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VECTOR FIELDS FROM LOCALLY INVERTIBLE POLYNOMIAL MAPS IN $\mathbb{C}^n$

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Abstract. Let $(F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a locally invertible polynomial map. We consider the canonical pull-back vector fields under this map, denoted by $\partial/\partial F_1, \dots, \partial/\partial F_n$. Our main result is the following: if $n-1$ of the vector fields $\partial/\partial F_j$ have complete holomorphic flows along the typical fibers of the submersion $(F_1, \dots, F_{j-1}, F_{j+1}, \dots, F_n)$, then the inverse map exists. Several equivalent versions of this main hypothesis are given.

1. Introduction and statement of results. We consider n-webs of polynomial vector fields in $\mathbb{C}^n$ which can be obtained from the euclidean n-web $\mathcal{W}$ in $\mathbb{C}^n$ by pull-back under a polynomial map

$$F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n \quad \text{with} \quad \det(DF) = 1. \tag{1.1}$$

Recall that the Jacobian Conjecture in $\mathbb{C}^n$ asserts the existence of the inverse map F^{-1}. Each of the polynomial vector fields

$$\frac{\partial}{\partial F_i} = (F_1, \dots, F_n)^* \frac{\partial}{\partial w_i}, \qquad i = 1, \dots, n, \tag{1.2}$$

has a restriction to the fibers $\mathcal{A}_{i,c} = (F_1, \dots, \widehat{F_i}, \dots, F_n)^{-1}(c)$ of the submersion; as usual, $\widehat{\ }$ over the ith coordinate indicates that it is omitted.

It is a classical result that the following assertions are equivalent (see [MO87], [Me92], [Cam97] and [Bus03]):

- The inverse map exists.
- $\partial/\partial F_1, \dots, \partial/\partial F_n$ are complete, i.e. their flows are defined for all complex times $t \in \mathbb{C}$ at every initial condition $p \in \mathbb{C}^n$.
- The web of affine curves $\{\mathcal{A}_{1,c}, \dots, \mathcal{A}_{n,c}\}$ is topologically trivial, i.e. every $\mathcal{A}_{i,c}$ is biholomorphic to $\mathbb{C}$.

The map F produces a collection of pairs

$$\{(\mathcal{A}_{i,c}, \partial/\partial F_i) \mid i = 1, \dots, n,\ c \in \mathbb{C}^{n-1}\}. \tag{1.3}$$

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Looking at the foliations $\mathcal{F}_i = \{A_{i,c}\}$, the last point has many facets, very roughly speaking: every $\mathcal{F}_i$ has trivial monodromy, its global Ehresmann connections are well-defined, no atypical fibers appear in all the submersions $(F_1, \ldots, \widehat{F_i}, \ldots, F_n)$. By studying this, we can deduce:

Main Theorem. *Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map as in* (1.1). *If $\partial/\partial F_2, \ldots, \partial/\partial F_n$ are complete on the typical fibers $\mathcal{A}_{2,c}, \ldots, \mathcal{A}_{n,c}$ of $(F_1, \ldots, \widehat{F_j}, \ldots, F_n)$, $j = 2, \ldots, n$, then F^{-1} exists.*

The proof of the main theorem is in two stages. In Lemma 4, we show that the completeness on typical fibers implies the same property on all the fibers $\mathcal{A}_{2,c}, \ldots, \mathcal{A}_{n,c}$. Secondly in Theorem 1, we consider a global Ehresmann conection in the directions of $\partial/\partial F_2, \ldots, \partial/\partial F_n$ to get the result. Furthermore, in Theorem 1, several equivalences of the completeness hypothesis are described.

The invertibility of F has been considered from many points of view (see [Ess00]). We start mainly from the algebraic point of view of [A77], [NS83]. For $n = 2$, invertibility from completeness in just one pair $(\mathcal{A}_{2,c}, \partial/\partial F_2)$ follows from the Abhyankar–Moh–Suzuki Theorem (see [Dru91], [Cam97] and the references therein, as well as [Dun08]). Actually, our study uses Riemann surfaces ideas and several complex variables.

The content of the work is as follows. In Section 2 we study the pull-back vector fields on Riemann surfaces from meromorphic maps. Section 3 contains the study of the pairs (1.3). The proof of the main result is in Section 4.

2. Meromorphic maps and vector fields on compact Riemann surfaces. Let $\mathbb{CP}^1 = \mathbb{C}_w \cup \{\infty\}$ be the projective line, with affine coordinate w. The vector field $\partial/\partial w$ induces a holomorphic vector field in $\mathbb{CP}^1$ having a double zero at $\infty \in \mathbb{CP}^1$. Let $\mathcal{L}$ be a compact Riemann surface.

Lemma 1. *Let $f : \mathcal{L} \to \mathbb{CP}^1$ be a non-constant meromorphic function. The non-identically zero meromorphic vector field*

$$\frac{\partial}{\partial f} := f^* \left(\frac{\partial}{\partial w} \right)$$

is well-defined on $\mathcal{L}$. Moreover, f has a canonically associated meromorphic one-form ω such that the diagram

$$\begin{tikzcd}
 & \frac{\partial}{\partial f} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\omega \arrow[rr, leftrightarrow] & & f(p) = \int^p \omega
\end{tikzcd}$$

commutes. $\partial/\partial f$ and ω are non-identically zero.

The Riemann surface-vector field pairs are denoted by $(\mathcal{L}, \partial/\partial f)$.

The diagram in the lemma comes from the theory of quadratic differentials (see [Str84], [Muc02]).

Proof of Lemma 1. Given f, we define $\omega = df$ and $\omega(X) \equiv 1$. In addition, ω is called the one-form of time for X, since for $p_0, p \in \mathcal{L}$ we have

$$f(p) - f(p_0) = \int_{p_0}^{p} \omega = \left\{ \begin{array}{l} \text{complex time to travel from} \\ p_0 \text{ to } p \text{ under the local flow of } \partial/\partial f \end{array} \right\}. \tag{2.1}$$

The map from f to ω is elementary. A non-identically zero meromorphic one-form ω determines a univalued meromorphic function $f(p) = \int^p \omega$ if and only if the periods and residues of ω vanish, i.e.

$$\int_{\gamma} \omega = 0 \quad \text{for each } [\gamma] \in H_1(\mathcal{L} - \{\text{poles of } \omega\}, \mathbb{Z}).$$

This is the case of the horizontal arrow in the diagram, all the correspondences are bijections. ∎

For everything that follows, the hypotheses of Lemma 1 are fulfilled.

We relate the poles and singular points of f to the zeros and poles of $\partial/\partial f$, respectively. Recall that the order of a zero $p \in \mathcal{L}$ of a meromorphic vector field X on a compact Riemann surface $\mathcal{L}$ is $s \geq 2$ if and only if its associated real vector field $\Re e(X)$ has $2s-2$ elliptic sectors at p. Additionally, X has a pole of order $-k \leq -1$ at some p if and only if $\Re e(X)$ has $k+2$ hyperbolic sectors (see [Muc02, p. 232]). We have the following result.

Remark 1. Let $(\mathcal{L}, \partial/\partial f)$ be a pair as in Lemma 1.

(1) $\partial/\partial f$ has a pole of order $-\kappa + 1 \leq -1$ at $p \in \mathcal{L}$ if and only if p is a ramification point of f of order $\kappa \geq 2$, and $f(p) = q \in \mathbb{CP}^1 - \{\infty\}$.
(2) $\partial/\partial f$ has a zero of order $\sigma + 1 \geq 2$ at $p \in \mathcal{L}$ if and only if p is a ramification point of f of order $\sigma \geq 1$, and $f(p) = \infty \in \mathbb{CP}^1$.
(3) $\partial/\partial f$ has zeros.
(4) $\partial/\partial f$ does not have simple zeros.

Lemma 2. *Let $(\mathcal{L}, \partial/\partial f)$ be a pair as in Lemma 1. Assume that $\partial/\partial f$ has zeros of orders $\{s_1, \ldots, s_r\}$. Then*

$$\deg(f) = (s_1 - 1) + \cdots + (s_r - 1) \geq 1.$$

Proof. Several proofs are available, depending on the reader's background.

Case 1. Assume that $\infty \in \mathbb{CP}^1$ is a regular value for f. The cardinality r of the fiber $f^{-1}(\infty) = \{p_1, \ldots, p_r\} \subset \mathcal{L}$ is the degree of f. Near each p_ν the function f is a local biholomorphism. Hence at each point p_ν, $\partial/\partial f$ has a double zero, and the assertion follows.

CASE 2. Assume that $\infty \in \mathbb{CP}^1$ is a critical value (ramification value) for f. Let p_ν be a ramification point over ∞ having $j_\nu \geq 2$ as its ramification index. Locally at p_ν, the function f is j_ν-to-one, and we get $j_\nu =$ (order of the zero of $\partial/\partial f) - 1$. ∎

The analogous formula using the poles of $\partial/\partial f$ requires more information.

COROLLARY 1. *Let* $(\mathcal{L}, \partial/\partial f)$ *be a pair as in Lemma* 1.

(1) *The genus of* $\mathcal{L}$ *is* g, *and* $\partial/\partial f$ *has zeros of order* $\{s_1, \ldots, s_r\}$ *and poles of* (*negative*) *order* $\{-k_1, \ldots, -k_\tau\}$. *Then*
$$\deg(f) = 2 - 2g - r + k,$$
where $k = k_1 + \cdots + k_\tau$.

(2) *In addition,* $\deg(f) \geq 2$ *if and only if* $\partial/\partial f$ *has at least one pole* (*of any order*).

Proof. We have:
$$s_1 + \cdots + s_r - k_1 - \cdots - k_\tau = 2 - 2g.$$
Hence,
$$(s_1 - 1) + \cdots + (s_r - 1) = 2 - 2g - r + k_1 + \cdots + k_\tau. \ ∎$$

We say that $\partial/\partial f$ is *complete* if its flow is well-defined for all complex times $t \in \mathbb{C}$ and every initial condition.

COROLLARY 2. *Let* $(\mathcal{L}, \partial/\partial f)$ *be a pair as in Lemma* 1. *The following assertions are equivalent:*

(1) $\partial/\partial f$ *is complete.*
(2) *The pair is* $(\mathbb{CP}^1, \partial/\partial z)$ *up to biholomorphism.*
(3) $\deg(f) = 1$.

Proof. The non-identically zero complete vector fields X on compact Riemann surfaces are classified as follows (see [LM00, p. 179]): $\mathcal{L}$ is a torus and X has no poles or zeros, or $\mathcal{L}$ is $\mathbb{CP}^1$ and X is holomorphic. Using Remark 1, only the second case is possible with a double zero. ∎

3. Tomography

3.1. Foliations by curves and pull-back vector fields. Following equation (1.1), let
$$F = (F_1, \ldots, F_n) : \mathbb{C}^n_z \to \mathbb{C}^n_w$$
be a polynomial map having $\det(DF) = 1$. Note that to avoid confusion, we use $\mathbb{C}^n_z$ and $\mathbb{C}^n_w$ to denote the domain and the target, together with the variables that we use in each of them.

Let us consider the affine coordinate lines
$$\mathbb{C}_{i,c} := \{(c_1, \ldots, c_{i-1}, w_i, c_{i+1}, \ldots, c_n)\} \subset \mathbb{C}^n_w,$$

where $w_i \in \mathbb{C}$ and $c := (c_1, \ldots, \widehat{c_i}, \ldots, c_n) \in \mathbb{C}_w^{n-1}$. We have the canonical web in $\mathbb{C}_w^n$,

$$\mathcal{W} = \{\mathbb{C}_{i,c} \mid i = 1, \ldots, n,\ c \in \mathbb{C}_w^{n-1}\}$$

that they define. Under pull-back, we get a new web $F^*\mathcal{W}$ of affine curves in $\mathbb{C}_z^n$. A first description of it using algebraic geometry is as follows.

Given one direction $i \in \{1, \ldots, n\}$, and fixing $c = (c_1, \ldots, \widehat{c_i}, \ldots, c_n) \in \mathbb{C}_w^{n-1}$, we define

$$\mathcal{A}_{i,c} = (F_1, \ldots, \widehat{F_i}, \ldots, F_n)^{-1}(c).$$

REMARK 2. (1) Each $\mathcal{A}_{i,c}$ is an affine smooth algebraic curve (a complete intersection) in $\mathbb{C}_z^n$, possibly with several connected components.

(2) For fixed $i \in \{1, \ldots, n\}$, the curves $\{\mathcal{A}_{i,c} \mid c \in \mathbb{C}_w^{n-1}\}$ define a non-singular polynomial foliation having $n-1$ first integrals on $\mathbb{C}_z^n$.

For the rest of this subsection, we consider the ith direction in the web $\mathcal{W}$, and the ideas that we develop are valid for any other choice of $i \in \{1, \ldots, n\}$.

Given the curve $\mathcal{A}_{i,c}$, we will consider the associated projective curve $\mathcal{P}_{i,c} \subset \mathbb{CP}_z^n$ and its desingularization (normalization)

$$\pi : \mathcal{L}_{i,\mu,c} \to \mathcal{P}_{i,c}. \tag{3.1}$$

To simplify the notation, we omit the reference to the number of connected components of the desingularization given by μ. Therefore, we consider $\mathcal{L}_{i,c}$ (the disjoint union of the connected components $\mathcal{L}_{i,\mu,c}$, for all μ, where c is fixed) as a compact Riemann surface, a priori with several connected components.

We compactify the affine space $\mathbb{C}_w^n$ in the ith direction, so that we get $\mathbb{CP}^1 \times \mathbb{C}_w^{n-1}$ (to be precise, the $\mathbb{CP}^1$-factor should be in the ith place). Note that by the definition of $\mathcal{A}_{i,c}$, the function F induces non-constant holomorphic maps

$$F_{i,c} : \mathcal{A}_{i,c} \to \mathbb{C}_{i,c}. \tag{3.2}$$

We can summarize all this as follows:

$$\begin{array}{ccccccccc}
\mathcal{L}_{i,c} & \xrightarrow{\pi} & \mathcal{P}_{i,c} & \xrightarrow{\nu} & \mathcal{A}_{i,c} & \xrightarrow{F_{i,c}} & \mathbb{C}_{i,c} & \to & \mathbb{CP}^1 \\
 & & \cap & & \cap & & \cap & & \cap \\
 & & \mathbb{CP}_z^n & \supset & \mathbb{C}_z^n & \xrightarrow{F} & \mathbb{C}_w^n & \subset & \mathbb{CP}^1 \times \mathbb{C}_w^{n-1}
\end{array} \tag{3.3}$$

Here π is the normalization map, from the compact Riemann surface $\mathcal{L}_{i,c}$ to the projective curve $\mathcal{P}_{i,c}$. By abuse of notation, the map ν is the immersion of the projective curve $\mathcal{P}_{i,c}$ minus its points at infinity into the affine curve $\mathcal{A}_{i,c}$. The "$\cap$" are obvious vertical inclusions. In particular, the rightmost one is given by $(w_i) \mapsto (c_1, \ldots, w_i, \ldots, c_n)$.

Thus, when c varies, (3.3) gives the following objects:

(i) The extensions of the *functions* in (3.3) to their normalizations

$$\{F_{i,c} : \mathcal{L}_{i,c} \to \mathbb{CP}^1 \mid i = 1, \dots, n,\ c \in \mathbb{C}_w^{n-1}\} \tag{3.4}$$

induced by F are a well-defined family of non-constant meromorphic functions.

(ii) The associated n-tuples

$$\{\partial/\partial F_i = F^*(\partial/\partial w_i) \mid i = 1, \dots, n\} \tag{3.5}$$

of commuting *polynomial vector fields* on $\mathbb{C}_z^n$. We learned this interesting idea from [NS83].

(iii) The vector field $\partial/\partial F_i$ is well-defined, non-identically zero and meromorphic on the Riemann surface $\mathcal{L}_{i,c}$; we get a family of *pairs*

$$\{(\mathcal{L}_{i,c}, \partial/\partial F_i) \mid i = 1, \dots, n,\ c \in \mathbb{C}_w^{n-1}\}. \tag{3.6}$$

Summing up, each function $F_{i,c}$ and its corresponding vector field $\partial/\partial F_{i,c}$ on $\mathcal{L}_{i,c}$ satisfy Lemma 1.

Remark 3. For $n = 2$, given a map (F_1, F_2) satisfying (1.1), the associated vector field

$$\frac{\partial}{\partial F_1} = \frac{\partial F_2}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial F_2}{\partial z_1}\frac{\partial}{\partial z_2}$$

coincides with the usual Hamiltonian vector field of F_2. In addition, $\partial/\partial F_1$ is tangent to the corresponding affine curves $\mathcal{A}_{1,c} = \{F_2(z_1, z_2) = c\}$.

We examine one fiber again and consider its decomposition

$$\mathcal{L}_{i,c} = \{e_1, \dots, e_v\} \cup (\mathcal{A}_{i,c}),$$

where $\{e_1, \dots, e_v\}$ is the finite non-empty collection of points that emerge from the normalization of points at infinity of $\mathcal{P}_{i,c}$, following (3.3), so that $\pi(e_\beta) \in \mathbb{CP}_\infty^{n-1} \subset \mathbb{CP}_z^n$ for $\beta \in \{1, \dots, v\}$. For simplicity we omit the dependence on i and c in the notation for the points "e".

A priori, the behavior of each $F_{i,c}$ in (3.4) is reflected in cases (i)–(v) in the table below.

Table 1

	Finite value in $\mathbb{C}_{i,c} \subset \mathbb{C}_w^n$	Value at infinity in $\{\infty\} \times \mathbb{C}_w^{n-1}$
finite point $p \in \mathcal{A}_{i,c}$	(i) local biholomorphism	
point at infinity	(ii) local biholomorphism	(iv) local biholomorphism
$e \in \{e_1, \dots, e_v\}$	(iii) ramification index ≥ 2	(v) ramification index ≥ 2

We recall that under the assumption (1.1), F is a local biholomorphism, so the empty places in the table are impossible for each function $F_{i,c}$.

To analyze (ii)–(v), we use the vector fields in (3.3), since they describe $F^*\mathcal{W}$ (recall the definitions and notation introduced at the beginning of this section) accurately. We get the following.

COROLLARY 3.

(1) (Regular point of $\partial/\partial F_i$) *At an affine point* $p \in \mathcal{A}_{i,c} \subset \mathbb{C}^n_z$, $F_{i,c}$ *is a local biholomorphism and* $\partial/\partial F_i$ *has a regular point at* p (*i.e.* $\partial/\partial F_i(p) \neq 0$). *See case* (i) *in Table* 1.
(2) (Removable point of $\partial/\partial F_i$) *A non-affine point* $e \in \{e_1, \dots, e_v\}$ *is such that* $F_{i,c}$ *is a local biholomorphism and its value* $F_{i,c}(e)$ *is finite if and only if* $\partial/\partial F_i$ *extends at* e *as a non-zero regular point. See case* (ii), *ibid.*
(3) (Pole of $\partial/\partial F_i$) *A non-affine point* $e \in \{e_1, \dots, e_v\}$ *is such that* $F_{i,c}$ *has a finite value* $F_{i,c}(e)$ *and ramification index* $\kappa \geq 2$ *if and only if* $\partial/\partial F_i$ *has a pole of order* $-\kappa + 1 \leq -1$ *at* e. *See case* (iii), *ibid.*
(4) (Zero of $\partial/\partial F_i$) *A non-affine point* $e \in \{e_1, \dots, e_v\}$ *is such that* $F_{i,c}$ *has infinite value* $F_{i,c}(e)$ *with ramification index* $\sigma \geq 1$ *if and only if* $\partial/\partial F_i$ *has a zero of order* $(\sigma + 1) \geq 2$ *at* e. *The point* e *is a zero of order* 2 *for case* (iv) *or of order at least* 3 *for case* (v), *ibid.* ∎

The classification in the corollary is very close to the ideas of Drużkowski [Dru91], but our description with vector fields is more explicit.

Note that if e_α, e_β are two points in $\{e_1, \dots, e_v\}$ such that $\pi(e_\alpha) = \pi(e_\beta) = \varrho \in \mathcal{P}_{i,c}$ is a singular point of $\mathcal{P}_{i,c}$, the behavior of $F_{i,c}$ and $\partial/\partial F_i$ at ϱ depends on the choice of the branch of $\mathcal{P}_{i,c}$, i.e. on the choice of e_α, e_β and not only on the singular point ϱ itself.

COROLLARY 4. *If for the value* $c \in \mathbb{C}^{n-1}_w$ *we have* $\mathcal{A}_{i,c} \neq \emptyset$, *then the zeros of* $(\mathcal{L}_{i,c}, \partial/\partial F_i)$ *form non-empty sets and have orders greater than or equal to* 2, *simple zeros are impossible.* ∎

3.2. Asymptotic values of F and the flows of $\partial/\partial F_i$. Now we will describe the interplay between pathological behavior of F, satisfying (1.1), and the local or global flows of $\{\partial/\partial F_i\}$.

The *set of asymptotic values of* F, $\mathcal{AV}(F) \subset \mathbb{C}^n_w$, is the locus where F fails to be proper; this means that there is no compact neighborhood U of $q \in \mathcal{AV}(F) \subset \mathbb{C}^n_w$ such that $F^{-1}(U)$ is compact in $\mathbb{C}^n_z$.

For dominant polynomial maps in $\mathbb{C}^n$, the structure of the set of asymptotic values is studied in many papers (see for example [Jel93], [Jel99], [Per98] and references therein).

Fixing a direction i, we look at the complete collection

$$\{F_{i,c} : \mathcal{L}_{i,c} \to \mathbb{CP}^1 \mid c \in \mathbb{C}^{n-1}_w\},$$

and construct the images of the points (ii)–(v) as subsets of $\mathbb{CP}^1 \times \mathbb{C}^{n-1}_w$, considering the $\mathbb{CP}^1$ factor as the compactification in the ith direction. Let us define the images as follows:

$$\begin{aligned}
R_i &:= \{F_{i,c}(\{\text{removable points of } \partial/\partial F_i\}) \mid c \in \mathbb{C}_w^{n-1}\},\\
P_i &:= \{F_{i,c}(\{\text{poles of } \partial/\partial F_i\}) \mid c \in \mathbb{C}_w^{n-1}\},\\
Z_i &:= \{F_{i,c}(\{\text{zeros of } \partial/\partial F_i\}) \mid c \in \mathbb{C}_w^{n-1}\}.
\end{aligned}$$

Therefore we have

$$R_i, P_i \subset \mathbb{C}_w^n \subset \mathbb{CP}^1 \times \mathbb{C}_w^{n-1},$$
$$Z_i = \{\infty\} \times \mathbb{C}_w^{n-1} \subset \mathbb{CP}^1 \times \mathbb{C}_w^{n-1}.$$

A priori, R_i and/or P_i could be empty, but Z_i is never empty. Let us define

$$R = \bigcup_{i=1}^{n} \overline{R}_i \quad \text{and} \quad P = \bigcup_{i=1}^{n} \overline{P}_i$$

with the closure taken in $\mathbb{C}_w^n$ (with the usual topology). Note that a priori $R \cap P \subset \mathbb{C}_w^n$ can be non-empty.

Remark 4. $\mathcal{AV}(F) = R \cup P$ and by Z. Jelonek's result [Jel93], $\mathcal{AV}(F)$ is an algebraic hypersurface or the empty set.

We want to give an interpretation of $R \cup P$ using local flows. Given $\{\partial/\partial F_i\}$ we can denote by

$$\Psi_i(t,p) : \Omega_i \to \mathbb{C}_z^n, \qquad i \in \{1, \dots, n\},$$

their local flows where t is the complex time. They are holomorphic maps on suitable open $(n+1)$-dimensional complex manifolds Ω_i, their maximal domain of definition. We have a dichotomy:

If $\Omega_i = \mathbb{C}_t \times \mathbb{C}_z^n$ then $\partial/\partial F_i$ is a *complete vector field*, and $\Psi_i(t,p)$ is a flow or a $(\mathbb{C},+)$-action using algebraic language.

If $\Omega_i \neq \mathbb{C}_t \times \mathbb{C}_z^n$ then $\partial/\partial F_i$ is an *incomplete vector field.*

Let $\Delta^n(p,\varepsilon)$ be the n-dimensional open polydisk with center p and radius $\varepsilon > 0$.

Remark 5. For an initial condition $p_0 \in \mathcal{A}_{i,c}$ the local flow Ψ_i can be written using a suitable branch of $F^{-1} : \Delta^n(F(p_0), \epsilon) \subset \mathbb{C}_w^n \to \mathbb{C}_z^n$ of the local inverse as follows:

$$\Psi_i(t,p_0) = F^{-1}(F(p_0) + (0, \dots, t, \dots, 0)). \tag{3.7}$$

This follows from equation (2.1).

Lemma 3. *Let F be a polynomial map with* $\det(DF) = 1$, *and let Ψ_i be its ith pull-back local flow as above.*

(1) *Let* $\Delta^n(q,\varepsilon)$ *be a polydisk inside* $\mathbb{C}^n_w - (R \cup P)$. *Then the local holomorphic flows*

$$\Psi_i(t,p_0) : \Delta(0,\varepsilon) \to \mathbb{C}^n_z, \qquad i \in \{1,\dots,n\},$$

that start at any $p_0 \in F^{-1}(q)$ *are well-defined for* $t \in \Delta(0,\varepsilon)$.

(2) *Assume that* $\Psi_i(t,p_0)$ *exists for* $t \in \Delta(0,\varepsilon)$ *at initial conditions* p_0 *in an open connected set* $B \subset \mathbb{C}^n_z$. *Then the diagram*

$$\begin{tikzcd} B \subset \mathbb{C}^n_z \arrow[r, "\Psi_i(t,\ )"] \arrow[d, "F"'] & \mathbb{C}^n_z \arrow[d, "F"] \\ \mathbb{C}^n_w \arrow[r, "T_i(t,\ )"'] & \mathbb{C}^n_w \end{tikzcd} \tag{3.8}$$

commutes for $T_i(t, w_1,\dots,w_n) = (w_1,\dots,w_i + t,\dots,w_n)$.

(3) $\partial/\partial F_1,\dots,\partial/\partial F_n$ *are complete if and only if* $\mathcal{AV}(F) = \emptyset$.

An advantage of our construction is the splitting of the asymptotic values $\mathcal{AV}(F)$ into two sets: the image of removable points R and poles P. We will apply this distinction in Theorem 1.

Proof. The assertions derive from the fact that F sends $\partial/\partial F_i$ to $\partial/\partial w_i$. Hence, by (2.1) and (3.7) the t in each local flow $\Psi_i(t,p)$ is in local correspondence with the variable $\{w_i\}$. Part (3), as far as we know, was first proved in [MO87]. The reader can also find proofs in [Cam97] and [Bus03]. ∎

Recall that $\Delta^n(q,\varepsilon) \cap \mathcal{AV}(F) = \emptyset$ in Lemma 3(1) is a sufficient but not necessary condition in order that $\Psi_i(t,p)$ starting at $p \in \{F^{-1}(q)\}$ are defined for every time $t \in \Delta(0,\varepsilon)$. A priori, $\{F^{-1}(q)\}$ can have two or more points.

Now we will examine the polynomial submersion defined by $(F_2,\dots,F_n)$ coming from (1.1). The second and third assertions in Lemma 3 are of particular interest when we search for a map between open plaques in the fibers of the submersions $(F_1,\dots,\widehat{F_i},\dots,F_n)$, as follows.

Corollary 5 (Local Ehresmann connections). *For* $t \in \Delta(0,\varepsilon)$ *as in* Lemma 3(2)–(3), *there exist biholomorphic maps*

$$\Psi_j(t,\ ) : U \subset \mathcal{A}_{1,c} \to V \subset \mathcal{A}_{1,c(t)}, \qquad j \in \{2,\dots,n\},$$

such that U, *an open plaque, goes to* V *and* $c(t) = (c_2,\dots,c_j + t,\dots,c_n)$.

Proof. Note that the length of the time and the size of U are bounded as in Lemma 3(1). ∎

A priori the study ot the local bifurcations $(\mathcal{A}_{i,c}, \partial/\partial F_i)$ with respect to $\{c\}$ is a hard problem. The local behavior of non-bifurcation pairs can be seen in the next result and the main theorem will give global non-bifurcation conditions.

COROLLARY 6. *Let $p_0 \in \mathcal{L}_{i,c}$ be such that $F(p_0) \in \mathcal{AV}(F)$.*

(1) *If p_0 is a removable point of $\partial/\partial F_i$, then $\Psi_i(t, p_0)$ can be extended to an open neighborhood $V(p_0) \subset \mathcal{L}_{i,c}$ as a holomorphic flow.*
(2) *If p_0 is a pole of order $-k$, then the local flow does not exist (even as a C^0 map). $F^{-1} : \mathbb{C}^n_w \to \mathbb{C}^n_z$ does not exist.*

Proof. For the first assertion, note that the flow is along the complex trajectory. The second assertion follows from Corollary 1. ∎

4. Invertible polynomial maps. A curve

$$\mathcal{A}_{i,c} = (F_1, \dots, \widehat{F_i}, \dots, F_n)^{-1}(c),$$

coming from a map satisfying (1.1), is a *typical fiber* if there is an open neighborhood $\mathcal{U}$ of $c \in \mathbb{C}^{n-1}_w$ such that the restriction $(F_1, \dots, \widehat{F_i}, \dots, F_n) : (F_1, \dots, \widehat{F_i}, \dots, F_n)^{-1}(\mathcal{U}) \to \mathcal{U}$ is a topologically trivial fiber bundle; otherwise $\mathcal{A}_{i,c'}$ is an *atypical fiber*.

For $n = 2$ the set of atypical fibers is always empty or finite (see [Bro83]). For $n \geq 3$, the set of atypical fibers of $(F_1, \dots, \widehat{F_i}, \dots, F_n)$ can be a hypersurface, probably reducible.

LEMMA 4. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n_z \to \mathbb{C}^n_w$ be a polynomial map with* $\det(DF) = 1$. *If $\partial/\partial F_2, \dots, \partial/\partial F_n$ are complete on typical $\mathcal{A}_{2,c}, \dots, \mathcal{A}_{n,c}$ of $(F_1, \dots, \widehat{F_j}, \dots, F_n)$, $j = 2, \dots, n$, then they are also complete on their atypical fibers.*

Note that in the hypothesis, a priori a typical $\mathcal{A}_{j,c}$ can be reducible and also support a complete $\partial/\partial F_j$. In this case $\mathcal{A}_{j,c}$ is a union of copies of $\mathbb{C}$.

Proof of Lemma 4. We will study the flow Ψ_2, and the same considerations will be true for $\Psi_3, \dots, \Psi_n$.

The atypical fibers of F_i determine a hypersurface $\mathbf{A}_i \subset \mathbb{C}^n_z$, probably reducible. There is a finite set Γ_i of values such that F_i is a locally trivial fiber bundle over $(\mathbb{C}_{w_j} - \Gamma_j)$ (see [Bro83]).

The atypical fibers of $(F_1, F_3, \dots, F_n)$ satisfy

$$\{\mathcal{A}_{2,c'}\} \subset \bigcup_{j \neq 2,\, j=1}^{n} \mathbf{A}_j \tag{4.1}$$

since clearly the intersection of typical hypersurfaces

$$\mathcal{A}_{2,c} = \bigcap_{j \neq 2,\, j=1}^{n} \{F_j = c_j\}$$

produces a typical fiber of $(F_1, F_3, \dots, F_n)$. Here, we are using the fact that each polynomial $F_1, F_3, \dots, F_n$ determines a locally trivial fiber bundle at every $p_0 \in \mathcal{A}_{2,c}$, and the transversality condition between $F_1, F_3, \dots, F_n$ from equation (1.1).

Let p_0 be a point in an atypical $\mathcal{A}_{2,c'}$. The vector field $(\partial/\partial F_2)(p_0)$ is nonzero, and hence at p_0 the vector field admits a local flow box. The atypical fibers $\{\mathcal{A}_{2,c'}\}$ are contained in the union of hypersurfaces, probably singular, at p_0 given by (4.1). Moreover if p_0 is a singular point of the union in (4.1), by the transversality condition from equation (1.1), locally at p_0, the hypersurface $\bigcup_{j\neq 2,\, j=1}^{n} \mathbf{A}_j$ admits a local model of the shape $\{\tilde{z}_1\tilde{z}_3\cdots\tilde{z}_n = 0\}$, where at most $n-1$ local coordinates $\tilde{z}_\iota$ appear, but not necessarily all the $n-1$ coordinates.

As a result, there exists a holomorphic embedding, of a one-dimensional disk, $E : \Delta_s(0,\varepsilon) \to \mathbb{C}^n_z$ such that

(i) $E(0) = p_0$ and the image $E((\Delta_s(0,\varepsilon))$ intersects the atypical fibers of $(F_1, F_3, \ldots, F_n)$ only in p_0,
(ii) at each point, $\partial/\partial F_2$ and the tangent vectors to the embedded disk are linearly independent.

Consider the flow $\Psi_2(t,\ ) := (\Psi_{2,1}, \ldots, \Psi_{2,i}, \ldots, \Psi_{2,n})(t,\ )$ of $\partial/\partial F_2$ starting at the initial conditions in the image $E(\Delta_s(0,\varepsilon))$.

Towards a contradiction, assume that Ψ_2 is not holomorphic on an atypical fiber; we then look at their components. Thus for at least one index i, $\Psi_{2,i}(t, e(0))$ exists, and it is holomorphic for some disk $\Delta(0,r) \subset \mathbb{C}_t$, but not for a complex t_0 with $r := |t_0|$.

Without loss of generality we reparametrize E, and assume that the new domain is $\Delta_s(0,2r)$, but preserving the same image and (i) and (ii).

On the other hand, by the completeness hypothesis for $s \in \Delta_s(0,2r) - \{0\}$, $\Psi_{2,i}(t, E(s))$ exists and it is holomorphic on $\Delta_s(0,2r)$, since this is fulfilled for any radius.

With this in mind we construct the following Hartogs figure:

$$H = \{(t,s) \in \Delta^2(0,2r) \mid |t| < r \text{ or } |s| > r\} \subset \mathbb{C}^2.$$

By the Hartogs Theorem (see [FG02, pp. 25–26]), $\Psi_{2,i}$ has a unique holomorphic extension to the whole $\Delta^2(0,2r)$. That is a contradiction to the existence of a pole of $\Psi_{2,i}(t, E(s))$ at $(t_0, 0) \in \Delta^2(0,2r)$. The flow of $\partial/\partial F_2$ exists for all complex t at every initial condition $p_0 \in \mathbb{C}^n_z$. ∎

The above result seems to be proved by the ideas of other authors; compare [For95] and [Reb04, Proposition 2.8] for the case $n = 2$.

Some results on the invertibility of polynomial maps of $\mathbb{C}^n$ can be proved using $\{\partial/\partial F_i\}$. The second stage for our main theorem is as follows.

THEOREM 1. *Let* $F = (F_1, \ldots, F_n) : \mathbb{C}^n_z \to \mathbb{C}^n_w$ *be a polynomial map with* $\det(DF) = 1$. *The following assertions are equivalent:*

(a) *The inverse map* $F^{-1} : \mathbb{C}^n_w \to \mathbb{C}^n_z$ *exists.*

(b) (Global Ehresmann connections) $\{\partial/\partial F_2, \ldots, \partial/\partial F_n\}$ *are complete on* $\mathbb{C}^n_z$.
(c) *The pairs* $\{(\mathcal{L}_{j,c},\ \partial/\partial F_j)\}$ *for* $j = 2, \ldots, n$ *and all* $c \in (F_1, \ldots, \widehat{F}_j, \ldots, F_n)(\mathbb{C}^n_z)$, *are biholomorphic to* $(\mathbb{CP}^1, \partial/\partial z_j)$; *the vector fields have only a double zero and no poles on* $\mathcal{L}_{j,c}$.
(d) *The projective curves* $\{\mathcal{P}_{j,c}\} \subset \mathbb{CP}^n_z$ *for* $j = 2, \ldots, n$ *and all* $c \in (F_1, \ldots, \widehat{F}_j, \ldots, F_n)(\mathbb{C}^n_z)$ *have only one (irreducible) branch at the hyperplane at infinity of* $\mathbb{CP}^n_z$.
(e) *The polynomial submersion* $(F_2, \ldots, F_n) : \mathbb{C}^n_z \to \mathbb{C}^{n-1}_w$ *is a globally trivial topological fiber bundle (no atypical fibers* $\mathcal{A}_{1,c'}$ *appear).*
(f) *The degree of* F *is one, and* F *is injective.*

Example 1. For the dominant map $(F_1, F_2)(z_1, z_2) = (z_1^d, z_2)$, $d \geq 2$, the critical set $\{\det(D(F_1, F_2)) = 0\}$ is a curve. However, the pull-back $\partial/\partial F_2 = \partial/\partial z_1$ is complete and the typical $\mathcal{A}_{2,c}$ has d connected components. Therefore, we cannot avoid $\det(DF) = 1$ in Theorem 1.

We point out below the new contributions in this paper:

(i) We work in any dimension $n \geq 2$ and use only $j = 2, \ldots, n$ as directions in (b)–(e).
(ii) The equivalence between (a) and the completeness of all $\{\partial/\partial F_1, \ldots, \partial/\partial F_n\}$ was shown by G. H. Meisters and C. Olech [MO87]. A simple proof is also given by A. Bustinduy [Bus03]. Our present assertion is only for $j = 2, \ldots, n$.
(iii) The equivalence between (a) and (d), in case $n = 2$, is a classical result of S. S. Abhyankar (see [A77] or [Ess00, Thm. 10.2.23(1), p. 253)].
(iv) (b)$\Rightarrow$(a) is a kind of cancellation theorem for $\mathbb{C}^n_z$ in the presence of $\det(DF) = 1$. For cancellation problems see [Kr89].
(v) Recall that (f)$\Rightarrow$(a) is the celebrated theorem by Newman, Białynicki-Birula and Rosenlicht [BB-R62].

Proof of Theorem 1. (a)$\Rightarrow$(b). By using [MO87] or [Bus03], the vector fields $\{\partial/\partial F_1, \ldots, \partial/\partial F_n\}$ are holomorphic and complete on $\mathbb{C}^n_z$.

(b)$\Rightarrow$(a). If we assume that the set of asymptotic values $\mathcal{AV}(F)$ is empty, then F is invertible. Therefore, we must assume $\mathcal{AV}(F) \neq \emptyset$.

The completeness of $\{\partial/\partial F_2, \ldots, \partial/\partial F_n\}$ imposes that $\mathcal{AV}(F)$ is invariant under the flows of $\{\Psi_2, \ldots, \Psi_n\}$ on $\mathbb{C}^n_w$. Thus, $\mathcal{AV}(F) = \bigcup_\alpha \{w_1 = c_{1\alpha}\}$ is a union of parallel hyperplanes.

Consider an affine typical $\mathcal{A}_{1,c}$, so that $F(\mathcal{A}_{1,c}) = \{(z_1, c_2, \ldots, c_n)\}$. Every point $p \in \mathbb{C}^n_z$ has a unique canonically associated $\Pi(p) \in \mathcal{A}_{1,c}$ using the Ehresmann connection from Corollary 5 and the completeness of the vector fields as follows. Given the image $F(p) = (w_1, \ldots, w_n) \in \mathbb{C}^n_w$:

- move p following the flow of $\partial/\partial F_2$ for $t_2 = w_2 - c_2$, and get p_2;
- move p_2 following the flow of $\partial/\partial F_3$ for $t_3 = w_3 - c_3$, and get p_3; $\dots$;
- move p_{n-1} following the flow of $\partial/\partial F_n$ for $t_n = w_n - c_n$, and get p_n.

As a result, then $\Pi(p) \in \mathcal{A}_{1,c} = p_n$ is well-defined and unique since the complete vector fields $\partial/\partial F_2, \dots, \partial/\partial F_n$ commute. We have constructed a holomorphic fiber bundle

$$\Pi : \mathbb{C}^n_z \to \mathcal{A}_{1,c}. \tag{4.2}$$

Each fiber $\Pi^{-1}(p_0) \subset \mathbb{C}^n_z$, $p_0 \in \mathcal{A}_{1,c}$, is biholomorphic to $\mathbb{C}^{n-1}$.

To prove this last assertion, we use the fact that the fiber $\Pi^{-1}(p_0)$ supports $n-1$ complete commuting $\{\partial/\partial F_2, \dots, \partial/\partial F_n\}$. Hence, the fiber is biholomorphic to $\mathbb{C}^\ell \times (\mathbb{C}^*)^{n-\ell-1}$ using the ideas in [Bus03, Section 3]. Moreover, the vector fields have double zeros at infinity, since by Remark 1(iv) zeros of order one are forbidden. The $\mathbb{C}^*$ factors are impossible, and so the fiber looks like $\mathbb{C}^{n-1}$.

Concerning the number of components of $\mathcal{A}_{1,c}$, if we assume for a moment that $\mathcal{A}_{1,c}$ has several connected components, recalling that the fiber is $\mathbb{C}^{n-1}$ which is connected, then the total space of the fiber bundle will be disconnected. This contradicts the fact that the total space of the fiber bundle (4.2) is the original $\mathbb{C}^n_z$. Therefore, the typical $\mathcal{A}_{1,c}$ is irreducible.

We remark that the fiber bundle (4.2) has a section: namely the original $\mathcal{A}_{1,c}$ as a submanifold of the total space $\mathbb{C}^n_z$.

If $\mathcal{A}_{1,c} \subset \mathcal{L}_{1,c}$ has at least two punctures (some puncture(s) come from the zero(s) of the $\partial/\partial F_1$ and at least a second puncture from $F(\mathcal{A}_{1,c}) \cap \mathcal{AV}(F)$), then the fundamental group of this fiber is non-trivial.

On the other hand, the homotopy sequence for differentiable fiber bundles with a section (see [Eb07, Prop. 4.20, p. 221]) asserts that the fundamental group of the total space is isomorphic to the product of the fundamental groups of the base and the fiber. In our case $\pi_1(\mathcal{A}_{1,c}) \neq e$; however, $\pi_1(\mathbb{C}^n_z) = e$, which is a contradiction. Thus, $\mathcal{AV}(\mathcal{F})$ must be empty and $\{\partial/\partial F_i \mid i = 1, \dots, n\}$ are complete. Hence F^{-1} exists.

(b)$\Leftrightarrow$(c). "$\Leftarrow$" follows from Corollary 2. For the converse, the vector fields are complete and each $\mathcal{L}_{j,c}$ is at most a finite union of projective lines $\mathbb{CP}^1$. Moreover, using (a)$\Leftrightarrow$(b) when F^{-1} exists, the $\mathcal{L}_{j,c}$ have only one connected component, as is asserted in (c).

(c)$\Leftrightarrow$(d). If we assume (c), then assertion (d) follows from Corollary 2 and Table 1. Conversely, there is a one-to-one correspondence between branches of the projective fibers $\mathcal{P}_{j,c}$ at the hyperplane at infinity of $\mathbb{CP}^n_z$ and zeros, removable points and poles of $\partial/\partial F_j$ on $\mathcal{L}_{j,c}$. Recalling Corollary 2 and Table 1, we note that complete vector fields of the kind $F^*(\partial/\partial w_j)$ have only one double zero on each $\mathcal{P}_{j,c}$. The equivalence follows.

(b)⇔(e). We assume (b), thus we use the geometry of the set of asymptotic values as in the proof of (a)⇔(b): each $\mathcal{A}_{1,c}$ can be pushed by the Ehresmann connection of $\{\partial/\partial F_2, \dots, \partial/\partial F_n\}$ for every time. Thus, $(F_2, \dots, F_n) : \mathbb{C}^n_z \to \mathbb{C}^{n-1}_w$ determines a holomorphically trivial fiber bundle. For the converse assertion, if the fiber bundle determined by $(F_2, \dots, F_n)$ as in the line above is topologically trivial, then the fundamental group of the fiber $\mathcal{A}_{1,c}$ is trivial and $\partial/\partial F_1$ is complete. Therefore (b) is true.

(b)⇔(f). Using (b) as hypothesis, $(F_2, \dots, F_n)$ determines a holomorphically trivial fiber bundle with fiber $\mathbb{C}^{n-1}$, base $\mathcal{A}_{1,c}$ and total space biholomorphic to $\mathbb{C}^n_z$, as in (4.2). For topological reasons, $\mathcal{A}_{1,c}$ is a complex line. The degree of F equals the degree of $F_{1,c} : \mathcal{A}_{1,c} \to \mathbb{C}_{1,c}$ (because $\mathcal{A}_{1,c}$ is a typical fiber), and $F_{1,c}$ is a biholomorphism. Hence, the degree of F is one.

Assume (f); the asymptotic values are $\mathcal{AV}(F) = R \cup P$ as in Remark 4.

We note that P is empty: otherwise one pair $(\mathcal{L}_{i,c}, \partial/\partial F_i)$, $i \in \{1, \dots, n\}$, has a pole; then by Remark 1(1), F would be of degree greater than or equal to 2, contrary to hypothesis (f).

As a result, $\mathcal{AV}(F) = R$, and it is empty or a hypersurface (see Remark 4 and [Jel93]).

If $R = \emptyset$ then F is bijective and we can conclude that $\{\partial/\partial F_1, \dots, \partial/\partial F_n\}$ are complete.

If $R \neq \emptyset$ then let us use a slight modification of the original idea in the Newman–Białynicki-Birula–Rosenlicht Theorem (see [BB-R62] or more recently [Gr99, Section 3.B]).

We note that $F : \mathbb{C}^n_z \to \mathbb{C}^n_w - R$ is a local biholomorphism of degree 1 (since $P = \emptyset$). Therefore,

$$H_1(\mathbb{C}^n_w - R, \mathbb{Z}) = \mathbb{Z}^{\oplus \nu},$$

where ν is the number of irreducible components of R; for the computation of this homology (see [Dim92, p. 103]). That contradicts $H_1(\mathbb{C}^n_z, \mathbb{Z}) = 0$. Thus R is empty, and assertion (b) holds. ∎

COROLLARY 7. *If one $(\mathcal{L}_{i,c}, \partial/\partial F_i)$ has a pole, then F^{-1} does not exist.*

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