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A VARIANT THEORY FOR THE GORENSTEIN FLAT DIMENSION

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Abstract. This paper discusses a variant theory for the Gorenstein flat dimension. Actually, since it is not yet known whether the category $\mathcal{GF}(R)$ of Gorenstein flat modules over a ring R is projectively resolving or not, it appears legitimate to seek alternate ways of measuring the Gorenstein flat dimension of modules which coincide with the usual one in the case where $\mathcal{GF}(R)$ is projectively resolving, on the one hand, and present nice behavior for an arbitrary ring R, on the other. In this paper, we introduce and study one of these candidates called the generalized Gorenstein flat dimension of a module M and denoted by $\mathrm{GGfd}_R(M)$ via considering exact sequences of modules of finite flat dimension. The new entity stems naturally from the very definition of Gorenstein flat modules. It turns out that the generalized Gorenstein flat dimension enjoys nice behavior in the general setting. First, for each R-module M, we prove that $\mathrm{GGfd}_R(M) = \mathrm{Gid}_R(\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}))$ whenever $\mathrm{GGf}_R(M)$ is finite. Also, we show that $\mathcal{GF}(R)$ is projectively resolving if and only if the Gorenstein flat dimension and the generalized Gorenstein flat dimension coincide. In particular, if R is a right coherent ring, then $\mathrm{GGfd}_R(M) = \mathrm{Gfd}_R(M)$ for any R-module M. Moreover, the global dimension associated to the generalized Gorenstein flat dimension, called the generalized Gorenstein weak global dimension and denoted by $\text{GG-wgldim}(R)$, turns out to be the best counterpart of the classical weak global dimension in Gorenstein homological algebra. In fact, it is left-right symmetric and it is related to the cohomological invariants $\text{r-sfli}(R)$ and $\text{l-sfli}(R)$ by the formula

$$\text{GG-wgldim}(R) = \max\{\text{r-sfli}(R), \text{l-sfli}(R)\}.$$

1. Introduction. Throughout this paper, R denotes an associative ring with identity element. All modules, if not otherwise specified, are assumed to be left R-modules. Also, given an R-module M, we denote by $\mathrm{fd}_R(M)$ the *flat dimension* of M, that is, the least positive integer n such that there is a short exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ with $F_0, F_1, \ldots, F_n$ flat modules, and ∞ if no such short exact sequence exists.

Recall that Gorenstein projective (resp., Gorenstein injective, Gorenstein flat) modules originate from the classical notions of projective (resp., injective, flat) modules by standing as images and kernels of the differentials of

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complete projective (resp., injective, flat) resolutions. Effectively, a module M is said to be *Gorenstein projective* if there exists an exact sequence of projective modules, called a *complete projective resolution*,

$$\mathbf{P} := \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$$

such that $\mathbf{P}$ remains exact after applying the functor $\mathrm{Hom}_R(-, P)$ for each projective module P and $M := \mathrm{Im}(P_0 \to P_{-1})$. Gorenstein injective modules are defined dually. Also, a module M is said to be *Gorenstein flat* if there exists an exact sequence of flat modules

$$\mathbf{F} := \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$$

such that $\mathbf{F}$ remains exact after applying the functor $I \otimes_R -$ for each injective right R-module I and $M := \mathrm{Im}(F_0 \to F_{-1})$. These new concepts allowed Enochs and Jenda [19, 20] to introduce new (Gorenstein homological) dimensions in order to extend the G-dimension defined by Auslander and Bridger [1, 2]. It turns out, in particular, that these Gorenstein homological dimensions are refinements of the classical dimensions of a module M, in the sense that $\mathrm{Gpd}_R(M) \leq \mathrm{pd}_R(M)$, $\mathrm{Gid}_R(M) \leq \mathrm{id}_R(M)$ and $\mathrm{Gfd}_R(M) \leq \mathrm{fd}_R(M)$ with equality each time the corresponding classical homological dimension is finite. The reader is referred to [3, 6, 12, 14, 15, 19, 20, 23–25, 28–30] for basics and recent investigations on Gorenstein homological theory, as well as some topics related to resolutions of flat modules.

It remains one of the key open problems of Gorenstein homological algebra whether the category $\mathcal{GF}(R)$ of Gorenstein flat modules is projectively resolving or not. The absence of this latter property makes it intricate to deal with the Gorenstein flat dimension except in the setting of right coherent rings where this property is satisfied. Motivated by the absence of this latter property, we started exploring alternative ways of measuring the Gorenstein flat dimension of a module M which coincide with the usual one in the setting of a left GF-closed ring R. In [10], we introduced and studied a new invariant called the cover Gorenstein flat dimension of a module M, denoted by $\mathrm{CGfd}_R(M)$. In fact, given a module M, for any exact sequence $0 \to M \to E \to G \to 0$ such that $\mathrm{fd}_R(E) < \infty$ and G is Gorenstein flat, we pointed out that $\mathrm{fd}_R(E)$ is an invariant which depends only on M and not on the choice of the exact sequence. This allowed us to define the new cover Gorenstein dimension of a module M as follows: $\mathrm{CGfd}_R(M) =: n$ if there exists an exact sequence $0 \to M \to E \to G \to 0$ such that $\mathrm{fd}_R(E) = n$ and G is Gorenstein flat, and $\mathrm{CGfd}_R(M) = \infty$ if no such exact sequence exists. We proved that, for any R-module M,

$$\mathrm{Gid}_R(M^+) \leq \mathrm{Gfd}_R(M) \leq \mathrm{CGfd}_R(M) \leq \mathrm{fd}_R(M)$$

with $\mathrm{Gid}_R(M^+) = \mathrm{Gfd}_R(M) = \mathrm{CGfd}_R(M)$ if $\mathrm{CGfd}_R(M) < \infty$. Moreover, we proved that $\mathcal{GF}(R)$ is projectively resolving if and only if the cover Gorenstein flat dimension and the Gorenstein flat dimension coincide. We gave many properties of the new dimension as well as its corresponding global dimension called the cover Gorenstein weak global dimension.

In the present paper, in which we continue our work started in [10, 11], we introduce and study a new invariant called the generalized Gorenstein flat dimension of a module M, denoted by $\mathrm{GGfd}_R(M)$, via considering exact sequences of modules of finite flat dimension. This entity stems naturally from the very definition of Gorenstein flat modules via replacing the concept of a complete flat resolution by the new one of a generalized complete flat resolution. The new dimension turns out to behave better than the Gorenstein flat dimension. First, we show that, for each R-module M,

$$\mathrm{Gid}_R(\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})) \leq \mathrm{GGfd}_R(M) \leq \mathrm{fd}_R(M)$$

with

$$\mathrm{Gid}_R(\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})) = \mathrm{GGfd}_R(M) \quad \text{whenever } \mathrm{GGfd}_R(M) \text{ is finite.}$$

Also, we prove that the following assertions are equivalent:

- *$\mathcal{GF}(R)$ is projectively resolving;*
- $\mathrm{Gfd}_R(M) = \mathrm{GGfd}_R(M)$ *for each R-module M;*
- $\mathrm{Gfd}_R(M) = \mathrm{Gid}_R(\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}))$ *for each R-module M such that* $\mathrm{Gfd}_R(M) < \infty$.

On the other hand, we prove that the finitistic dimension associated to the generalized Gorenstein flat dimension of an arbitrary ring R, defined by

$$\mathrm{FGGFD}(R) := \sup\{\mathrm{GGfd}_R(M) : M \text{ is an } R\text{-module with } \mathrm{GGfd}_R(M) < \infty\},$$

coincides with the known finitistic flat dimension $\mathrm{FFD}(R)$ of R. It is worth reminding, in this regard, that the finitistic Gorenstein flat dimension

$$\mathrm{FGFD}(R) := \sup\{\mathrm{Gfd}_R(M) : M \text{ is an } R\text{-module with } \mathrm{Gfd}_R(M) < \infty\}$$

coincides with $\mathrm{FFD}(R)$ in the restricted setting of a right coherent ring [25, Theorem 3.24]. Further, the generalized Gorenstein global weak dimension of R, defined by $\text{l-GG-wgldim}(R) := \sup\{\mathrm{GGfd}_R(M) : M \text{ is a left } R\text{-module}\}$, manifests itself as the best counterpart of the classical global weak dimension in Gorenstein homological algebra. Actually, it is left-right symmetric, and it is connected to to the cohomological invariants $\text{l-sfli}(R) := \{\mathrm{fd}_R(I) : I \text{ is an injective left } R\text{-module}\}$ and $\text{r-sfli}(R) := \{\mathrm{fd}_R(I) : I \text{ is an injective right } R\text{-module}\}$ by the equality

$$\text{l-GG-wgldim}(R) = \text{r-GG-wgldim}(R) = \max\{\text{l-sfli}(R), \text{r-sfli}(R)\}.$$

$\text{GG-wgldim}(R)$ will denote the common value of $\text{l-GG-wgldim}(R)$ and $\text{r-GG-wgldim}(R)$. Moreover, if R is left Noetherian (resp., right Noetherian), we prove that

$$\text{GG-wgldm}(R) = \text{l-G-gldim}(R)$$

(resp., GG-wgldim(R) = r-G-gldim(R)). These two results are the Gorenstein versions of [27, Theorems 9.15 and 9.22], the latter giving the relation between the global dimension and the weak global dimension.

In the end, it may be interesting to note that the problems we investigate in this paper are closely related to some problems on periodic resolutions of flat modules studied by D. Benson and K. Goodearl [9] and D. Simson [32], and recently generalized by the author and M. Khaloui [13].

2. Generalized Gorenstein flat dimension. The goal of this section is to introduce and study the generalized Gorenstein flat dimension. The newly introduced invariant behaves better than the known Gorenstein flat dimension in the general setting.

First, it is worth reminding the reader of the adjointness isomorphism for derived functors

$$\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Tor}_n^R(A,B),\mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}_R^n(A,\operatorname{Hom}_{\mathbb{Z}}(B,\mathbb{Q}/\mathbb{Z}))$$

for any left R-module B and any right R-module A. Throughout, for any (left) R-module M, we denote by M^+ the Pontryagin dual $\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ of M.

For the convenience of the reader, we begin by giving a brief account on resolving classes of modules and basic properties of copure injective dimension and copure flat dimension. Recall that a class Γ of R-modules is called *projectively resolving* if Γ includes all projective modules and for any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \Gamma$ we have $X \in \Gamma$ if and only if $X' \in \Gamma$. Similarly, Γ is called *injectively resolving* if Γ includes all injective modules and for any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \Gamma$ we have $X \in \Gamma$ if and only if $X'' \in \Gamma$. In this context, recall that the category of Gorenstein projective (resp., injective) modules, denoted by $\mathcal{GP}(R)$ (resp., $\mathcal{GI}(R)$), is projectively (resp., injectively) resolving. As to the category of Gorenstein flat modules $\mathcal{GF}(R)$, it is still an open problem whether it is projectively resolving or not.

Moreover, recall that the notions of copure injective and copure flat module were introduced and studied by Enochs and Jenda [22]. A module M is said to be *copure injective* (resp., *copure flat*) if $\operatorname{Ext}_R^1(I,M) = 0$ (resp., $\operatorname{Tor}_1^R(I,M) = 0$) for any injective left (resp., right) R-module I. A module M is said to be *strongly copure injective* (resp., strongly copure flat) if $\operatorname{Ext}_R^n(I,M) = 0$ (resp., $\operatorname{Tor}_n^R(I,M) = 0$) for any injective left (resp., right) R-module I and any integer $n \geq 1$. Also, Enochs and Jenda introduced the copure injective dimension and the copure flat dimension as follows: Let M

be an R-module. Then

$\mathrm{cid}_R(M) = \sup\{n \in \mathbb{N} : \mathrm{Ext}_R^n(I, M) \neq 0$ for some injective module $I\}$,

$\mathrm{cfd}_R(M) = \sup\{n \in \mathbb{N} : \mathrm{Tor}_n^R(I, M) \neq 0$ for some injective right module $I\}$.

Notice that these two dimensions are refinements of the injective dimension and flat dimension respectively in the sense that $\mathrm{cid}_R(M) \leq \mathrm{id}_R(M)$ (resp., $\mathrm{cfd}_R(M) \leq \mathrm{fd}_R(M)$) with equality if $\mathrm{id}_R(M) < \infty$ (resp., $\mathrm{fd}_R(M) < \infty$).

Next, we collect the basic properties of the copure flat dimension.

Proposition 2.1.

(1) *Let M be an R-module and $n \geq 1$ an integer. Then the following assertions are equivalent:*
   (a) $\mathrm{cfd}_R(M) \leq n$.
   (b) *For each exact sequence $0 \to K \to E_{n-1} \to \cdots \to E_1 \to E_0 \to M \to 0$ such that the E_i are strongly copure flat modules, K is strongly copure flat.*
   (c) *For each exact sequence $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ such that the F_i are flat modules, K is strongly copure flat.*

(2) *Let $0 \to N \to E \to M \to 0$ be an exact sequence of R-modules.*
   (a) *If E is strongly copure flat and $\mathrm{cfd}_R(M) \geq 1$, then $\mathrm{cfd}_R(M) = 1 + \mathrm{cfd}_R(N)$.*
   (b) $\mathrm{cfd}_R(M) \leq 1 + \max\{\mathrm{cfd}_R(E), \mathrm{cfd}_R(N)\}$.
   (c) $\mathrm{cfd}_R(E) \leq \max\{\mathrm{cfd}_R(M), \mathrm{cfd}_R(N)\}$.

(3) *Let $\cdots \to E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \to \cdots$ be an exact sequence of R-modules with $M_i := \mathrm{Im}(d_i)$ for each integer i. Then*

$$\sup\{\mathrm{cfd}_R(E_i) : i \in \mathbb{Z}\} \leq \sup\{\mathrm{cfd}_R(M_i) : i \in \mathbb{Z}\}$$

with equality if $\sup\{\mathrm{cfd}_R(M_i) : i \in \mathbb{Z}\}$ is finite.

Proof. The proofs of (1) and (2) are routine. Let us prove (3). By shifting and summing we get the periodic exact sequence

$$\cdots \xrightarrow{\bigoplus d_i} \bigoplus_{i\in\mathbb{Z}} E_i \xrightarrow{\bigoplus d_i} \bigoplus_{i\in\mathbb{Z}} E_i \xrightarrow{\bigoplus d_i} \bigoplus_{i\in\mathbb{Z}} E_i \xrightarrow{\bigoplus d_i} \cdots$$

with $\mathrm{Im}(\bigoplus d_i) = \bigoplus_i M_i$. Considering the derived short exact sequence

$$0 \to \bigoplus_i M_i \to \bigoplus_i E_i \to \bigoplus_i M_i \to 0$$

and using (2)(c), we get $\mathrm{cfd}_R(\bigoplus_i E_i) \leq \mathrm{cfd}_R(\bigoplus_i M_i)$, that is, $\sup\{\mathrm{cfd}_R(E_i) : i \in \mathbb{Z}\} \leq \sup\{\mathrm{cfd}_R(M_i) : i \in \mathbb{Z}\}$.

Now, assume that $\sup\{\mathrm{cfd}_R(M_i) : i \in \mathbb{Z}\}$ is finite. Then $\mathrm{cfd}_R(\bigoplus_i M_i)$ is finite. Via considering the associated long exact sequence to $0 \to \bigoplus_i M_i \to \bigoplus_i E_i \to \bigoplus_i M_i \to 0$ with respect to the functor $\mathrm{Tor}(I, -)$ for each injective right R-module I, we get $\mathrm{cfd}_R(\bigoplus_i E_i) = \mathrm{cfd}_R(\bigoplus_i M_i)$ yielding the desired equality. ∎

DEFINITION 1. Let R be a ring.

(1) Let $\mathbf{E} = \cdots \to E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \to \cdots$ be an exact sequence of R-modules, and $M_i := \mathrm{Im}(d_i)$ for each integer i. The sequence $\mathbf{E}$ is called a *generalized complete flat resolution* if the sets $\{\mathrm{fd}_R(E_i) : i \in \mathbb{Z}\}$ and $\{\mathrm{cfd}_R(M_i) : i \in \mathbb{Z}\}$ are bounded.

(2) An R-module M is called a *generalized Gorenstein flat module* if M is the kernel or the image of a differential of a generalized complete flat resolution.

REMARK 1. Let $\mathbf{E} = \cdots \to E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \to \cdots$ be a generalized complete flat resolution and let $M_i := \mathrm{Im}(d_i)$ for each integer i. Then, as $\mathrm{cfd}_R(E_i) = \mathrm{fd}_R(E_i)$ for each integer i and applying Proposition 2.1(3), we get

$$\sup\{\mathrm{fd}_R(E_i) : i \in \mathbb{Z}\} = \sup\{\mathrm{cfd}_R(M_i) : i \in \mathbb{Z}\}.$$

DEFINITION 2.

(1) Let $\mathbf{E} = \cdots \to E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \to \cdots$ be a generalized complete flat resolution and let $M_i := \mathrm{Im}(d_i)$ for each integer i. The common value $\sup\{\mathrm{fd}_R(E_i) : i \in \mathbb{Z}\} = \sup\{\mathrm{cfd}_R(M_i) : i \in \mathbb{Z}\}$ is called the *degree* of $\mathbf{E}$.
(2) Let $n \geq 0$ be an integer. An exact sequence $\mathbf{E} = \cdots \to E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \to \cdots$ is called a *complete n-flat resolution* if it is a generalized complete flat resolution of degree n.
(3) Let $n \geq 0$ be an integer. An R-module M is called a *Gorenstein n-flat R-module* if M is the kernel or image of a differential of a complete n-flat resolution.

REMARK 2. It is clear from Definition 2 that if $\mathbf{E} = \cdots \to E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \to \cdots$ is a complete n-flat resolution, then $\mathrm{fd}_R(E_i) \leq n$ and $\mathrm{Tor}^R_{k+1}(I, M_i) = 0$ for each $M_i := \mathrm{Im}(d_i)$, each integer $k \geq n$ and each injective right R-module I. Complete 0-flat resolutions coincide with the known complete flat resolutions.

If R is a ring, then l-sfli(R) is defined to be the supremum of the flat lengths of injective left R-modules, that is,

$$\text{l-sfli}(R) := \sup\{\mathrm{fd}_R(I) : I \text{ is an injective left } R\text{-module}\}.$$

Similarly,

$$\text{r-sfli}(R) := \sup\{\text{fd}_R(I) : I \text{ is an injective right } R\text{-module}\}.$$

The following result characterizes complete flat resolutions in the setting of a ring R such that $\text{r-sfli}(R) < \infty$.

Proposition 2.2. *Let R be a ring.*

(1) *Assume that* $\text{r-sfli}(R) < \infty$. *Let* $\mathbf{E} = \cdots \to E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \to \cdots$ *be an exact sequence and* $M_i = \text{Im}(d_i)$ *for each integer* i. *Then* $\mathbf{E}$ *is a generalized complete flat resolution if and only* $\sup\{\text{fd}_R(E_i) : i \in \mathbb{Z}\} < \infty$.
(2) *Assume that* $\text{r-sfli}(R) < \infty$ *and* $\text{l-sfli}(R) < \infty$. *Then any left* (*resp., right*) *R-module M is a left* (*resp., right*) *generalized Gorenstein flat module.*

Proof. (1) It suffices to observe that $\text{cfd}_R(M) \leq \text{r-sfli}(R) < \infty$ for each left R-module M.

(2) Assume that $\text{r-sfli}(R) < \infty$ and $\text{l-sfli}(R) < \infty$. Let M be a left R-module. Consider the exact sequence $\mathbf{E} = \cdots \to F_1 \to F_0 \to I_0 \to I_1 \to \cdots$, where $\cdots \to F_1 \to F_0 \to M \to 0$ is a flat resolution of M and $0 \to M \to I_0 \to I_1 \to \cdots$ is an injective resolution of M. As $\text{l-sfli}(R) < \infty$, we get $\sup\{\text{fd}_R(I_j) : j \geq 0 \text{ an integer}\} \leq \text{l-sfli}(R) < \infty$. It follows, by (1), that E is a generalized complete flat resolution, and thus M is a left generalized Gorenstein flat R-module. A similar argument shows that any right R-module M is a right generalized Gorenstein flat module, as desired. ■

We next introduce the generalized Gorenstein flat dimension.

Definition 3.

(1) Let M be an R-module. We define the *generalized Gorenstein flat dimension* of M as follows:

$$\text{GGfd}_R(M) = \begin{cases} \text{cfd}_R(M) & \text{if } M \text{ is a generalized Gorenstein flat } R\text{-module,} \\ \infty & \text{otherwise.} \end{cases}$$

(2) An R-module M is called G-*Gorenstein flat* if $\text{GGfd}_R(M) = 0$. We denote by $\mathcal{GGF}(R)$ the category of G-Gorenstein flat modules.

If $\mathcal{A}$ is a class of modules, we denote by $\mathcal{A}^+$ the class of all dual modules M^+ such that $M \in \mathcal{A}$. Recall that, given a module M, if M^+ is injective, then M is necessarily flat. The corresponding property in Gorenstein homological algebra, that is,

$$M^+ \text{ is Gorenstein injective } \Rightarrow M \text{ is Gorenstein flat,}$$

is not yet verified, while its converse is known to be true (see [25, Proposition 3.11]). In this context, our next result shows that the Pontryagin dual of any G-Gorenstein flat module is Gorenstein injective. This means that if the above implication "M^+ is Gorenstein injective $\Rightarrow M$ is Gorenstein flat" is true, then the notions of Gorenstein flat module and G-Gorenstein flat module coincide.

PROPOSITION 2.3. *Let R be a ring. Then:*

(1) $\mathcal{GF}(R) \subseteq \mathcal{GGF}(R)$.
(2) $\mathcal{GF}(R)^+ \subseteq \mathcal{GGF}(R)^+ \subseteq \mathcal{GI}(R)$.

Proof. (1) It is straightforward.

(2) Let M be a G-Gorenstein flat module. Then there exists a complete n-flat resolution (for some positive integer n)

$$\mathbf{E} = \cdots \to E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \to \cdots$$

such that $M = \mathrm{Im}(d_0)$. Let us consider the dual exact sequence

$$\mathbf{E}^+ = \cdots \to E_{-1}^+ \to E_0^+ \to E_1^+ \to \cdots .$$

Then $\mathrm{id}_R(E_i^+) = \mathrm{fd}_R(E_i) \leq n$ and, by [22, Lemma 3.4], $\mathrm{cid}_R(M_i^+) = \mathrm{cfd}_R(M_i) \leq n$ for each integer i. Hence $\mathbf{E}$ is a complete n-injective resolution [11, Definition 2.1], and thus, by [11, Theorem 2.15], $\mathrm{Gid}_R(M^+) < \infty$. Note that, as $\mathrm{GGfd}_R(M) = 0$, we get $\mathrm{cfd}_R(M) = \mathrm{cid}_R(M^+) = 0$ and it follows, by [25, Theorem 2.22], that $\mathrm{Gid}_R(M^+) = \mathrm{cid}_R(M^+) = 0$. Consequently, M^+ is Gorenstein injective. ∎

We will prove that the categories $\mathcal{GF}(R)$ and $\mathcal{GGF}(R)$ coincide whenever R is left GF-closed, in particular when R is right coherent. Next, we list various properties of generalized Gorenstein flat modules.

PROPOSITION 2.4. *Let M be an R-module.*

(1) *If M is Gorenstein n-flat for some positive integer n, then* $\mathrm{GGfd}_R(M) \leq n$.
(2) *The following assertions are equivalent:*
   (a) *M is a generalized Gorenstein flat module.*
   (b) $\mathrm{GGfd}_R(M) < \infty$.
(3) *If* $\mathrm{fd}_R(M) = n < \infty$, *then M is a Gorenstein n-flat module and* $\mathrm{GGfd}_R(M) = n$.
(4) *Let* $0 \to N \to E \xrightarrow{d} M \to 0$ *be an exact sequence such that* $\mathrm{fd}_R(E) < \infty$ *and M is a generalized Gorenstein flat module. Then N is a generalized Gorenstein flat module.*

Proof. (1) and (2) are straightforward.

(3) If $\mathrm{fd}_R(M) = n$, then it suffices to note that the exact sequence of R-modules $0 \to M \to M \to 0$ is a complete n-flat resolution.

(4) Let $0 \to N \to E \xrightarrow{d} M \to 0$ be an exact sequence such that $\mathrm{fd}_R(E) < \infty$ and M is a generalized Gorenstein flat module. Let $\mathbf{E} = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots$ be a Gorenstein n-flat resolution for some positive integer n with $M := \mathrm{Im}(E_0 \to E_{-1})$ and let $m := \max\{\mathrm{fd}_R(E), n\}$. Observe that $\mathrm{GGfd}_R(M) = \mathrm{cfd}_R(M) \leq n$, and thus $\mathrm{cfd}_R(N) \leq m$. Let $\cdots \to F_1 \to F_0 \to N \to 0$ be a flat resolution of N. Then it is readily checked that $\cdots \to F_1 \to F_0 \to E \to E_{-1} \to E_{-2} \to \cdots$ is a generalized complete flat resolution of degree $r := \sup\{\mathrm{fd}_R(E), \mathrm{fd}_R(E_i) : i \leq -1\} \leq m$, so that N is a generalized Gorenstein flat module, as desired. ∎

It is well known that when the Gorenstein projective dimension (resp., the Gorenstein injective dimension) of a module M is finite, one might express $\mathrm{Gpd}_R(M)$ (resp., $\mathrm{Gid}_R(M)$) in terms of the vanishing of the functor Ext, or in terms of the Gorenstein projectivity (resp., Gorenstein injectivity) of syzygies (resp., cosyzygies) of projective (resp., injective) resolutions of M [25, Theorems 2.20 and 2.22]. In this regard, note that for the Gorenstein flat dimension this property still resists proof. Our next theorem shows that the generalized Gorenstein flat dimension behaves better in this respect.

THEOREM 2.5. *Let M be a generalized Gorenstein flat R-module issued from a generalized complete flat resolution of degree r. Then:*

(1) $\mathrm{Gfd}_R(M) < \infty$, *and more precisely,* $\mathrm{GGfd}_R(M) \leq \mathrm{Gfd}_R(M) \leq r$.

(2) *Let $n \geq 0$ be an integer. The following assertions are equivalent:*

  (a) $\mathrm{GGfd}_R(M) \leq n$.

  (b) $\mathrm{Tor}^R_{k+1}(I, M) = 0$ *for each injective right R-module I and each integer $k \geq n$.*

  (c) *For each exact sequence $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ such that the F_i are flat modules, the nth yoke K is a G-Gorenstein flat module.*

Proof. (1) First, as $\mathrm{GGfd}_R(M) < \infty$, we have

$$\mathrm{GGfd}_R(M) = \mathrm{cfd}_R(M) \leq \mathrm{Gfd}_R(M).$$

Moreover, there exists a complete r-flat resolution

$$\mathbf{E} = \cdots \to E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \to \cdots$$

such that $M = \mathrm{Im}(d_0)$. Let $M_i := \mathrm{Im}(d_i)$ (with $M = M_0$) for each integer i. Fix i and consider the exact sequence $0 \to M_{i+1} \to E_i \to M_i \to 0$ and the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M'_{i+1} & \to & F_i & \to & M'_i & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_{i+1,r-1} & \to & P_{i+1,r-1} \oplus P_{i,r-1} & \to & P_{i,r-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_{i+1,0} & \to & P_{i+1,0} \oplus P_{i,0} & \to & P_{i,0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_{i+1} & \to & E_i & \to & M_i & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the $P_{i,j}$ and $P_{i+1,j}$ are projective modules. As $\mathrm{fd}_R(E_i) \leq r$, F_i is a flat R-module for each integer i. Also, as $\mathrm{cfd}_R(M_i) \leq r$ and $\mathrm{cfd}_R(M_{i+1} \leq r$, we get $\mathrm{cfd}_R(M'_i) = \mathrm{cfd}_R(M'_{i+1}) = 0$. It follows that the derived exact sequence

$$\mathbf{F} = \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} F_{-1} \to \cdots$$

is a complete flat resolution, and thus each M'_i is Gorenstein flat over R. Hence $\mathrm{Gfd}_R(M) \leq r$ since $0 \to M'_0 \to P_{0,r-1} \to P_{0,r-2} \to \cdots \to P_{0,0} \to M_0 = M \to 0$ is an exact sequence with M'_0 Gorenstein flat and the $P_{0,j}$ projective modules.

(2) (a)⇔(b) holds by definition.

(c)⇒(a). Consider a flat resolution $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ of M with nth yoke K. Note that $\mathrm{GGfd}_R(M) = \mathrm{cfd}_R(M)$ and $\mathrm{GGfd}_R(K) = \mathrm{cfd}_R(K) = 0$. Then, applying Proposition 2.1(1), we get $\mathrm{GGfd}_R(M) \leq n$.

(a)⇒(c). Suppose that $\mathrm{GGfd}_R(M) \leq n$. Let $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ be an exact sequence of modules such that the F_i are flat. A successive application of Proposition 2.4(4) shows that K is a generalized Gorenstein flat module. Then $\mathrm{GGfd}_R(K) = \mathrm{cfd}_R(K)$. As $\mathrm{GGfd}_R(M) = \mathrm{cfd}_R(M) \leq n$, we get, by Proposition 2.1, $\mathrm{cfd}_R(K) = 0$. It follows that $\mathrm{GGfd}_R(K) = 0$, that is, K is G-Gorenstein flat, as desired. ∎

Recall that, for each R-module M, $\mathrm{id}_R(M^+) = \mathrm{fd}_R(M)$. As to the Gorenstein dimensions, it is only known that $\mathrm{Gid}_R(M^+) \leq \mathrm{Gfd}_R(M)$ with equality if R is a right coherent ring (cf. [25, Proposition 3.11]). Next, we prove that $\mathrm{Gid}_R(M^+) = \mathrm{GGfd}_R(M)$ for each R-module M with finite generalized Gorenstein flat dimension without restrictions on the ring R.

COROLLARY 2.6. *Let M be an R-module. Then:*

(1) $\mathrm{cfd}_R(M) \leq \mathrm{Gid}_R(M^+) \leq \mathrm{GGfd}_R(M) \leq \mathrm{fd}_R(M)$.

(2) *If* $\mathrm{GGfd}_R(M) < \infty$, *then*

$$\mathrm{cfd}_R(M) = \mathrm{Gid}_R(M^+) = \mathrm{GGfd}_R(M).$$

(3) *If* $\mathrm{fd}_R(M) < \infty$, *then*

$$\mathrm{cfd}_R(M) = \mathrm{Gid}_R(M^+) = \mathrm{Gfd}_R(M) = \mathrm{GGfd}_R(M) = \mathrm{fd}_R(M).$$

(4) *If R is right coherent, then, for each R-module M,* $\mathrm{GGfd}_R(M) = \mathrm{Gfd}_R(M)$.

Proof. (1) Via the above adjointness isomorphism, we get

$$\mathrm{Tor}_n^R(I, M)^+ \cong \mathrm{Ext}_R^n(I, M^+)$$

for each right injective module I. Then

$$\begin{aligned} \mathrm{cfd}_R(M) &= \sup\{k : \mathrm{Ext}_R^k(I, M^+) \neq 0 \text{ for some injective right module } I\} \\ &\leq \mathrm{Gid}_R(M^+). \end{aligned}$$

Assume that $\mathrm{GGfd}_R(M) \leq n$ for some positive integer n. Let $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ be an exact sequence of modules such that the F_i are flat. Then the sequence $0 \to M^+ \to F_0^+ \to F_1^+ \to \cdots \to F_{n-1}^+ \to K^+ \to 0$ is exact with the F_i^+ injective modules. By Theorem 2.5, K is G-Gorenstein flat, and thus, by Proposition 2.3, K^+ is Gorenstein injective. Hence $\mathrm{Gid}_R(M^+) \leq n$. It follows that $\mathrm{Gid}_R(M^+) \leq \mathrm{GGfd}_R(M)$. Also, by Proposition 2.4(3), $\mathrm{GGfd}_R(M) \leq \mathrm{fd}_R(M)$.

(2) If $\mathrm{GGfd}_R(M) < \infty$, then, by Definition 1, $\mathrm{GGfd}_R(M) = \mathrm{cfd}_R(M)$. Hence, by (1), the desired equalities follow.

(3) If $\mathrm{fd}_R(M) < \infty$, then $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$, so that, by (1),

$$\mathrm{cfd}_R(M) = \mathrm{Gid}_R(M^+) = \mathrm{Gfd}_R(M) = \mathrm{GGfd}_R(M) = \mathrm{fd}_R(M).$$

(4) Assume that R is right coherent. Let M be an R-module such that $\mathrm{Gfd}_R(M) < \infty$. Then, by [25, Theorem 3.14], $\mathrm{Gfd}_R(M) = \mathrm{cfd}_R(M)$ and, by [15, Lemma 2.19], there exists an exact sequence $0 \to M \to E \to G \to 0$ such that $\mathrm{fd}_R(E) = \mathrm{Gfd}_R(M)$ and G is Gorenstein flat. Applying Proposition 2.4(4), we find that M is generalized Gorenstein flat. It follows that $\mathrm{GGfd}_R(M) = \mathrm{cfd}_R(M) = \mathrm{Gfd}_R(M)$. On the other hand, assume that $\mathrm{GGfd}_R(M) < \infty$. Then, by Theorem 2.5(1), $\mathrm{Gfd}_R(M) < \infty$. Hence, by the first step, $\mathrm{GGfd}_R(M) = \mathrm{Gfd}_R(M)$. ∎

The following theorem explores the relations between the generalized Gorenstein flat dimension of the different differential images within a complete n-flat resolution. First, we give the following lemma relative to the behavior of the generalized Gorenstein flat dimension vis-à-vis short exact sequences.

LEMMA 2.7. *Let* $0 \to N \to E \to M \to 0$ *be an exact sequence of* R*-modules.*

(1) *If* E *is a generalized Gorenstein flat module, then*
$$\mathrm{GGfd}_R(E) \le \max\{\mathrm{GGfd}_R(M), \mathrm{GGfd}_R(N)\}.$$
(2) *If* M *is a generalized Gorenstein flat* R*-module, then*
$$\mathrm{GGfd}_R(M) \le 1 + \max\{\mathrm{GGfd}_R(E), \mathrm{GGfd}_R(N)\}.$$

Proof. (1) Observe that, as E is generalized Gorenstein flat,
$$\mathrm{GGfd}_R(E) = \mathrm{cfd}_R(E).$$
Also, by Proposition 2.1(2)(c),
$$\mathrm{cfd}_R(E) \le \max\{\mathrm{cfd}_R(N), \mathrm{cfd}_R(M)\}.$$
Since, by Corollary 2.6(1),
$$\max\{\mathrm{cfd}_R(N), \mathrm{cfd}_R(M)\} \le \max\{\mathrm{GGfd}_R(N), \mathrm{GGfd}_R(M)\},$$
it follows that
$$\mathrm{GGfd}_R(E) = \mathrm{cfd}_R(E) \le \max\{\mathrm{GGfd}_R(N), \mathrm{GGfd}_R(M)\},$$
as desired.

(2) The proof is similar to that of (1), applying Corollary 2.6 and using Proposition 2.1(2)(b). ∎

THEOREM 2.8.

(1) *Let* $0 \to N \to E \xrightarrow{d} M \to 0$ *be an exact sequence such that* $\mathrm{fd}_R(E) < \infty$ *and* M *is a generalized Gorenstein flat module.*
   (a) *If* $\mathrm{GGfd}_R(M) \le \mathrm{fd}_R(E)$*, then* $\max\{\mathrm{GGfd}_R(N), \mathrm{GGfd}_R(M)\} = \mathrm{fd}_R(E)$*.*
   (b) *If* $\mathrm{GGfd}_R(M) > \mathrm{fd}_R(E)$*, then* $\mathrm{GGfd}_R(M) = 1 + \mathrm{GGfd}_R(N)$*.*

(2) *Let*
$$\mathbf{E} = \cdots \to E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \to \cdots$$
be a generalized complete flat resolution. Let $M_i := \mathrm{Im}(d_i)$ *for each integer* i*. Then*
$$\begin{aligned}\sup\{\mathrm{Gfd}_R(M_i) : i \in \mathbb{Z}\} &= \sup\{\mathrm{GGfd}_R(M_i) : i \in \mathbb{Z}\}\\ &= \sup\{\mathrm{fd}_R(E_i) : i \in \mathbb{Z}\}.\end{aligned}$$

Proof. First, note that, by Proposition 2.4(4), N is a generalized Gorenstein flat module.

(1)(a) Assume that $\mathrm{GGfd}_R(M) \le \mathrm{fd}_R(E)$. As $\mathrm{GGfd}_R(M) = \mathrm{cfd}_R(M)$ and $\mathrm{GGfd}_R(N) = \mathrm{cfd}_R(N)$, we easily get $\mathrm{GGfd}_R(N) \le \mathrm{fd}_R(E)$, so that

$$\max\{\mathrm{GGfd}_R(N), \mathrm{GGfd}_R(M)\} \le \mathrm{fd}_R(E).$$

Now, as $\mathrm{GGfd}_R(E) = \mathrm{fd}_R(E)$, Lemma 2.7(1) establishes the reverse inequality yielding the desired equality.

(b) The proof is routine since $\mathrm{GGfd}_R(M) = \mathrm{cfd}_R(M)$, $\mathrm{GGfd}_R(E) = \mathrm{cfd}_R(E)$ and $\mathrm{GGfd}_R(N) = \mathrm{cfd}_R(N)$.

(2) First, as $\mathrm{GGfd}_R(M_i) = \mathrm{cfd}_R(M_i)$ for each integer i, by Remark 1 we have

$$\sup\{\mathrm{GGfd}_R(M_i) : i \in \mathbb{Z}\} = \sup\{\mathrm{fd}_R(E_i) : i \in \mathbb{Z}\}.$$

Also, by Theorem 2.5(1), $\mathrm{GGfd}_R(M_i) \le \mathrm{Gfd}_R(M_i) \le \sup\{\mathrm{fd}_R(E_k) : k \in \mathbb{Z}\}$ for each integer i. Hence the desired equality follows. ∎

The following two results are consequences of Theorem 2.8. The first one exhibits a class of modules for which the new generalized Gorenstein flat dimension and Gorenstein flat dimension coincide.

Corollary 2.9. *Let $0 \to M \to E \to M \to 0$ be an exact sequence such that $\mathrm{fd}_R(E) < \infty$. Then*

$$\mathrm{GGfd}_R(M) = \mathrm{Gfd}_R(M).$$

Moreover, if $\mathrm{cfd}_R(M) < \infty$, then

$$\mathrm{GGfd}_R(M) = \mathrm{Gfd}_R(M) = \mathrm{cfd}_R(M) = \mathrm{fd}_R(E).$$

Proof. Denote by d the surjective homomorphism $d : E \to M$ with kernel M. Assume that $\mathrm{cfd}_R(M) < \infty$. Then the exact sequence $\cdots \stackrel{d}{\to} E \stackrel{d}{\to} E \stackrel{d}{\to} E \stackrel{d}{\to} \cdots$ is a generalized complete flat resolution. Hence, by Theorem 2.8(2), $\mathrm{GGfd}_R(M) = \mathrm{Gfd}_R(M) = \mathrm{fd}_R(E) = \mathrm{cfd}_R(M)$. If $\mathrm{cfd}_R(M) = \infty$, then, as $\mathrm{cfd}_R(M) \le \mathrm{Gfd}_R(M)$ and $\mathrm{cfd}_R(M) \le \mathrm{GGfd}_R(M)$, we get $\mathrm{GGfd}_R(M) = \mathrm{Gfd}_R(M) = \infty$. ∎

Corollary 2.10. *Let $\mathbf{F} = \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} F_{-1} \to \cdots$ be an exact sequence of R-modules such that each F_i is flat. Then $\mathbf{F}$ is a complete flat resolution if and only if $\mathbf{F}$ is a generalized complete flat resolution.*

Proof. Apply Theorem 2.8(2). ∎

Next, we aim at finding conditions on our introduced generalized Gorenstein flat dimension for the category of Gorenstein flat modules $\mathcal{GF}(R)$ to be projectively resolving.

Lemma 2.11 ([10, Lemma 1]). *Let R be a ring. Let $n \ge 0$ be an integer. Then the following assertions are equivalent:*

(1) *$\mathcal{GF}(R)$ is projectively resolving.*
(2) *Given a short exact sequence $0 \to K \to N \to M \to 0$, if M and K are Gorenstein flat, then so is N.*

LEMMA 2.12 ([10, Lemma 2]). *Let $0 \to K \to N \to M \to 0$ be an exact sequence of R-modules such that M and K are Gorenstein flat modules. Then $\mathrm{Gfd}_R(N) \le 1$.*

Recall that a ring R is said to be *left GF-closed* if $\mathcal{GF}(R)$ is projectively resolving. In particular, a right coherent ring or a ring of finite weak global dimension are left GF-closed. Note that the class of GF-closed rings properly contains the class of right coherent rings (see [4]).

THEOREM 2.13. *Let R be a ring. Then the following assertions are equivalent:*

(1) *R is left GF-closed.*
(2) *$\mathcal{GF}(R)$ is projectively resolving.*
(3) *$\mathrm{Gfd}_R(M) = \mathrm{GGfd}_R(M)$ for each R-module M.*
(4) *Given an R-module M, if $\mathrm{Gfd}_R(M) < \infty$, then $\mathrm{Gfd}_R(M) = \mathrm{Gid}_R(M^+)$.*
(5) *Given an R-module M, if $\mathrm{Gfd}_R(M) < \infty$ and M^+ is Gorenstein injective, then M is Gorenstein flat.*

Proof. (1)⇔(2) holds by definition.

(2)⇒(3). Let M be an R-module such that $\mathrm{Gfd}_R(M) = n < \infty$. Then, by [5, Lemma 2.2], there exists an exact sequence $0 \to M \to E \to G \to 0$ such that $\mathrm{fd}_R(E) = n$ and G is Gorenstein flat. It follows, by Proposition 2.4(4), that M is a generalized Gorenstein flat module, and thus $\mathrm{GGfd}_R(M) = \mathrm{cfd}_R(M)$. Moreover, as $\mathrm{cfd}_R(G) = 0$, we get $\mathrm{cfd}_R(M) = \mathrm{cfd}_R(E) = \mathrm{fd}_R(E) = n$. Hence $\mathrm{GGfd}_R(M) = n = \mathrm{Gfd}_R(M)$. Now, if $\mathrm{GGfd}_R(M) < \infty$, then, by Theorem 2.5(1), $\mathrm{Gfd}_R(M) < \infty$, and thus by the first step, we get $\mathrm{GGfd}_R(M) = \mathrm{Gfd}_R(M)$, as desired.

(3)⇒(4). Apply Corollary 2.6(2).

(4)⇒(5). It is straightforward.

(5)⇒(1). Assume that (5) holds and let $0 \to K \to N \to M \to 0$ be an exact sequence of R-modules such that M and K are Gorenstein flat. Then, by Lemma 2.12, $\mathrm{Gfd}_R(N) \le 1$. Moreover, by [25, Proposition 3.11], M^+ and K^+ are Gorenstein injective. Hence, considering the dual exact sequence

$$0 \to M^+ \to N^+ \to K^+ \to 0,$$

and since the category $\mathcal{GI}(R)$ of Gorenstein injective modules is injectively resolving, we deduce that N^+ is Gorenstein injective. Therefore, by (5), N is Gorenstein flat. It follows that R is left GF-closed, as desired. ∎

We end this section by discussing upper bounds of the Gorenstein projective dimension of Gorenstein flat modules under some conditions on the cardinality of the base ring R.

REMARK 3. It is known from Simson's notes [30] and [31] that the upper bound of the projective dimension of flat R-modules can depend on the cardinality of R. In particular, when R is countable this upper bound is one [31, Theorem]. As for Gorenstein flat modules, the upper bound of the projective dimension of Gorenstein flat modules might be infinite even if R is countable. In fact, let R be the countable quasi-Frobenius ring $\mathbb{Q}[X]/(X^2)$. By [7, Example 1.5], the ideal $(\overline{X})$ is a strongly Gorenstein projective module, thus a strongly Gorenstein flat module which is not a projective module. Therefore, taking into account the short exact sequence $0 \to (\overline{X}) \to R \to (\overline{X}) \to 0$, we deduce that $\mathrm{pd}_R((\overline{X})) = \infty$.

A more pertinent question is: Let R be a countable ring and M be any Gorenstein flat module. Is $\mathrm{Gpd}_R(M) \leq 1$? To answer this question, it suffices to consider strongly Gorenstein flat modules since any Gorenstein flat module is a direct summand of some strongly Gorenstein flat modules. By [11, Theorem 2.4] and Simson's theorem above, we are reduced to proving that $\mathrm{Gpd}_R(M) < \infty$ for any strongly Gorenstein flat module M. This holds for instance when R is a countable ring such that the cohomological invariant l-silp(R) is finite, in particular, when the Gorenstein global dimension l-G-gldim(R) is finite by [11, Theorem 3.3]. A general answer to the above question depends heavily on answering the following open question which is, in a sense, a copure version of the above result on upper bounds of the projective dimension of flat modules: Let R be a countable ring and M be a strongly copure flat module. Is the copure projective dimension of M less than or equal to one?

Finally, it is worth pointing out that the generalized Gorenstein flat dimension can be arbitrarily large for modules over a countable ring R. Actually, for any positive integer n, $R = \mathbb{Z}[X_1, \ldots, X_n]$ is a countable ring of weak global dimension $n+1$. Then the generalized Gorenstein flat dimension coincides with the flat dimension, and thus the former invariant can take any value between 0 and $n + 1$.

3. Generalized Gorenstein weak global dimension. In this section, we give some applications of results of Section 2 and we study properties of the global dimension related to the generalized Gorenstein flat dimension. This new global dimension behaves better than the Gorenstein weak global dimension, G-wgldim(R), of R, and manifests itself to be the best candidate for the counterpart of the classical weak global dimension in Gorenstein homological algebra.

We define the *left finitistic generalized Gorenstein flat dimension* of R to be the invariant

$$\text{l-FGGFD}(R) := \sup\{\text{GGfd}_R(M) : M \text{ is a left } R\text{-module of finite generalized Gorenstein flat dimension}\}.$$

It is easily seen that, for an arbitrary ring R, $\text{FFD}(R) \leq \text{FGFD}(R)$, where $\text{FFD}(R)$ denotes the (left) finitistic flat dimension of R, and $\text{FGFD}(R)$ denotes the (left) finitistic Gorenstein flat dimension of R. Holm proved that $\text{FGFD}(R) = \text{FFD}(R)$ for a right coherent ring R [25, Theorem 3.24]. On the other hand, in the case of the (left) finitistic Gorenstein projective dimension, $\text{FGPD}(R)$, and the (left) finitistic Gorenstein injective dimension, $\text{FGID}(R)$, it is known that $\text{FGPD}(R) = \text{FPD}(R)$ and $\text{FGID}(R) = \text{FID}(R)$ for any ring R [25, Theorems 2.28 and 2.29]. Next, we provide the analog of these two theorems for the generalized Gorenstein flat dimension, proving that $\text{FGGFD}(R) = \text{FFD}(R)$ for an arbitrary ring R, and extending the above theorem [25, Theorem 3.24] to left GF-closed rings.

THEOREM 3.1. *Let R be a ring. Then* $\text{FGGFD}(R) = \text{FFD}(R) \leq \text{FGFD}(R)$. *Moreover, if R is left* GF*-closed, then* $\text{FGGFD}(R) = \text{FFD}(R) = \text{FGFD}(R)$.

Proof. One may easily prove, by Corollary 2.6(1) and Theorem 2.5(1), that $\text{FFD}(R) \leq \text{FGGFD}(R) \leq \text{FGFD}(R)$. Let M be an R-module such that $n := \text{GGfd}_R(M) < \infty$. Then there exists a generalized complete flat resolution of R-modules

$$\cdots \to E_1 \to E_0 \to E_{-1} \to \cdots$$

such that $M = \text{Im}(E_0 \to E_{-1})$. By Theorem 2.8(2),

$$\text{GGfd}_R(M) = n \leq \sup\{\text{fd}_R(E_i) : i \in \mathbb{Z}\} \leq \text{FFD}(R),$$

so that $\text{FGGFD}(R) \leq \text{FFD}(R)$, and equality follows. The last assertion easily follows since, via Theorem 2.13, the Gorenstein flat dimension and the generalized Gorenstein flat dimension coincide when R is left GF-closed. ∎

Let us define the *left generalized Gorenstein weak global dimension* of R to be the invariant

$$\text{l-GG-wgldim}(R) := \sup\{\text{GGfd}_R(M) : M \text{ is a left } R\text{-module}\}.$$

Our next theorem characterizes the generalized Gorenstein weak global dimension of R in terms of the cohomological invariants $\text{r-sfli}(R) := \sup\{\text{fd}_R(I) : I \text{ is an injective right } R\text{-module}\}$ and $\text{l-sfli}(R) := \sup\{\text{fd}_R(I) : I \text{ is an injective left } R\text{-module}\}$ for an arbitrary ring R. It turns out that the generalized Gorenstein weak global dimension seems to be the best counterpart of the classical weak global dimension in Gorenstein homological algebra. Actually, the following theorem represents the analog in Gorenstein homological theory of [27, Theorem 15] which states that the left weak global dimension coincides with the right weak global dimension. To prove this theorem, we need the following two lemmas. The first one generalizes [18, Lemma 5.1] which asserts that if $\mathrm{Gfd}_R(N) < \infty$ for any R-module N, then $\text{r-sfli}(R) < \infty$.

Lemma 3.2 ([10, Lemma 2]). *Let R be a ring. Then*

$$\text{r-sfli}(R) \leq \text{l-G-wgldim}(R), \qquad \text{l-sfli}(R) \leq \text{r-G-wgldim}(R).$$

Proof. Assume that $\text{l-G-wgldim}(R) \leq n$ for some positive integer n. Then $\mathrm{Gfd}_R(M) \leq n$ for each left R-module M. Hence $\mathrm{Tor}^R_{n+1}(I, M) = 0$ for each injective right module I and each left module M, yielding $\mathrm{fd}_R(I) \leq n$ for injective right module I. This means that $\text{r-sfli}(R) \leq n$. It follows that $\text{r-sfli}(R) \leq \text{l-G-wgldim}(R)$, as desired. ∎

The next lemma proves that the newly introduced generalized Gorenstein flat dimension and the flat dimension coincide for injective modules. This property is reminiscent of a similar behavior of the Gorenstein projective dimension (resp., Gorenstein injective dimension) when applied to injective modules (resp., projective modules).

Lemma 3.3. *Let R be a ring. Then*

$$\mathrm{GGfd}_R(I) = \mathrm{fd}_R(I)$$

for each injective R-module I.

Proof. Let I be an injective module. If $\mathrm{GGfd}_R(I) = \infty$, then, by Corollary 2.6(1), $\mathrm{fd}_R(I) = \infty = \mathrm{GGfd}_R(I)$. Now, assume that $\mathrm{GGfd}_R(I) = n < \infty$. Then there exists a generalized complete flat resolution $\mathbf{E} = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots$ such that $I := \mathrm{Im}(E_0 \to E_{-1})$. Hence, in particular, $0 \to I \to E_{-1} \to M_{-1} \to 0$ is an exact sequence, where $M_{-1} := \mathrm{Im}(E_{-1} \to E_{-2})$. Therefore, the latter exact sequence splits, implying that $\mathrm{fd}_R(I) \leq \mathrm{fd}_R(E_{-1}) < \infty$. It follows, by Corollary 2.6(3), that $\mathrm{GGfd}_R(I) = \mathrm{fd}_R(I)$, as desired. ∎

Theorem 3.4. *Let R be a ring. Then the following coincide:*

(1) $\text{l-GG-wgldim}(R)$.
(2) $\text{r-GG-wgldim}(R)$.
(3) $\max\{\text{l-sfli}(R), \text{r-sfli}(R)\}$.
(4) $\max\{\text{l-G-wgldim}(R), \text{r-G-wgldim}(R)\}$.

Proof. Let $n \geq 0$ be an integer. Assume that $\text{l-GG-wgldim}(R) \leq n$. Then $\mathrm{GGfd}_R(M) \leq n$ for each left R-module M. Thus, $\mathrm{cfd}_R(M) \leq n$, that is, $\mathrm{Tor}^R_{r+1}(I, M) = 0$ for each R-module M, each injective right R-module I and each integer $r \geq n$. Hence $\mathrm{fd}_R(I) \leq n$ for each injective right R-module I, so that $\text{r-sfli}(R) \leq n$. Further, by Lemma 3.3, $\text{fd}_R(Q) = \text{GGfd}_R(Q) \leq n$ for each injective left R-module Q. Therefore $\text{l-sfli}(R) \leq n$. It follows that $\max\{\text{r-sfli}(R), \text{l-sfli}(R)\} \leq n$. Consequently, $\max\{\text{r-sfli}(R), \text{l-sfli}(R)\} \leq \text{l-GG-wgldim}(R)$.

Conversely, assume that $\max\{\text{r-sfli}(R), \text{l-sfli}(R)\} \leq n$. Let M be an R-module. Let $\cdots \to F_1 \to F_0 \to M \to 0$ and $0 \to M \to I_0 \to I_1 \to \cdots$ be, respectively, a flat resolution and injective resolution of M. Pasting these two resolutions yields the exact sequence

$$\mathbf{F} = \cdots \to F_1 \to F_0 \to I_0 \to I_1 \to \cdots$$

with $\text{Im}(F_0 \to I_0) = M$. Note that, as $\text{l-sfli}(R) \leq n$, $\text{fd}_R(I_j) \leq n$ for each integer $j \geq 0$. Let I be an injective right R-module. Then $\text{fd}_R(I) \leq \text{r-sfli}(R) \leq n$, so that $\text{Tor}^R_{r+1}(I, N) = 0$ for each left R-module N and each integer $r \geq n$. This means that $\text{cfd}_R(N) \leq n$ for each left R-module N. It follows that $\mathbf{F}$ is a complete n-flat resolution, and thus M is Gorenstein n-flat. Hence $\text{GGfd}_R(M) \leq n$. Therefore $\text{l-GG-wgldim}(R) \leq n$. It follows that $\text{l-GG-wgldim}(R) \leq \max\{\text{r-sfli}(R), \text{l-sfli}(R)\}$. Consequently, $\text{l-GG-wgldim}(R) = \max\{\text{l-sfli}(R), \text{r-sfli}(R)\}$. A similar argument establishes the second equality $\text{r-GG-wgldim}(R) = \max\{\text{r-sfli}(R), \text{l-sfli}(R)\}$, and therefore

$$\text{l-GG-wgldim}(R) = \text{r-GG-wgldim}(R) = \max\{\text{r-sfli}(R), \text{l-sfli}(R)\}.$$

Moreover, let M be a generalized Gorenstein flat module issued from a generalized complete flat resolution $\mathbf{E} = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots$ of degree $r := \sup\{\text{fd}_R(E_i) : i \in \mathbb{Z}\}$, and let $M_i := \text{Im}(E_i \to E_{i-1})$ for each integer i and with $M = M_0$. Then, by Theorem 2.5(1),

$$\text{GGfd}_R(M) \leq \text{Gfd}_R(M) \leq r := \sup\{\text{fd}_R(E_i) : i \in \mathbb{Z}\}.$$

Also, by Theorem 2.8(2),

$$\sup\{\text{fd}_R(E_i) : i \in \mathbb{Z}\} = \sup\{\text{GGfd}_R(M_i) : i \in \mathbb{Z}\} \leq \text{l-GG-wgldim}(R).$$

It follows that $\text{Gfd}_R(M) \leq \text{l-GG-wgldim}(R)$ for each left R-module M, so that $\text{l-G-wgldim}(R) \leq \text{l-GG-wgldim}(R)$. Similarly, $\text{r-G-wgldim}(R) \leq \text{r-GG-wgldim}(R)$. As, by the first step, $\text{l-GG-wgldim}(R) = \text{r-GG-wgldim}(R) = \max\{\text{r-sfli}(R), \text{l-sfli}(R)\}$, it follows from Lemma 3.2 that

$$\begin{aligned}\max\{\text{l-sfli}(R), \text{r-sfli}(R)\} &\leq \max\{\text{l-G-wgldim}(R), \text{r-G-wgldim}(R)\}\\ &\leq \text{l-GG-wgldim}(R) = \text{r-GG-wgldim}(R)\\ &= \max\{\text{r-sfli}(R), \text{l-sfli}(R)\}.\end{aligned}$$

Consequently,

$$\begin{aligned}\max\{\text{l-sfli}(R), \text{r-sfli}(R)\} &= \max\{\text{l-G-wgldim}(R), \text{r-G-wgldim}(R)\}\\ &= \text{l-GG-wgldim}(R) = \text{r-GG-wgldim}(R),\end{aligned}$$

completing the proof. ∎

Definition 4. Let R be a ring. We define the *generalized Gorenstein weak global dimension* of R, denoted by GG-wgldim(R), to be the common value r-GG-wgldim(R) = l-GG-wgldim(R).

We deduce the following result which compares the left and right Gorenstein weak global dimensions.

Corollary 3.5. *Let R be a ring. If R is left* GF*-closed* (*resp., right* GF*-closed*), *then*

$$\text{r-G-wgldim}(R) \le \text{l-G-wgldim}(R) = \text{GG-wgldim}(R)$$

(*resp.,* l-G-wgldim$(R) \le$ r-G-wgldim(R) = GG-wgldim(R)).

Consequently, if R left and right GF*-closed, then*

$$\text{r-G-wgldim}(R) = \text{l-G-wgldim}(R) = \text{GG-wgldim}(R).$$

Proof. Assume that R is a left GF-closed ring. Then, by Theorem 2.13, $\text{GGfd}_R(M) = \text{Gfd}_R(M)$ for each left R-module M, and thus, by Theorem 3.3, l-G-wgldim(R) = GG-wgldim$(R) \ge q$r-G-wgldim(R). The rest of the proof is now clear. ■

In [11], we prove that the Gorenstein global dimension of an arbitrary ring R is connected to the cohomological invariants l-silp$(R) = \max\{\text{id}_R(P) :$ P is a projective left R-module$\}$ and l-spli$(R) = \max\{\text{pd}_R(I) : I$ is an injective right R-module$\}$ via the formula

$$\text{l-G-gldim}(R) = \max\{\text{l-silp}(R), \text{l-spli}(R)\} \quad [11, \text{Theorem 3.3}].$$

Our next result compares the generalized Gorenstein weak global dimension with the Gorenstein global dimension.

Corollary 3.6. *Let R be a ring. Then*

$$\text{GG-wgldim}(R) \le \max\{\text{l-G-gldim}(R), \text{r-G-gldim}(R)\}.$$

Proof. Note that, by [11, Theorem 3.3],

$$\text{l-sfli}(R) \le \text{l-spli}(R) \le \text{l-G-gldim}(R),$$
$$\text{r-sfli}(R) \le \text{r-spli}(R) \le \text{r-G-gldim}(R).$$

Then, by Theorem 3.3,

$$\begin{aligned}\text{GG-wgldim}(R) &= \max\{\text{l-sfli}(R), \text{r-sfli}(R)\}\\ &\le \max\{\text{l-G-gldim}(R), \text{r-G-gldim}(R)\}. \quad ■\end{aligned}$$

Recall that if R is a left Noetherian (resp., right Noetherian) ring, then l-gldim(R) = wgldim (R) (resp., r-gldim(R) = wgldim(R)); and thus if R is left and right Noetherian, then l-gldim(R) = r-gldim(R) = wgldim(R) [27, Theorem 9.22]. Our final result provides the analog of this theorem in Gorenstein homological algebra.

THEOREM 3.7. *Let R be a ring. If R is left Noetherian (resp., right Noetherian), then*

$$\text{l-G-gldim}(R) = \text{GG-wgldim}(R)$$

(*resp.,* $\text{r-G-gldim}(R) = \text{GG-wgldim}(R)$).

Consequently, if R is left and right Noetherian, then

$$\text{l-G-gldim}(R) = \text{r-G-gldim}(R) = \text{GG-wgldim}(R) = \max\{\text{id}_R({}_RR), \text{id}(R_R)\}.$$

Proof. Let R be a left Noetherian ring. By [16, Corollary 3.9] and [26, Theorem 2.1],

$$\text{r-sfli}(R) = \text{id}_R({}_RR) = \text{Gid}_R({}_RR) \leq \text{l-G-gldim}(R).$$

Also,

$$\text{l-sfli}(R) \leq \text{l-spli}(R) = \sup\{\text{Gpd}_R(I) : I \text{ is an injective left } R\text{-module}\}$$

[26, Theorem 2.2]

$$\leq \text{l-G-gldim}(R).$$

Hence, by Theorem 3.3, $\text{GG-gldim}(R) \leq \text{l-G-gldim}(R)$.

Conversely, assume that $\text{GG-gldim}(R) = n < \infty$. Then, by Theorem 3.3, $\text{l-sfli}(R) \leq n$ and $\text{r-sfli}(R) \leq n$. Notice that, as R is left Noetherian, by [21, Proposition 9.1.2] and [16, Corollary 3.9],

$$\text{l-silp}(R) = \text{id}_R({}_RR) = \text{r-sfli}(R) \leq n.$$

Hence, as $\text{l-sfli}(R) \leq n$, by [17, Proposition 3.3 and Lemma 3.5],

$$\text{l-spli}(R) \leq \text{l-FPD}(R) \leq \text{l-silp}(R) \leq n.$$

It follows, by [11, Theorem 3.3], that $\text{l-G-gldim}(R) \leq n$. Hence $\text{l-G-gldim}(R) \leq \text{GG-gldim}(R)$, establishing the desired equality. Since the remaining part of the theorem easily follows from the first one, the proof is complete. ∎

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