$. Occasionally, we will also use $D(J, m)$ to refer to the heights of the m-descendants of J. (For more results using this notion of descendants, see [DGPS].) An important observation is the following:

LEMMA 3.1. *Suppose that T is a rank-one transformation on an infinite measure space X, and that J is a level of the jth column. Then $\mu(J \cap T^n J) > 0$ if and only if there exists m such that $n \in D(J, m) - D(J, m)$.*

Proof. First, suppose that $\mu(J \cap T^n J) > 0$ for some $n > 0$. Choose m such that $\max(D(J, m)) + n \leq h_m$. Then we can write $J = \bigcup_{D(J,m)} J^l$, where the union is over all the m-descendants of J. Further, write $J^l = T^{h(l)} I$, where I is the bottom level of C_m. Then the condition $\mu(J \cap T^n J) > 0$ becomes

$$\mu\left(\left(\bigcup T^{h(l)} I\right) \cap \left(\bigcup T^{n+h(l)} I\right)\right) > 0,$$

which implies that $D(J, m) \cap (n + D(J, m)) \neq \emptyset$, that is, $n \in D(J, m) - D(J, m)$. ∎

Observe that actually we can compute the exact value of $\mu(J \cap T^n J)$ by using this notion of descendants. Indeed, for large enough N there will be at least n spacers on top of the last column and therefore T^n will be defined in C_N on each of the descendants of J. Thus, we obtain

COROLLARY 3.2. *Suppose that T is a rank-one transformation on an infinite measure space X, and that J is a level of the jth column. Then if $\mu(J \cap T^n J) > 0$, then for all large N,*

$$\mu(J \cap T^n J) = \frac{|D(J, N) \cap (D(J, N) + n)|}{|D(J, N)|}.$$

DEFINITION 3.3. A transformation T is *v-α-type* if for all sets A of positive (finite) measure,

$$\limsup \mu(A \cap T^{v_1 n} A \cap \cdots \cap T^{v_d n} A) = \alpha \mu(A).$$

This definition is inspired by [HK64] and is useful for proving results allowing us to move from a sufficient semiring of intervals to all sets of positive measure. In fact, considering the larger class of transformations which have an α such that any set A of positive (finite) measure has

$$\limsup \mu(A \cap T^{v_1 n}A \cap \cdots \cap T^{v_d n}A) \geq \alpha\mu(A)$$

gives us similar results, but we stick to the precedent. The concept of α-type is also discussed in [OH71] and in the dissertation [L09].

Next, we have a basic analytic lemma. The exact statement of this lemma is from [BFMCS].

LEMMA 3.4 (Double Approximation Lemma). *Let* $\epsilon > 0$, $\delta > 0$, $1 > \tau > 0$ *and* I *be an interval* τ-full *of a measurable set* A. *If* $\{r_n\}$ *is an infinite sequence such that* $r_n > 1$ *for large enough* n, *there exists an* $N \in \mathbb{N}$ *such that if*

$$I_k = \left[\frac{k}{r_1 \cdots r_N}, \frac{k+1}{r_1 \cdots r_N} \right] \quad \text{for } 0 \leq k \leq r_1 \cdots r_N - 1,$$

then there exists a subset $K \subset \{0, 1, \ldots, r_1 \cdots r_N - 1\}$ *with* $|K| > (\tau - \delta)r_1 \cdots r_N$ *such that each* I_j, $j \in K$, *is* $(1 - \epsilon)$-full *of* A.

The next lemma is critical for verifying all of our later v-positive type examples.

LEMMA 3.5. *If* T *is a rank-one transformation that is* v-α-type *for all intervals that appear in a level of any of the defining columns, then* T *is* v-positive type.

Proof. If $I, T^{v_1 n}I, \ldots, T^{v_d n}I$ are in C_N and are each $(1 - \delta)$-full of A, then $I \cap T^{v_1 n}I \cap \cdots \cap T^{v_d n}I$ is $(1 - \delta(d+1))$-full of A. Then, given I that is $(1 - \delta)$-full of A, we can choose $\tau = 1 - 2\delta$ so that beyond an N, τ of the subintervals of I which appear as its descendants in later columns will be $(1 - \delta)$-full. Then, it is a simple calculation to see that if n is chosen so that $\mu(I \cap T^{v_1 n}I \cap \ldots \cap T^{v_d n}) > (1 - \epsilon)\alpha\mu(I)$, then the descendants will give

$$\mu(A \cap \cdots \cap T^{v_d n}A) \geq (1 - \epsilon)(\alpha - 2\delta(d+1))(1 - \delta(d+1))\mu(I).$$

Now, if we simply approximate all of A with intervals I that are $(1 - \delta)$-full of A (choosing δ small enough so that $2\delta(d+1) < \alpha$), we have shown that A has v-positive type. Of course we can do so and the result follows. ∎

LEMMA 3.6. *If* T *is a transformation that is* v-α-type *along a fixed sequence (i.e. the* $\limsup$ *occurs along the same subsequence) for all sets that are elements of a sufficient semiring, then* T *is* v-α-type *for all sets.*

The proof is essentially the same as the one given in [S08, Lemma 6.7.3]. This lemma will be used in all of the examples in this section, where the

sufficient semiring is the set of all levels, and the common subsequence will usually be some variant of $\{h_n\}$.

LEMMA 3.7. *If T is v-zero type for all columns C_m, then T is v-zero type.*

Proof. If T is infinite-measure-preserving, the columns grow so that $X = \bigcup_m C_m$. For any set A of positive finite measure and any $\epsilon > 0$, we can pick M such that $\mu(A \cap C_M^c) < \epsilon$. Then

$$\mu(A \cap T^{v_1 n} A \cap \cdots \cap T^{v_d n} A) < \mu(C_M \cap T^{v_1 n} C_M \cap \cdots \cap T^{v_d n} C_M) + (d+1)\epsilon.$$

From this, we see that in fact A is v-zero type. ∎

We can now exhibit examples of rank-one transformations that are v-positive type and others that are not, which we do in Sections 4 and 5.

Note what these basic lemmas do not allow us to do. Moving from a sufficient semiring (usually composed of unions of intervals) to all finite-positive-measure sets for v-positive type transformations is difficult without the additional control of v-α-type.

4. Rank-one examples. Many of these examples are most easily seen visually, and we encourage the drawing of diagrams such as the ones shown below.

EXAMPLE 4.1. For every vector $(v_1, \ldots, v_d)$, we can create a transformation T that is v-positive type very simply.

An example for $d = 2$ is given below. Let T be a cutting and stacking transformation starting with C_0 consisting of one level, $[0, 1]$. Then, for infinitely many n, cut the column C_n into $d + 1$ subcolumns and for $0 \leq i \leq d - 1$, on top of the $(d - i)$th subcolumn (called C_n^{d-i+1}) put $h_n(v_{i+1} - v_i - 1)$ spacers ($v_0 = 0$). The effect of the procedure is to put the first level of the $(d - i)$th subcolumn $h_n v_i$ levels away from the first level of the $(d + 1)$th subcolumn. As long as this is done infinitely often,
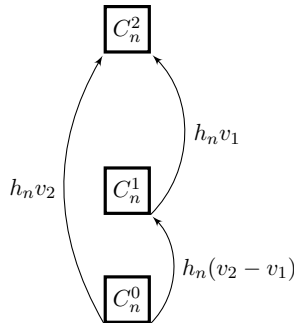

Fig. 1. $d = 2$ example

at $\{n_k\}$, then $\mu(I \cap T^{h_{n_k}v_1}I \cap \cdots \cap T^{h_{n_k}v_d}I) > \frac{\mu(I)}{d+1}$, when I is any level of any of the defining columns of T. By the lemma above, as this gives that T is v-$\frac{1}{d+1}$-type for all of these intervals, we find that in fact T is v-positive for all sets of positive measure.

This argument is very flexible, because we can add however many spacers we want after the last subcolumn and we need only perform this procedure infinitely often, not at every step. For example, the construction given in [DGMS] describes how to make a rank-one power weakly mixing transformation with a procedure that needs similarly only occur infinitely often, and this procedure could be combined with the procedure in this example to create a rank-one transformation that is both power weakly mixing and v-positive type. Furthermore, as the set of finite integer-valued vectors is countable, we can use this same basic construction to create a rank-one transformation that is v-positive type for each $v \in \mathbb{Z}^d$ (for all values of $d > 1$) and additionally power weakly mixing.

Next, we show that v-positive type and v-multiplicative-positive type are not equivalent. Before tackling a general example, here is a specific one.

EXAMPLE 4.2. Let $v = (1, 2)$. We construct a rank-one transformation T that is v-multiplicative positive type (hence positive type) but not v-positive type.

Start out with $C_0 = I = [0, 1]$. At each step cut into four subcolumns. On the first subcolumn, put no spacers. On the next column, put $3h_n$ spacers. On the third subcolumn, put h_n spacers. On the last subcolumn, put $8h_n$ spacers. Now, this is clearly v-multiplicative-positive type for I (just take overlap $T^{v_i m}$ for $m = h_n$) and similarly for all interval levels. As the construction has overlap occurring at $\frac{1}{4}\mu(I)$, T is v-multiplicative-positive type for all positive measure sets.

To show T is not v-positive type, we show this for I. For each m, consider n where $h_{n-1} < m \leq h_n = 16h_{n-1}$. Then, to show that $\mu(I \cap T^m I \cap T^{2m}I) = 0$, we look at the four subcolumns A, B, C, D of C_{n-1} (not including spacers). It is clear that any nonzero intersection of $I, T^m I$, and $T^{2m}I$ will happen between two or more of the above subcolumns but in no case purely within one subcolumn. Therefore, it is just a matter of tracking down the different possible combinations of subcolumns and showing that there can be no overlap. We show these different scenarios with corresponding pictures. To start, we note that $\mu(D \cap T^m C \cap T^{2m}I) = 0$, as levels in D and C that are both in I are at most $2h_{n-1} + 8h_{n-2}$ apart and levels in D and B that are both in I are at least $5h_{n-1} + 8h_{n-2} > 2(2h_{n-1} + 8h_{n-2})$ apart. Thus no possible m can give this type of overlap. See Figure 2.

For the second scenario, we consider $B \cap T^m A \cap T^{2m}I$ overlap. Although $m \leq h_n$, we might have overlap occurring in a larger picture, as illustrated
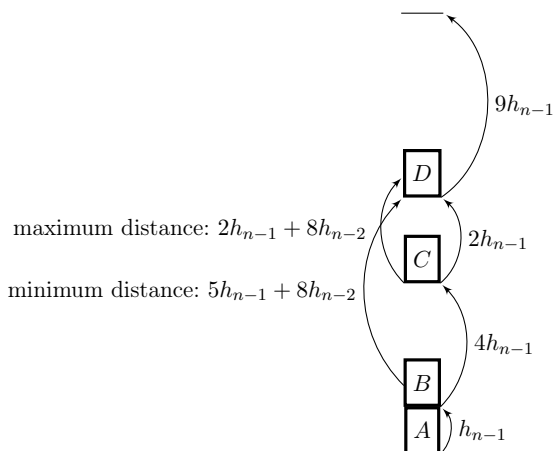
Fig. 2. Disproving nontrivial $D \cap T^m C \cap T^{2m} I$ overlap

below with Figure 3. Here the maximum distance available $(h_{n-1} + 8h_{n-2})$ is even smaller than that considered in the first case, and the minimum distance $(h_{n-1} + 8h_{n-2})$ is even larger, so the same argument as before shows that there must be zero overlap incorporating a smaller subcolumn.
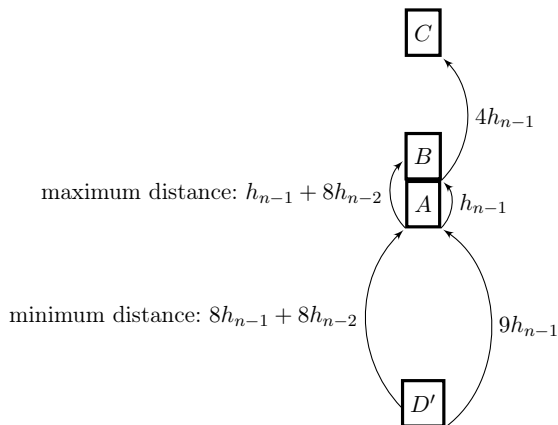

Fig. 3. Disproving nontrivial $B \cap T^m A \cap T^{2m} I$ overlap

The impossibility of the third scenario, requiring $\mu((C \cup D) \cap T^m(A \cup B) \cap T^{2m} I) = 0$, follows very similarly and is shown in Figure 4.

Tracking down each of these scenarios gives a contradiction requiring the intersection to have measure 0, and so we conclude that for $h_{n-1} \leq m \leq n$, $\mu(I \cap T^m I \cap T^{2m} I) = 0$. Unlike the previous example, there is little flexibility, because if one were to perform cutting and stacking steps of a different type, one would need to check that these additional steps do not incidentally
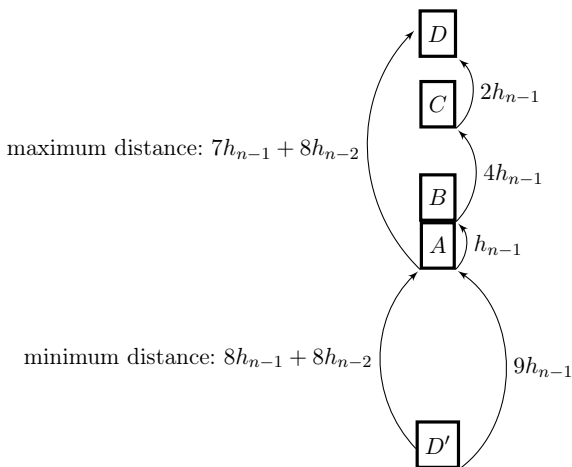
Fig. 4. Disproving nontrivial $(C \cup D) \cap T^m(A \cup B) \cap T^{2m}I$ overlap

bestow v-positive type. However, note that the proof above actually shows that T is v-positive type for no A of finite measure, as we can choose a finite union of levels $J \subset C_n$ such that $A \subset J$ (within ϵ) and then for $m > h_n$, we have $\mu(J \cap T^m J \cap T^{2m}J) = 0$ by the argument above.

Now, the above example generalizes to the following example:

EXAMPLE 4.3. For every vector v with $v_d/v_1 \geq d$, we can create a rank-one transformation T that is v-multiplicative-positive type but not v-positive type.

This transformation is in a spirit similar to the previous one. To create T with cutting and stacking, let $C_0 = [0,1]$. At *each* step, cut into $2d$ subcolumns. Then, on top of the $(2i - 1)$th subcolumn $1 \leq i \leq d$, place $(v_i - 1)h_n$ spacers. On top of the $2i$th subcolumn $(i < d)$, place Mh_n spacers, where $M \geq v_d(v_d + 1)/v_1$. On top of the last column, put kh_n spacers so that the spacers added on top of the last subcolumn have measure at least v_d times the sum of the measures of each subcolumn and of the spacers put on top of each of the previous subcolumns. It is fairly clear that T is v-multiplicative-positive type. Whenever $n = h_m$, the desired inequality will be achieved for any level interval. And by the lemma above that means T is v-multiplicative-positive type for any set of positive measure.

To see that T is not v-positive type, we primarily concentrate on the beginning level $I = C_0$. (Note that a failure for this interval will imply a failure for all other levels but perhaps not for other sets.) For any m, pick n so $h_{n-1} < m \leq h_n$. Then, considering subcolumns X_{2i} and X_{2i-1}, we note that if $\mu(X_{2i} \cap T^{mv_1} X_{2i-1}) > 0$, then the maximum value for $v_1 m$ is $v_d h_{n-1} + kh_{n-2} \leq (v_d + 1)h_{n-1}$ and then $T^{-v_d m}X_{2i}$ will be entirely contained within

the spacers above X_{2i-2}. Similarly, in the other case, if $\mu(X_i \cap T^{v_1 m} X_j) > 0$, then $T^{-v_d m} X_i$ is contained in spacers, by the choice of M detailed above.

In fact, the above two examples are examples of rank-one transformations that are clearly positive type (as they are v-multiplicative-positive type), but in fact we can show that they are v-zero type. The argument above showed that for $I = C_0$,

$$\mu(I \cap T^{v_1 n} I \cap \cdots \cap T^{v_d n} I) \to 0.$$

This argument generalizes easily to $I = C_k$ for any k. In fact, the argument given holds if we confine ourselves to $m \geq h_N$ where $N \geq k$, and so the limit for any $I = C_k$ will still be zero, and by Lemma 3.7, we can conclude that T is v-zero type.

5. Multiple recurrence and positive type. Just as the rank-one examples helped us distinguish v-positive type from v-multiplicative-positive type, Markov shift examples allow us to distinguish between v-recurrence and v-positive type. Multiple recurrence is a property famously studied in the finite measure case, culminating in Furstenberg's Multiple Recurrence Theorem. More recently, it has been studied in the infinite measure case, the reader may refer to [AN00] for Markov shift examples and [EHH98] for odometer (rank-one) examples.

DEFINITION 5.1. A transformation T is *k-recurrent* if for every measurable set A of positive measure, there exists an integer $n > 0$ such that

$$\mu(A \cap T^n A \cap \cdots \cap T^{kn} A) > 0,$$

and T is *multiply recurrent* if it is k-recurrent for every positive integer k.

Analogously, we define *v-recurrence* as the notion that for a given positive measure set A and vector v, there exists an $n > 0$ such that $\mu(A \cap T^{v_1 n} A \cap \cdots \cap T^{v_d n} A) > 0$. We must conclude that v-recurrence and v-positive type, although the latter implies the former, are different notions. An example showing this is given by a conservative ergodic Markov shift such that $T \times T$ is conservative ergodic, while $T \times T \times T$ is not conservative [KP63]. Then T cannot have positive type, otherwise, by [AN00, Proposition 2.2], $T \times T \times T$ will also have positive type, so, in particular, it will be conservative (or more generally, such a T is remotely infinite [BF64], so of zero type [KS69]). T is therefore not positive type and thus clearly not $(1,2)$-positive type. However, by [AN00, Theorem 1.1], $T \times T$ conservative is equivalent, for a general Markov shift T, to $(1,2)$-recurrence. This means that this above example is $(1,2)$-recurrent but not $(1,2)$-positive type.

6. Results for more general sequences. This section contains some basic facts that encourage us to believe that the above definitions for v-pos-

itive type and v-multiplicative-positive type are some of the most natural properties to consider. Specifically, this section explores the quantities $\mu(A \cap T^{n_i} A)$ for a variety of different, more general sequences $\{n_i\}$.

First, we present a simple result indicating that considering sequences other than $\{kn\}$ may prove difficult:

FACT 6.1. *For any h there is an ergodic T of positive type, measure-preserving and invertible, such that for some A of positive measure and for any c not sharing all prime factors of h, $\lim_{n\to\infty} \mu(T^{hn+c} A \cap A) = 0$.*

Proof. To see this, we will construct a generalization of the 2-point extension. First, pick a prime factor k of h such that k does not divide c. Then let $S : X \to X$ be any measure-preserving invertible positive type transformation. (Example: the canonical Hajian–Kakutani skyscraper.) Then, create k copies of this measure space, labeling them X_i and using the same coordinate system on each. Define T as follows: for any $x_1 \in X_1$, let $T(x_1) = x_1 \in X_2$, using the same coordinates as mentioned. Similarly, continue up to $T^{k-1}(x_1) = x_1 \in X_k$ and then let $T^k(x_1) = S(x_1) \in X_1$. Because k is prime, T is conservative ergodic by S's ergodicity. Then, if A is a positive measure set so that $A \subset X_1$, it is very clear that for any n, $\mu(T^{kn+c} A \cap A) = 0$, and so the same assertion with h replacing k must also hold. ∎

FACT 6.2. *For any k there exist T, A, c such that*

$$\limsup \mu(A \cap T^n A \cap T^{kn} A) > 0 \quad but \quad \lim \mu(T^c A \cap T^n A \cap T^{kn} A) = 0.$$

Proof. To obtain an example, just take a rank-one construction where $r_n = k$ and add no spacers infinitely often to get the first inequality. However, pick c to be some integer that is not expressible as $\sum_{n \leq N} a_n h_n$ where each a_n is in $\{-(k-1), \ldots, k+1\}$. If we choose h_n to be increasing quickly enough, there are some c fitting this description. ∎

Given these two facts, it is natural to restrict to considering intersections merely in the form $T^{kn} A \cap A$ without additional constants. Intersections of this form are also already studied in the context of recurrence, so this further seemed a natural choice. However, when still considering sequences in general, we did obtain the following two results:

PROPOSITION 6.3. *If $\lim_{n\to\infty} \mu(A \cap T^{a_n} A \cap T^{a_n+b_n} A) = 0$ for every pair of infinite sequences $\{a_n\}, \{b_n\}$ and every set A of finite positive measure, then T is zero type.*

Proof. The proof proceeds by using the contrapositive. Assume that T is of positive type. Specifically, for any A of positive finite measure we can pick a sequence $\{a_k\}$ so that

$$\lim_{n\to\infty} \mu(A \cap T^{a_n} A) = c > 0.$$

Then, we need to simply pick $\{b_n\}$ and $\{a'_n\}$ such that

$$\lim_{n\to\infty} \mu(A \cap T^{a'_n} A \cap T^{a'_n + b_n} A) > 0.$$

To see that this is possible, note that we may assume that $\mu(T^{a_n} A \cap A) \geq c' = \mu(A)/M$ for some large M. Then, for any beginning N, we note that $\mu(\bigcup_{n=N}^{N+2M} A \cap T^{a_n} A) \leq \mu(A)$ and additionally

$$\mu\left(\bigcup_{n=N}^{N+2M} A \cap T^{a_n} A \right)$$
$$\geq \sum_{n=N}^{N+2M} \mu(A \cap T^{a_n} A) - \sum_{N \leq n,m \leq N+2M} \mu(A \cap T^{a_n} A \cap T^{a_m} A)$$
$$\geq 2\mu(A) - \sum_{N \leq n,m \leq N+2M} \mu(A \cap T^{a_n} A \cap T^{a_m} A),$$

and so

$$\sum_{N \leq n,m \leq N+2M} \mu(A \cap T^{a_n} A \cap T^{a_m}) \geq \mu(A);$$

then there exist some n, m with $n \geq m$ in the given range such that $\mu(A \cap T^{a_n} A \cap T^{a_m} A) \geq \mu(A)/\binom{2M+1}{2}$. We pick $a'_i = a_m$ and $b_i = a_n - a_m$ and then choose successive terms a_{i+1} and b_{i+1} by increasing N. Then, consequently,

$$\limsup_{n\to\infty} \mu(A \cap T^{a'_n} A \cap T^{a'_n + b_n} A) \geq \mu(A)/\binom{2M+1}{2} > 0. \quad \blacksquare$$

PROPOSITION 6.4. *For any pair of infinite sequences $\{a_n\}, \{b_n\}$, there is a T of positive type such that $\lim_{n\to\infty} \mu(A \cap T^{a_n} A \cap T^{a_n + b_n} A) = 0$. In fact, this T can be chosen to be a half-rigid rank-one transformation.*

Proof. The transformation constructed will be a variant of the Hajian–Kakutani skyscraper. At each step of the construction, we divide the column into two equal pieces each of height h_n and stack an additional c_n pieces on top of the rightmost column, where c_n is to be determined. Then, we stack the columns, left under right, and continue. Thus, $h_{n+1} = 2h_n + c_n$, and so our only task is to pick a sequence $\{h_n\}$ so that the transformation satisfies the above condition. We note that if $I = [0, 1]$ is the initial column in the 0th step, then $\mu(I \cap T^{h_n} I) \geq \frac{1}{2}\mu(I)$, and so T is clearly of positive type. In fact, in the nth column, the levels that are contained in I occur at the following heights: $0, h_0, h_1, h_1 + h_0, h_2, h_2 + h_0, h_2 + h_1, h_2 + h_1 + h_0, \ldots, h_n + h_{n-1} + \cdots + h_1$ and therefore moving up by h_n moves the bottom half of these levels up to the top half of these intervals. Let I_n represent the levels in I that occur in the nth column C_n.

Now, we pick $I = [0,1]$. Then $\lim_{n \to \infty} \mu(I \cap T^{a_n} I \cap T^{a_n+b_n} I) = 0$ will hold if for each n, the difference set of the descendants of I in the N column—i.e. the set $\Delta(I,N) = D(I,N) - D(I,N)$ of numbers, each expressible as $\sum_{i=0}^{N} c_i h_i$ with each c_i in $\{-1,0,1\}$—contains for any k at most two of $\{a_k, b_k, a_k + b_k\}$. For example, if for a particular k the set $\Delta(I,N)$ contains a_k and $a_k + b_k$ but not b_k, then $\mu(I \cap T^{a_k} I) \neq 0$ and $\mu(I \cap T^{a_k+b_k} I) \neq 0$ but $\mu(T^{a_k} I \cap T^{a_k+b_k} I) = \mu(I \cap T^{b_k} I) = 0$ by the above discussion, and so $\mu(I \cap T^{a_k} I \cap T^{a_k+b_k} I) = 0$. We made this argument for I, but to expand this conclusion to a general set A of finite measure, we need to make this argument for any column C_m. Numbers in $D(C_m, N) - D(C_m, N)$ are not of the form $\sum_{i=0}^{N} c_i h_i$; instead they are of the form $\sum_{i=m}^{N} a_i h_i + b$ where b's only restriction is that $|b| \leq h_m$. So it is impossible to guarantee the given statement for k with $a_k \leq h_m$. However, because we care about the $\lim_{n \to \infty}$, we concern ourselves with k large enough that $a_k, b_k \geq h_m$. So long as there are at most two of $\{a_k, b_k, a_k + b_k\}$ for these large k, the above argument gives us the statement for C_m. In other words, we will be operating at such a large scale that the column C_m acts like a level. Now, it follows that the statement holds for any finite union of levels, and thus, by an approximation argument, for any finite measure set A.

To choose h_n to avoid all of these individual terms, we first require that $h_n \geq 4h_{n-1}$ (i.e. $c_n \geq 2h_n$). This condition forces all numbers in $\Delta = \bigcup_n \Delta(I,N)$ to have a unique representation in the form $\sum_{i=0}^{N} a_i h_i$ where $a_i \in \{-1,0,1\}$ for each i. Additionally, it means that if $a_k \notin \Delta(I, N-1)$ and $a_k \leq 3h_N$, then $a_k \notin \Delta(I,N)$. Relatedly, this tells us that if $a_k, b_k \in \Delta(I, N-1)$, then either $a_k + b_k \in \Delta(I, N-1)$ or $a_k + b_k \notin \Delta(I,N)$. Then, if $a_k \in \Delta(I, N-1)$ but $b_k \notin \Delta(I, N-1)$, we vary c_{N-1}, starting at $3h_{N-1}$ and increasing if necessary, so that $b_k \notin \Delta(I,N)$. As we increase, we will certainly reach a value of c_{N-1} such that $b_k \leq 3h_N = 3(2h_{N-1} + c_{N-1})$. But there are a variety of k that we have to worry about. Specifically, let

$$K_a = \{k \mid a_k \in \Delta(I, N-1) \text{ but } b_k \notin \Delta(I, N-1)\},$$
$$K_b = \{k \mid b_k \in \Delta(I, N-1) \text{ but } a_k \notin \Delta(I, N-1)\}.$$

Then K_a and K_b are both finite and depend only on h_i, $i \leq N-1$. As we vary c_{N-1}, these sets will not change. Since they are both finite, we can increase c_N without worrying about adding to these sets. In particular, we can make c_{N-1} big enough to miss all a_k for $k \in K_b$ and b_k for $k \in K_a$. Now, we might introduce new a_j, but either we introduce only a_j or we introduce both $a_j, b_j \in \Delta(I,N)$ by increasing c_{n-1}. In the first case, we have not violated any conditions and we will deal with that value of j when we move on to c_n or larger. In the second case, as $a_j, b_j \notin \Delta(I,N)$, we may conclude that $a_j, b_j \geq c_{N-1} \geq 2h_{n-1}$ so that both a_j and b_j have h_N in

their descendent expansions. Thus, $a_j + b_j \geq 2h_N - 2h_{N-1} \geq h_n + 2h_{N-1}$, telling us that $a_j + b_j \notin \Delta(I, N-1)$. Repeating this process, it is clear that for no k will all three of $\{a_k, b_k, a_k + a_k\}$ be in Δ. Consequently, $\lim_{n \to \infty} \mu(A \cap T^{a_n} A \cap T^{a_n + b_n} A) = 0$. ∎

In fact, the above constructed T also has

$$\lim_{n \to \infty} \mu(A \cap T^{a_n}) \mu(A \cap T^{b_n} A) \mu(A \cap T^{a_n + b_n} A) = 0.$$

The above examples show that v-positive type, due partly to its similarity to k-recurrence and partly due to the examples above, is a natural generalization to consider.

7. Further explorations: positive type for different v. We now want to discuss vectors v, w in pairs. If T is v-positive type, is it w-positive type? We start with a new definition for comparing vectors. Unfortunately, we will require different definitions for v-positive type and v-multiplicative-positive type.

DEFINITION 7.1. Given two vectors $v \in \mathbb{Z}^d$ and $w \in \mathbb{Z}^{d'}$, if there exist positive integers n, m, c such that the set of components of nv is included in the set of components of $mw - c$, where $c = 0$ or $c = mw_i$ for some i, we say that v is *positive type less than* w, from now on denoted $v \leq_p w$. (Note that this property requires automatically that $d \leq d'$.)

Furthermore, if there exist positive integers n, m, c such that each $nv_i = \sum_{j \in I(i)} mw_j - \sum_{j' \in J(i)} mw_{j'}$, then we say that v is *multiplicative-positive type less than* w, denoted $v \leq_m w$. Note that $v \leq_p w \Rightarrow v \leq_m w$ but the reverse direction does not hold.

One note about the above definitions: in the following theorems using $\leq_m$ and $\leq_p$, we note that these constructions often hold only for vectors v in the accepted form, that is, with v_i distinct. Any transformation T of positive type has v-positive type for any $v = (\ell, \ldots, \ell) \in \mathbb{Z}^d$. These or other such "redundant vectors" need to be considered differently from the vectors in standard format.

Observe also that if $v \leq_m w$, and T is w-multiplicative-positive type, then T is also v-multiplicative-positive type. Indeed, suppose that nv has all its components contained in mw. If T has w-multiplicative-positive type, then T^m has w-multiplicative-positive type, so T has mw-multiplicative-positive type, hence T has nv-multiplicative-positive type. If we suppose that T has v-multiplicative-zero type, then T will have trivially nv-multiplicative-zero type. As this is a contradiction, T also has v-multiplicative-positive type.

THEOREM 7.2. *Suppose* $v \in \mathbb{Z}^d$. *Then there exists a rank-one transformation* T *which has* v-*multiplicative-positive type, but which does not have* w-*multiplicative-positive type for any* $w \not\leq_m v$.

Proof. We will construct a rank-one transformation which satisfies the above conditions as follows. Let $I = C_0 = [0,1]$. For each m, we construct C_{m+1} by cutting C_m in $d+1$ parts and adding spacers on the first d sub-columns such that the number of levels between the first $(d+1)$th of the bottom level of C_m and the bottom $(d+1)$ths of the other levels are

$$D' := \{0, v_1 h_m, v_2 h_m, \ldots, v_d h_m\}.$$

On the last subcolumn, we add a lot of spacers. We add sufficiently many spacers s_n so that there is a unique representation of elements in $D(I, N) - D(I, N)$ in base (h_i) and in fact so that where $H_m = \sum_{k \leq m} \sum_{1 \leq i \leq d} v_i h_k$ we have the ratio h_{m+1}/H_m grow without bound. Then we stack from left to right. See Figure 5 for an illustration of this construction where $d = 3$.
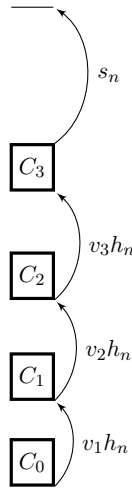

Fig. 5

Now, to see that T is v-multiplicative-positive type we note that for any $n = h_m$ we have (using the notion of descendants)

$$\mu(I \cap T^{v_1 h_m} I) \cdots \mu(I \cap T^{v_d h_m} I)$$

$$\geq \left(\frac{\mu(I)}{d+1} |D' \cap D' + v_1 h_m| \right) \cdots \left(\frac{\mu(I)}{d+1} |D' \cap D' + v_d h_m| \right)$$

$$\geq \left(\frac{1}{d+1} \right)^d |D' \cap D' + v_1 h_m| \cdots |D' \cap D' + v_d h_m| = \left(\frac{1}{d+1} \right)^d.$$

In particular, this proves that

$$\limsup \mu(I \cap T^{v_1 n} I) \cdots \mu(I \cap T^{v_d n} I) > 0.$$

By the lemmas in earlier sections, as we have this constant $\left(\frac{1}{d+1}\right)^d$ delineating the positive overlap, this argument shows T is v-multiplicative-positive type for all A of positive measure.

Now, given any $w \in \mathbb{Z}^{d'}$, if for every M there exists an $n > M$ such that $\mu(I \cap T^{v_1 n} I) \cdots \mu(I \cap T^{v_d n} I) > 0$, we can show that $w \leq_m v$ as described above. First, pick an M large enough so that

$$h_M > \frac{w_{d'}}{w_1} H_{M-1}.$$

Then, if we have $n > M$ and the minimum $N(n)$ such that

$$w_1 n, \ldots, w_c n \in D(I, N) - D(I, N),$$

by the construction of the columns C_n, there is a unique representation of elements in $D(I, N) - D(I, N)$ in base (h_j). For each $1 \leq i \leq d'$ let $A_i h_{\ell_i}$ be the biggest term in the expansion of $w_i n$. Then, for any i, j derived this way, we note that because $h_{m+1} \gg h_m$ and by our choice of M, we must have $h_{\ell_i} = h_{\ell_j} = h_N$. (Thus $\ell_i = \ell_j = N$.) In fact, we note that when we express w_i / w_j using the expansions, the h_N dominate all other terms. For large enough n, as $w_1 n, \ldots, w_{d'} n$ must all occur in the same scale ($h_m \geq w_i n \geq h_{m+1}$), and as the construction gives the same method for cutting and stacking at every n, there are a finite number of possibilities for A_i, A_j as we increase n. As h_{m+1}/H_M grows without bound, for some choice of A_i, A_j, as we increase n we see that

$$\frac{A_i}{A_j} - \lim \frac{H_{n-1}}{h_n} \leq \frac{w_i}{w_j} \leq \frac{A_i}{A_j} + \lim \frac{H_{n-1}}{h_n}$$

with both limits zero, so that

$$\frac{w_i}{w_j} = \frac{A_i}{A_j} \quad \text{or} \quad w_i = A_i\left(\frac{w_1}{A_1}\right).$$

However, for each i,

$$A_i \in \left\{ 0, v_1, v_2, \ldots, v_d, v_2 - v_1, \ldots, v_{d-1} - v_d, \ldots, \sum_I v_j - \sum_J v_i \right\}$$

for certain partitions I and J by the construction of T. Substituting these terms in explicitly, we see that $w_i = \left(\sum v_k - \sum v_k'\right)\frac{w_1}{\sum v_j - \sum v_j'}$, so $w_i \sum v_j - w_i \sum v_j' = \sum v_k w_1 - \sum v_k' w_1$. Note that the v_j, v_j' (derived from A_1) are constant and the v_k, v_k' (derived from A_i) are not. Thus by definition, $w \leq_m v$.

One note: not all $w \leq_m v$ will have w-multiplicative-positive type for this T. However, variations S on T can be constructed so that S will be w-multiplicative-positive type. These variations are obtained by putting the spacers on the subcolumns so that the order is varied, $v_2 h_n$ before $v_1 h_n$, for example. ∎

THEOREM 7.3. *The same statement as Theorem 7.2 holds for v-positive type, replacing $\leq_m$ with $\leq_p$.*

Proof. We construct a similar transformation, with the spacer pattern seen in Figure 6 so that there are enough spacers s_n that all elements of $D(I, N) - D(I, N)$ have a unique representation in (h_i).
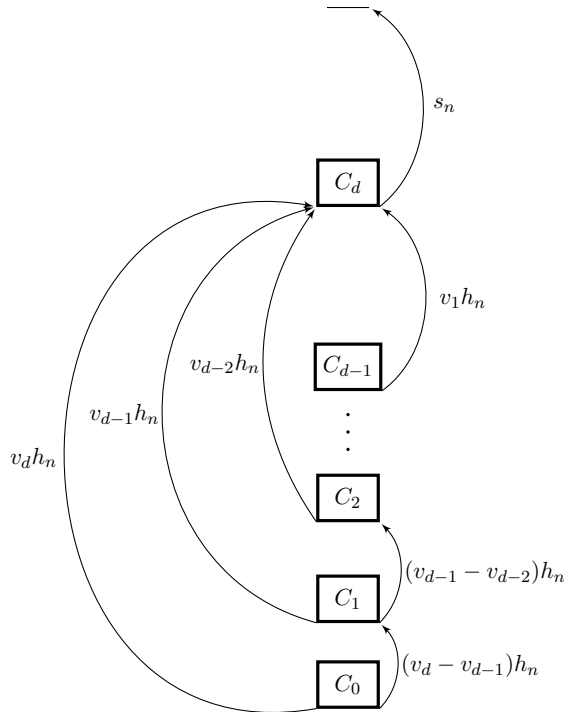

Fig. 6

Then it is clear that this transformation is v-positive type. To see that it is only v-positive type for $w \leq_p v$, if it is w-positive type, we must have an infinite sequence of n with $w_i n \in M(n) - D(I, N)$, where $M(n)$ is fixed and in $D(I, N)$. Say $w_i n$ is expanded with coefficients A_i in (h_i). Then note again that through the unique representation of $w_i n$, we pick out the largest term $A_i h_{\ell_i}$ for each $1 \leq i \leq d'$ and B, the largest term of $M(n)$ (that is, no larger than the largest term of w_i). All h_{ℓ_i} must be equal, once we choose n large enough compared to the terms of w. Then, as before, $w_i/w_j = A_i/A_j$ but furthermore by our construction, $A_i, A_j \in \{v_1 - v_J, v_2 - v_J, \ldots v_d - v_J\}$ where v_J depends only on the value of B, and then the equation $w_i = w_1(A_i)/A_1$ actually implies that $w_i = w_1(v_k - v_J)/(v_j - v_J)$; this gives the correct result, as the only term that varies with w_i is v_k. ∎

COROLLARY 7.4. *Let $\mathcal{V} = \{v\}$ be a finite family of finite vectors. Then there exists a rank-one transformation T which is v-multiplicative-positive type for all $v \in \mathcal{V}$, but which is not w-multiplicative-positive type for any finite vector w with $w \not\leq_m v$ for all $v \in \mathcal{V}$. Similarly for positive type.*

Proof. We will use the same process as in the proof of the theorem above to produce a transformation which has the required properties. If $\mathcal{V} = V_1, \ldots, V_D$ we define the transformation T as follows: for each $m > 1$ with largest prime power p_j we divide C_m into $d+1$ columns where $V_i \in \mathbb{Z}^{d(V)}$ so that $V_i = (V_{i,1}, \ldots, V_{i,d(V)})$ for $i = j$ mod D and then we add spacers exactly as above except for the last column, where there are enough spacers added so that $h_{m+1} \geq H_M$ where $H_M = \sum_{k \leq m} \sum_{i \leq m} \sum_{1 \leq j \leq d(V)} V_{i,j} h_k$.

Because for any prime there are an infinite number of powers p^i, for any $V_j \in \mathcal{V}$, the corresponding prime p gives that

$$\mu(I \cap T^{h_{p^i} V_{j,1}} I) \cdots \mu(I \cap T^{h_{p^i} V_{j,d(V_j)}} I) \geq \left(\frac{\mu(I)}{d(V_j) + 1} \right)^{d(V_j)},$$

and so T is V_J-positive type.

However, using an argument similar to the above it can be seen that if T is w-positive type, there must be a $v \in \mathcal{V}$ such that $w \leq_m v$.

Extension onto Theorem 7.3 is done exactly analogously. ∎

This section essentially showed that except for the "natural" dependencies, different vectors $v, w \in (\mathbb{Z}^+)^d$ behave independently for v-positive type.

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Tudor Pădurariu   Cesar E. Silva
University of California   Department of Mathematics
Los Angeles, CA 90095-1555, U.S.A.   Williams College
E-mail: tudor_pad@yahoo.com   Williamstown, MA 01267, U.S.A.
                                            E-mail: csilva@williams.edu
Evangelie Zachos
Princeton University
Princeton, NJ 08544, U.S.A.
E-mail: evangelie.zachos@gmail.com