COLLOQUIUM MATHEMATICUM

VOL. 140 2015 NO. 2

TOPOLOGICAL ASPECTS OF INFINITUDE OF PRIMES IN ARITHMETIC PROGRESSIONS

BY

FRANTIŠEK MARKO (Hazleton, PA) and ŠTEFAN PORUBSKÝ (Praha)

Abstract. We investigate properties of coset topologies on commutative domains with an identity, in particular, the $\mathcal{S}$-coprime topologies defined by Marko and Porubský (2012) and akin to the topology defined by Furstenberg (1955) in his proof of the infinitude of rational primes. We extend results about the infinitude of prime or maximal ideals related to the Dirichlet theorem on the infinitude of primes from Knopfmacher and Porubský (1997), and correct some results from that paper. Then we determine cluster points for the set of primes and sets of primes appearing in arithmetic progressions in $\mathcal{S}$-coprime topologies on $\mathbb{Z}$. Finally, we give a new proof for the infinitude of prime ideals in number fields.

1. Introduction. Our primary motivation comes from Furstenberg [Fu] who in 1955 used an elegant topological idea to prove the infinitude of primes. He used a topology on the set $\mathbb{Z}$ of all integers induced by the system of all two-sided infinite arithmetic progressions $\{an+b\}_{n=-\infty}^{\infty}$. Subsequently, Golomb [Go1], [Go2] used one-sided infinite arithmetic progressions with $(a,b)=1$ to introduce a topology on the set $\mathbb{N}$ of positive integers with the aim of applying a similar topological approach to the Dirichlet theorem on the infinitude of primes in arithmetic progressions. Furstenberg's and Golomb's ideas were then analyzed in [KP] in a more general setting of commutative rings with identity and without zero divisors. In [Po], the Furstenberg proof was extended to generalized ideal systems, the so-called x-ideals (cf. [Au]). Actually, it seems that it was Golomb [Go2] who foreshadowed the possibility of extending the technique of Furstenberg's proof to more abstract algebraic structures. However, his ideas were not always transparent or precise. In 2003 Broughan [Br] came with another generalization of the original Furstenberg idea.

Our direct motivation comes mainly from papers [Br], [KP] and [MP]. Our investigation is also based on the interplay of the concepts of (topolog-

2010 *Mathematics Subject Classification*: Primary 11B05; Secondary 11N80, 11N25, 11A25, 22A99, 22A15.

Key words and phrases: coset topology, topological semigroup, topological density, Dirichlet theorem on primes, arithmetic progression, maximal ideal, ring of finite character, residually finite ring, infinitude of primes, pseudoprime.

DOI: 10.4064/cm140-2-5 © Instytut Matematyczny PAN, 2015

ical) density and strong density (cf. Section 4.1 for more details) which was developed in [KP] as a result of an analysis of some previous proofs given in the domain of integers. A deeper analysis of these ideas shows that they are applicable not only to primes in arithmetic progressions but also to a wider class of sequences; for instance, various types of pseudoprimes, coprime elements, etc. in arithmetic sequences (we direct the interested reader to [KP] for further details).

In Section 2 we recall the definition of coset topologies and properties of induced topologies. In Section 3 we study $\mathcal{S}$-coprime topologies and their properties in the context of topological semigroups. In Section 4 we discuss the infinitude of primes in arithmetic progressions, and clarify and correct some results of [KP]. In Section 5 we generalize results of [Br], and compute the cluster points of arithmetic progressions with respect to various $\mathcal{S}$-coprime topologies on $\mathbb{Z}$. In Section 6 we use $\mathcal{S}$-coprime topologies to find a certain topological condition that guarantees the infinitude of prime ideals in rings and leads to a new short proof of the infinitude of prime ideals in number fields.

2. Coset topologies

2.1. Definition. For the convenience of the reader we recall the definition of coset topology from [KP]. All rings R under consideration will be commutative with an identity $1 = 1_R \neq 0$, and with no proper zero divisors.

Given two subsets A, B of R, let $A + B = \{a + b : a \in A,\ b \in B\}$ and $AB = \{ab : a \in A,\ b \in B\}$. Two ideals $\mathfrak{A}, \mathfrak{B}$ of R are called *coprime* if $\mathfrak{A} + \mathfrak{B} = R$.

We call $\mathfrak{A}$ a *proper ideal* if it is a proper subset of R, that is, $\mathfrak{A} \neq R$. The ideal R is called the *unit ideal* and (0) is the *zero ideal*. Let $\mathcal{I} = \mathcal{I}_R$ be the set of all proper nonzero ideals of R. The set $\mathcal{I} \cup \{R\}$ is closed under the operation of addition of ideals, and the set $\mathcal{I}$ itself is closed under the operation of intersection and multiplication defined above.

The set of proper maximal ideals will be denoted by $\mathcal{M}$, and the set of proper nonzero prime ideals by $\mathcal{P}$.

Suppose that we have assigned to each element $a \in R$ a nonempty subset $\mathcal{S}_a$ of $\mathcal{I}$ with the following property:

$$\text{(2.1)} \qquad \text{If } \mathfrak{A}, \mathfrak{B} \in \mathcal{S}_a, \text{ then there is } \mathfrak{C} \in \mathcal{S}_a \text{ such that } \mathfrak{C} \subset \mathfrak{A} \cap \mathfrak{B}.$$

For each $a \in R$, the collection $\mathcal{C}_a = \{a + \mathfrak{A} : a \in R,\ \mathfrak{A} \in \mathcal{S}_a\}$ is then a filter base at a because for $\mathfrak{A}, \mathfrak{B} \in \mathcal{S}_a$ there is $\mathfrak{C} \in \mathcal{S}_a$ such that $\mathfrak{C} \subset \mathfrak{A} \cap \mathfrak{B}$ and hence $a + \mathfrak{C} \subset (a + \mathfrak{A}) \cap (a + \mathfrak{B})$. The set of cosets

$$\text{(2.2)} \qquad \mathcal{C}_{\mathcal{S}} = \{a + \mathfrak{A} : a \in R, \mathfrak{A} \in \mathcal{S}_a\}$$

forms a subbase of a topology $\tau_{\mathcal{S}}$ on R, called a *coset topology*.

The role of cosets in defining a topology is based on the following simple result:

LEMMA 2.1 ([KP, Lemma 1]). *Let* $a+\mathfrak{A}, b+\mathfrak{B}$ *be two cosets of ideals* $\mathfrak{A}, \mathfrak{B}$ *of a ring* R. *Then their intersection* $(a+\mathfrak{A}) \cap (b+\mathfrak{B})$ *is either empty or a single coset* $z+\mathfrak{A}\cap\mathfrak{B}$, *where* $z \in (a+\mathfrak{A}) \cap (b+\mathfrak{B})$.

Coset topologies were used and studied in several papers. For instance, Knopfmacher and Porubský [KP] investigated the following three coset topologies on R:

- the *linear topology* τ_1 generated by all cosets, where $\mathcal{S}_a = \mathcal{I}$ for every $a \in R$,
- the *nontrivial cosets topology* τ_2 with $\mathfrak{A} \in \mathcal{S}_a$ if and only if $a \notin \mathfrak{A}$,
- the *invertible cosets topology* τ_3, where $\mathfrak{A} \in \mathcal{S}_a$ if and only if (a) and $\mathfrak{A}$ are *coprime ideals*, that is, $(a)+\mathfrak{A} = R$.

2.2. Induced topologies. Let $R^0 = R \setminus \{0\}$. Two elements a, b of R^0 are called *associated*, written $a \sim b$, if the principal ideals (a) and (b) coincide. Of course, this means that $a = ub$ for a unit $u \in R$.

The aim of [KP] was to clarify and formalize some ideas indicated by Golomb and other authors in order to transfer the Furstenberg topology from $\mathbb{Z}$ to the set of positive (or nonnegative) integers which does not form a ring. A closer analysis of these ideas leads to the necessity of taking into account an induced topology on the factor set

$$G_R = R^0/{\sim}.$$

If $\overline{a}$ denotes the $\sim$-equivalence class of a, then G_R with the multiplication $\overline{a}.\overline{b} = \overline{ab}$ forms a commutative semigroup with zero. Given $X \subset R^0$, let $\overline{X} = \bigcup\{\overline{x} : x \in X\}$.

Following [KP], let

$$\theta : R^0 \to G_R, \qquad \theta(a) = \overline{a},$$

be the canonical semigroup epimorphism relative to the ring multiplication.

For the remainder of this section we shall suppose that the multiplicative semigroup $(R, \cdot)$ of the ring R is a topological semigroup relative to the topology $\tau_{\mathcal{S}}$. Consequently, $R^0 = (R \setminus \{0\}, \cdot)$ is a topological semigroup relative to the topology $\tau^*_{\mathcal{S}}$ induced on R^0 by $\tau_{\mathcal{S}}$.

REMARK 2.2. If the underlying ring R is infinite, then all elements of the subbase $\mathcal{C}_{\mathcal{S}}$ are also infinite, and consequently no nonempty finite subset of R or R^0 is open in the topology $\tau_{\mathcal{S}}$ or $\tau^*_{\mathcal{S}}$, respectively.

On the other hand, as was shown in [KP], the set G_R can be endowed with the quotient topology $\Delta^*_{\mathcal{S}}$ with respect to the canonical epimorphism θ

and the induced topology $\tau_{\mathcal{S}}^*$ on R^0. This quotient topology

$$\varDelta_{\mathcal{S}}^* = \left\{X \subset G_R : \theta^{-1}(X) = \bigcup\{\overline{x} : \overline{x} \in X\} \text{ belongs to } \tau_{\mathcal{S}}^*\right\} \tag{2.3}$$

is the greatest topology with respect to which the canonical epimorphism θ is continuous.

The proofs of the following two statements are analogous to those in [KP].

Lemma 2.3. *The canonical epimorphism $\theta : R^0 \to G_R$ is continuous and open with respect to the topologies $\tau_{\mathcal{S}}^*, \varDelta_{\mathcal{S}}^*$.*

Proposition 2.4. *G_R forms a topological semigroup relative to the quotient topology $\varDelta_{\mathcal{S}}^*$.*

3. $\mathcal{S}$-coprime topologies

3.1. Definition. The notion of $\mathcal{S}$-coprime topology was introduced in [MP]. We recall its definition below.

Fix a subset $\mathcal{S}$ of $\mathcal{P}$. For each $a \in R$ define the set $\mathcal{S}_a$ as follows:

F: An ideal $\mathfrak{A} \in \mathcal{I}$ belongs to $\mathcal{S}_a$ if and only if $(a) + \mathfrak{A} \not\subset \mathfrak{P}$ for every $\mathfrak{P} \in \mathcal{S}$, or equivalently, for every $\mathfrak{P} \in \mathcal{S}$ there exist $t_{\mathfrak{P}} \in R$ and $x_{\mathfrak{P}} \in \mathfrak{A}$ such that $t_{\mathfrak{P}} a + x_{\mathfrak{P}} \notin \mathfrak{P}$.

In particular, $\mathcal{S}_a$ contains all ideals coprime to the principal ideal (a). We can give another description of elements from $\mathcal{S}_a$ as follows. Let $\mathcal{H}_a$ be the set of those ideals from $\mathcal{S}$ that contain a. Then

$$\mathcal{S}_a = \{\mathfrak{Q} \in \mathcal{I} : \mathfrak{Q} \not\subset \mathfrak{B} \text{ for every } \mathfrak{B} \in \mathcal{H}_a\}.$$

If R is a Dedekind domain, then $\mathfrak{A} \in \mathcal{S}_a$ if and only if the greatest common divisor $(a, \mathfrak{A})$ is a product of prime ideals that do not belong to $\mathcal{S}$.

A coset $a + \mathfrak{A}$ such that $\mathfrak{A} \in \mathcal{S}_a$ is called *$\mathcal{S}$-coprime.*

The Krull theorem says that every commutative ring with a multiplicative identity has a maximal ideal. On the other hand, in a commutative ring with identity, every maximal ideal is a prime ideal. In the special case when $\mathcal{S}$ consists of maximal ideals, the above condition **F** can be reformulated as

F′: An ideal $\mathfrak{A} \in \mathcal{I}$ belongs to $\mathcal{S}_a$ if and only if for every $\mathfrak{M} \in \mathcal{S}$ we have $(a) + \mathfrak{A} + \mathfrak{M} = R$.

Here the condition holds for every ideal $\mathfrak{M}$, not necessarily maximal, for if $ta + x + m = 1$ for some $t \in R$, $x \in \mathfrak{A}$, $m \in \mathfrak{M}$, then $ta + x = 1 - m$, while clearly $1 - m \notin \mathfrak{M}$. The condition is sufficient because $ta + x = y$ with $y \notin \mathfrak{M}$ and $\mathfrak{M}$ maximal implies $(a) + \mathfrak{A} + \mathfrak{M} = R$.

3.2. Properties. In the following lemmas extending [KP, Lemmas 4–6] we list some useful properties of $\mathcal{S}$-coprime cosets.

LEMMA 3.1. *Let $\mathfrak{M}$ be a maximal ideal of a ring R and let $m \notin \mathfrak{M}$. Then the coset $m + \mathfrak{M}$ is $\mathcal{S}$-coprime for every $\mathcal{S}$.*

The proof is immediate. In the next lemmas we suppose that $\mathcal{S}$ is given.

LEMMA 3.2. *If $a + \mathfrak{A}$ and $b + \mathfrak{A}$ are two $\mathcal{S}$-coprime cosets, then so is $ab + \mathfrak{A}$.*

Proof. Let $\mathfrak{P} \in \mathcal{S}$. Then there exist $t_{\mathfrak{P}}, t'_{\mathfrak{P}} \in R$ and $x_{\mathfrak{P}}, x'_{\mathfrak{P}} \in \mathfrak{A}$ such that $t_{\mathfrak{p}}a + x_{\mathfrak{P}} \notin \mathfrak{P}$ and $t'_{\mathfrak{P}}b + x'_{\mathfrak{P}} \notin \mathfrak{P}$. Since $\mathfrak{P}$ is a prime ideal, we have $(t_{\mathfrak{p}}a + x_{\mathfrak{P}})(t'_{\mathfrak{P}}b + x'_{\mathfrak{P}}) \notin \mathfrak{A}$. Here $(t_{\mathfrak{p}}a + x_{\mathfrak{P}})(t'_{\mathfrak{P}}b + x'_{\mathfrak{P}}) = t_{\mathfrak{p}}t'_{\mathfrak{P}}ab + t'_{\mathfrak{P}}bx_{\mathfrak{P}} + t_{\mathfrak{p}}ax'_{\mathfrak{P}} + x_{\mathfrak{P}}x'_{\mathfrak{P}}$. The facts that $t_{\mathfrak{p}}t'_{\mathfrak{P}} \in R$ and $t'_{\mathfrak{P}}bx_{\mathfrak{P}} + t_{\mathfrak{p}}ax'_{\mathfrak{P}} + x_{\mathfrak{P}}x'_{\mathfrak{P}} \in \mathfrak{A}$ finish the proof. ■

LEMMA 3.3. *Let $a + \mathfrak{A}$ be $\mathcal{S}$-coprime. Then so is $a + \mathfrak{A}^n$ for all $n = 1, 2, \ldots$.*

Proof. Given $\mathfrak{P} \in \mathcal{S}$, there exist $t_{\mathfrak{p}} \in R$ and $x_{\mathfrak{p}} \in \mathfrak{A}$ such that $t_{\mathfrak{p}}a + x_{\mathfrak{P}} \notin \mathfrak{P}$. Since $\mathfrak{P}$ is a prime ideal, by induction on n we see that $(t_{\mathfrak{p}}a + x_{\mathfrak{P}})^n \notin \mathfrak{P}$. On the other hand $(t_{\mathfrak{p}}a + x_{\mathfrak{P}})^n = a\sum_{k=1}^{n}\binom{n}{k}a^{k-1}t_{\mathfrak{p}}^k x_{\mathfrak{p}}^{n-k} + x_{\mathfrak{p}}^n$, where $\sum_{k=1}^{n}\binom{n}{k}a^{k-1}t_{\mathfrak{p}}^k x_{\mathfrak{p}}^{n-k} \in R$ and $x_{\mathfrak{p}}^n \in \mathfrak{A}^n$. ■

Denote the collection of $\mathcal{S}$-coprime cosets by

$$\mathcal{C}_{\mathcal{S}} = \{a + \mathfrak{A} : a \in R,\, \mathfrak{A} \in \mathcal{S}_a\}.$$

By [MP, Lemma 2.8], the system $\mathcal{C}_{\mathcal{S}}$ forms a base of a topology $\tau_{\mathcal{S}}$ on R, called the *$\mathcal{S}$-coprime topology.*

PROPOSITION 3.4. *The topology $\tau_{\mathcal{S}}$ converts the multiplicative semigroup of the ring R into a topological semigroup. Also, R^0 is a topological semigroup relative to the topology $\tau^*_{\mathcal{S}}$ induced on R^0 by the topology $\tau_{\mathcal{S}}$.*

Proof. It remains to prove that the ring multiplication is continuous. Choose an $\mathcal{S}$-coprime coset $ab + \mathfrak{C}$. For every $\mathfrak{P} \in \mathcal{S}$ there are $t \in R$ and $x \in \mathfrak{C}$ such that $tab + x \notin \mathfrak{P}$. Since $tab + x = (tb)a + x = (ta)b + x$, the cosets $a + \mathfrak{C}$ and $b + \mathfrak{C}$ are also $\mathcal{S}$-coprime. Clearly $(a + \mathfrak{C})(b + \mathfrak{C}) \subset ab + \mathfrak{C}$, proving the claim. ■

Our topology $\tau_{\mathcal{S}}$ yields a new class of generalized topologies based on an idea introduced by Broughan [Br] in the case of the ring of integers. Note that the extreme cases $\mathcal{S} = \emptyset$ and $\mathcal{S} = \mathcal{P}$ produce the previously defined linear topology τ_1 and invertible cosets topology τ_3, respectively.

If $\mathcal{M} \subset \mathcal{S}$, then $\tau_{\mathcal{S}}$ is the invertible cosets topology, and by [KP, Theorem 12] the topological spaces $(R, \tau_{\mathcal{S}})$, $(R^0, \tau^*_{\mathcal{S}})$ and $(G_R, \Delta^*_{\mathcal{S}})$ are all connected. This is not the case when $\mathcal{M} \not\subset \mathcal{S}$.

PROPOSITION 3.5. *If $\mathcal{M} \not\subset \mathcal{S}$, then the topological spaces $(R, \tau_{\mathcal{S}})$, $(R^0, \tau^*_{\mathcal{S}})$ and $(G_R, \Delta^*_{\mathcal{S}})$ are disconnected.*

Proof. Let $\mathfrak{P} \in \mathcal{M} \setminus \mathcal{S}$. Then $\mathfrak{P} \in \mathcal{S}_a$ for each $a \in R$, and consequently all cosets $a + \mathfrak{P}$ are open in the topology $\tau_{\mathcal{S}}$. Since R is a disjoint union of these cosets, we conclude that $(R, \tau_{\mathcal{S}})$ is disconnected. This immediately implies that $(R^0, \tau^*_{\mathcal{S}})$ is disconnected too.

To handle the case of $(G_R, \Delta^*_{\mathcal{S}})$, assign to each $a \in R^0$ the set U_a of nonzero elements of all cosets modulo $\mathfrak{P}$ that contain an element associated to a. Observe that U_a and U_b are either disjoint or identical. Indeed, assume $c \in U_a \cap U_b$, and write $c = au_1 + p_1$ and $c = bu_2 + p_2$, where u_1, u_2 are units and $p_1, p_2 \in \mathfrak{P}$. Then any element x of U_a can be written as $x = au + p$, where u is a unit and $p \in \mathfrak{P}$. Then

$$xu^{-1}u_1 = au_1 + pu^{-1}u_1 = c + (pu^{-1}u_1 - p_1) = bu_2 + (pu^{-1}u_1 - p_1 + p_2),$$

showing that $x \in U_b$. Thus we get $U_a \subset U_b$, and analogously $U_b \subset U_a$. Each U_a is an open subset of $(R^0, \tau^*_{\mathcal{S}})$, and R^0 is a disjoint union of these sets. Moreover, since U_1 does not contain $\mathfrak{P} \setminus \{0\}$, we have $U_1 \neq R^0$ and the above union must contain at least two members. Finally, the sets $V_a = \theta(U_a)$ are open, because $\theta^{-1}(V_a) = U_a$ are open, and the space G_R is a disjoint union of at least two open sets V_a, showing that G_R is disconnected. ∎

4. Strong density and the Dirichlet condition

4.1. Definition. The following interesting observation was made by Sierpiński (1):

PROPOSITION 4.1 ([Si2, p. 124]). *The following two statements are equivalent:*

- **T:** *If a and b are positive integers such that $(a, b) = 1$, then there exist infinitely many primes of the form $ak + b$, where k is a positive integer.*
- **$\mathbf{T}_1$:** *If a and b are positive integers such that $(a, b) = 1$, then there exists at least one prime number p of the form $ak + b$, where k is a positive integer.*

As we have already mentioned, some earlier ideas involving arithmetic progressions are not always transparent. For instance, Golomb [Go2, p. 181] writes:

> In particular, if the proof that works for the rational integers should also be valid in other rings of algebraic integers (where the corresponding topology, based on residue classes of ideals, is introduced), the enrichment of number theory would be enormous. Thus, the corresponding theorem for the Gaussian integers would imply infinitely many Gaussian primes in the progression $\{n+i\}$, and hence infinitely many rational primes of the form $n^2 + 1$ is a classical unsolved problem.

(1) See also [Si1, Si3, Sp, Sc, Wo].

Unfortunately, the arithmetic progression $\{n+i\}$ does not form a residue class of an ideal in the ring of Gaussian integers.

For us, a more important point worth noting are ambiguities of using arithmetic progressions $\{ak+b\}$, where k runs over nonnegative (or positive) integers, as a topological basis of the set of nonnegative integers (which is a semigroup but not a ring); in particular, when it is not clear whether $b \leq a$.

Sierpiński's original proof in [Si1, p. 526] works with arithmetic progressions $\{ak+b\}$ with k running over nonnegative integers, and with b arbitrarily large. His proof by contradiction starts with the assumption that there is an arithmetic progression with $(a,b)=1$ containing only finitely many primes. The contradiction is reached using an arithmetic subprogression of $\{ak+b\}$ which has the initial term larger than the greatest prime contained in the original progression $\{ak+b\}$. (In later editions [Si2, p. 124] and [Si3, p. 129], he uses arithmetic progressions with k running over positive integers but with sufficiently large multiples of the original common difference.) Six years later, the equivalence of Proposition 4.1 was formulated by Spira [Sp] as an elementary problem in the American Mathematical Monthly. The first solution to this problem given in [Ha] was incorrect. This is because the assumption of the existence of a prime in the progression $\{ak+b\}$, where k runs over nonnegative integers, cannot guarantee that $ak+b$ is a prime for some $k>0$ if b itself is a prime. The correct proof given in [Ze] uses a subprogression with a common difference that is a higher power of the original common difference (in the spirit of Lemma 3.3). This subprogression contains an additional prime (note that Sierpiński uses a similar idea in his proof in [Si2, p. 124] and [Si3, p. 129]). Golomb [Go2, Theorem 6] reformulated the equivalence of Proposition 4.1 but for its proof he refers simply to [Sp].

These comments show an interesting facet of the above mentioned proofs of the equivalence between statements $\mathbf{T}$ and $\mathbf{T}_1$, namely, k must run over positive integers, and not over nonnegative integers only. The reason is that if b itself is a prime, then the arguments used may reproduce b and need not generate a new prime. That means that the simple density does not ensure that the next prime constructed is greater than b. This was the reason for introducing the concept of strong density in [KP]. Subject to certain further hypotheses on R, the $\mathcal{P}$-Dirichlet condition for a subset P of R defined below, called simply Dirichlet condition in [KP], is equivalent to the *strong* density of the set $\overline{P}$ in G_R.

Given a subset W of a topological space (Y,τ), call W *strongly dense* in Y if every nonempty open set in τ contains at least two elements of W. In case of ambiguity we shall use the term τ*-strongly dense*.

The following statement, which is Lemma 16 of [KP], shows that under certain assumptions, density and strong density are equivalent. This statement was given in [KP] without proof. Since it will be needed below, for the sake of completeness we provide its proof here.

LEMMA 4.2. *Assume that Y is a T_1-space. Assume W is dense in Y and no singleton $\{w\}$ for $w \in W$ is open in Y. Then W is strongly dense in Y.*

Proof. Take an arbitrary nonempty open subset X of Y. Since W is dense, there is $w \in X \cap W$. As Y is a T_1-space, the set $\{w\}$ is closed. Hence $X \setminus \{w\} = X \cap (Y \setminus \{w\})$ is an open set that cannot be empty because $\{w\}$ is not open. Since W is dense in Y, we conclude that there is another element $w' \in W \cap (X \setminus \{w\})$, showing that W is strongly dense in Y. ∎

As was shown in [KP], Golomb's proof of [Go2, Theorem 6] can be retrieved using the fact that the subspace topology $\mathfrak{D}$, defined on p. 135 of [KP], is Hausdorff. Thus as expected, there are situations when both types of densities are equivalent. We have the following analogue of [KP, Proposition 15].

PROPOSITION 4.3. *Assume that a ring R is such that $\mathfrak{A}^2 \neq \mathfrak{A}$ for every ideal $\mathfrak{A}$ of R different from R and the zero ideal. Then a set is strongly dense in the topology $\tau_{\mathcal{S}}$, respectively $\tau^*_{\mathcal{S}}$, if and only if it is dense in $\tau_{\mathcal{S}}$, respectively $\tau^*_{\mathcal{S}}$.*

Proof. We only need to prove sufficiency; necessity is obvious. Assume that A is dense in the topology $\tau_{\mathcal{S}}$ or $\tau^*_{\mathcal{S}}$, and consider an $\mathcal{S}$-coprime coset $a + \mathfrak{A}$. Since $\mathfrak{A}$ is a nonzero proper ideal of R, there is $y \in \mathfrak{A} \setminus \mathfrak{A}^2$. Then $a+y+\mathfrak{A}$ is also $\mathcal{S}$-coprime. This is clear because for $\mathfrak{A} \in \mathcal{S}$ and $t_{\mathfrak{A}}a+x_{\mathfrak{A}} \notin \mathfrak{A}$ we have $t_{\mathfrak{A}}(a+y)+(-t_{\mathfrak{A}}y+x_{\mathfrak{A}}) = t_{\mathfrak{A}}a+x_{\mathfrak{A}} \notin \mathfrak{A}$. By Lemma 3.3, the cosets $a+\mathfrak{A}^2$ and $a+y+\mathfrak{A}^2$ are $\mathcal{S}$-coprime. They are disjoint by the choice of y, and both are included in $a+\mathfrak{A}$. Therefore the density of A implies that A is strongly dense in the topology $\tau_{\mathcal{S}}$ or $\tau^*_{\mathcal{S}}$, respectively. ∎

The above condition $\mathfrak{A}^2 \neq \mathfrak{A}$ for proper nonzero ideals $\mathfrak{A}$ of R is satisfied if R is a Noetherian domain. A more general condition ensuring that $\mathfrak{A}^2 \neq \mathfrak{A}$ for proper nonzero ideals $\mathfrak{A}$ of R is $\bigcap_{n=1}^{\infty} I^n = (0)$ for every proper ideal I of R. The latter condition is satisfied for Noetherian domains. Another class of rings R for which the last condition is satisfied are the so-called almost Dedekind domains. In fact, almost Dedekind domains are characterized within the class of Prüfer domains by the fulfilment of the condition $\bigcap_{n=1}^{\infty} I^n = (0)$ for all proper ideals I (cf. [Gi2, (29.5) Theorem] or [Gi1, Corollary1]).

The condition $\bigcap_{n=1}^{\infty} I^n = (0)$ is clearly satisfied for every ideal contained in the Jacobson radical $J(R)$ (the intersection of all maximal ideals of R). Therefore this condition is valid for all proper ideals in a Noetherian local ring.

In this connection, we would like to mention two extreme classes of rings: idempotent-free rings and fully idempotent rings.

A ring R with an identity is called *idempotent-free* if the only idempotent ideals of R are the zero ideal and R itself. A commutative ring is idempotent-free only if the only idempotents of R are 0 and 1. The converse is false for general commutative rings but it is true for commutative Noetherian rings, since if I is a finitely generated idempotent ideal of a commutative ring R, then I is principal and is generated by an idempotent element [Gi3, Lemma 1].

A ring R (not necessarily commutative and with an identity) in which $I = I^2$ for each ideal I is referred to as a *fully idempotent ring*. It is proved in [Co, 1.2 Theorem] that a ring R is fully idempotent if and only if every factor ring of R is a semiprime ring (a ring is called *semiprime* if no nonzero ideal is nilpotent). In [JKL, Proposition 1.4], the authors prove that a commutative domain is fully idempotent if and only if it is a field. The assumption of being a domain is essential. For instance, a direct product of two fields has two ideals, both idempotent (and both maximal).

The previous proposition gives an impetus to the following modification of the Dirichlet condition defined in [KP]: Given an $\mathcal{S}$-coprime topology on a ring R, we say that R satisfies the *$\mathcal{S}$-Dirichlet condition for a subset $A \subset R$* if every $\mathcal{S}$-coprime coset in R (or, equivalently, every $\tau_{\mathcal{S}}$-open set) contains infinitely many pairwise nonassociated elements from A.

4.2. Corrections and extensions to [KP]. The paper [KP] includes an extensive discussion relating the Dirichlet condition to density, specifically to strong density. It partly depends on the following assumptions about the rings R under investigation:

(i) R admits a nonnegative integer-valued *norm* mapping N with the properties:
   (a) $N(x) = 0$ if and only if $x = 0$,
   (b) $N(x) = 1$ if and only if x is a unit,
   (c) $N(ab) = N(a)N(b)$ for all $a, b \in R$.

(ii) For any fixed $x, y \in R$ and any units u, v of R, $N(ux + vy)$ is bounded uniformly relative to $N(x)$ and $N(y)$.

(iii) G_R contains only finitely many elements $\overline{a}$ for which $N(a)$ takes any given, fixed value $k \in \mathbb{N}$.

Most of the rings appearing in number theory are Dedekind and residually finite. Here if I is a nonzero ideal of a ring R such that the ring R/I is finite, then I is said to be *residually finite.* The ring R is said to be *residually finite* if every nonzero ideal of R is residually finite. In this case, the positive integer $\mathcal{N}(I) = \operatorname{card}(R/I)$ is called the *norm* of I.

Unfortunately, Lemma 17 of [KP] is not completely true in the stated form, as was pointed out by W. Narkiewicz. The source of the incorrect conclusion is the fact that the norms $N(ux+vy)$ of ideals $(ux+vy)$ with fixed $x, y \in R$ but varying units u, v of R are not uniformly bounded relative to $N(x)$ and $N(y)$ in general. One obvious circumstance when condition (ii) is satisfied is when the ring R has only finitely many units (2). Therefore one possibility to correct Lemma 17 of [KP] is as follows.

Lemma 4.4. *Every residually finite Dedekind domain with a finite number of units satisfies assumptions* (i)–(iii) *with* $N = \mathcal{N}$ *and* $N(0) = 0$.

Theorem 19 of [KP] can be replaced by a special case (when $\mathcal{S} = \mathcal{P}$) of the next statement.

Theorem 4.5. *Suppose* $\mathcal{S}$ *contains at least one maximal ideal of* R, *and* R *satisfies assumptions* (i)–(iii). *If* P *is a subset of* R *such that* $\widetilde{P} = \{\overline{x} : x \in P\}$ *is infinite, then the* $\mathcal{S}$*-Dirichlet condition for* P *is valid if and only if* $\widetilde{P}$ *is* $\Delta^*_{\mathcal{S}}$*-strongly dense in* G_R.

Proof. For necessity, use the definition of the topology $\Delta^*_{\mathcal{S}}$ to infer that the $\mathcal{S}$-Dirichlet condition for P implies the $\Delta^*_{\mathcal{S}}$-strong density of $\widetilde{P}$ in G_R without any restriction on $\mathcal{S}$ or R.

Conversely, let $\widetilde{P}$ be $\Delta^*_{\mathcal{S}}$-strongly dense in G_R. Then for each $\mathcal{S}$-coprime coset $x+\mathfrak{A}$, where $\mathfrak{A} \neq \{0\}$, there exists $p \in P \cap (x+\mathfrak{A})$ such that $p \not\sim x$. Since $x+\mathfrak{A}$ is $\mathcal{S}$-coprime, there exist $t_{\mathfrak{M}} \in R$ and $x_{\mathfrak{M}} \in \mathfrak{A}$ such that $t_{\mathfrak{M}} x + x_{\mathfrak{M}} = 1$, where $\mathfrak{M}$ is a maximal ideal belonging to $\mathcal{S}$. Then $x + (x_{\mathfrak{M}})$ is also an $\mathcal{S}$-coprime coset, and Lemma 3.3 shows that so is $x + (x_{\mathfrak{M}})^n$ for every $n = 1, 2, \ldots$. The rest of the proof is analogous to that of [KP, Theorem 19]. ∎

Lemma 4.4 now implies the following consequence.

Corollary 4.6. *Assume that* R *is a residually finite Dedekind domain which has only a finite number of units, and* $\mathcal{S}$ *contains at least one maximal ideal of* R. *If* P *is a subset of* R *such that* $\widetilde{P} = \{\overline{x} : x \in P\}$ *is infinite, then the* $\mathcal{S}$*-Dirichlet condition for* P *is valid if and only if* $\widetilde{P}$ *is* $\Delta^*_{\mathcal{S}}$*-strongly dense in* G_R.

The following statements extend Theorems 20 and 21 of [KP], and they can be proved by adapting the corresponding proofs in [KP].

Theorem 4.7. *Suppose* $\mathcal{S}$ *contains at least one maximal ideal of* R. *If* R *satisfies* (i) *and* (ii), *then the set* P *of irreducible elements in* R *has empty interior in the topology* $\tau^*_{\mathcal{S}}$ *on* R^0, *and similarly for* $\widetilde{P}$ *in* $(G_R, \Delta^*_{\mathcal{S}})$.

(2) It is well-known that the only number fields with finitely many units are $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-D})$, where $D > 0$. Another such class of rings are polynomial rings $F[X]$ over a finite field F.

THEOREM 4.8. *Let P_1 be the set of all the irreducible elements $p \in R$ for which the principal ideal* (p) *belongs to $\mathcal{S}$, and suppose $\widetilde{P_1}$ is infinite.*

(1) *If R satisfies* (i) *and* (iii)*, then $(R^0, \tau_{\mathcal{S}}^*)$ is a Hausdorff space.*
(2) *If R satisfies* (i)–(iii)*, then $(G_R, \Delta_{\mathcal{S}}^*)$ is a Hausdorff space.*

Perhaps more transparent is the following adaptation of the last result.

THEOREM 4.9. *Assume every nonzero proper ideal of R is contained only in finitely many maximal ideals of R, and $\mathcal{S}$ contains infinitely many proper maximal ideals of R. Then both topological spaces $(R, \tau_{\mathcal{S}})$ and $(R^0, \tau_{\mathcal{S}}^*)$ are Hausdorff. Moreover, if R has only a finite number of units, then $(G_R, \Delta_{\mathcal{S}}^*)$ is also Hausdorff.*

Proof. Let $x, y \in R^0$, $x \neq y$. Then there exists a maximal ideal $\mathfrak{M} \in \mathcal{S}$ such that $\mathfrak{M} \not\subset (x)$, $\mathfrak{M} \not\subset (y)$ and $\mathfrak{M} \not\subset (x-y)$. Since $\mathfrak{M}$ is maximal, $x+\mathfrak{M}$ and $y+\mathfrak{M}$ are open in $\tau_{\mathcal{S}}$ and $\tau_{\mathcal{S}}^*$. Since $(x+\mathfrak{M}) \cap (y+\mathfrak{M}) = \emptyset$, the Hausdorff property for $(R, \tau_{\mathcal{S}})$ and $(R^0, \tau_{\mathcal{S}}^*)$ follows.

To prove the second statement, suppose that α, β are two distinct elements from G_R. If $x \in \alpha$ and $y \in \beta$, $x, y \in R^0$, then $x - uy \neq 0$ for every unit u. Let $\mathfrak{M} \in \mathcal{S}$ be such that $\mathfrak{M} \not\subset (x)$, $\mathfrak{M} \not\subset (y)$ and $\mathfrak{M} \not\subset (ux+vy)$ for all units u, v. Then no element of $x + M$ is associated with an element of $y + \mathfrak{M}$: if $x + m_1 = u(y + m_2)$ for some unit u and $m_1, m_2 \in \mathfrak{M}$, then $0 \neq x - uy = um_2 - m_1 \in \mathfrak{M}$, which is impossible due to the choice of $\mathfrak{M}$. ∎

The following statement and its corollaries extend Theorem 25 and Corollaries 25.1 and 25.3 of [KP].

THEOREM 4.10. *Let R be a residually finite Dedekind domain which has only a finite number of units such that the set $\widetilde{\mathcal{P}_1}$ of nonassociated prime elements of R is infinite. Assume $\mathcal{S}$ contains infinitely many prime ideals* (p)*, where $p \in \widetilde{\mathcal{P}_1}$. If P is a subset of R such that $\widetilde{P} = \{\overline{x} : x \in P\}$ is infinite, then the $\mathcal{S}$-Dirichlet condition for P is valid if and only if $\widetilde{P}$ is $\mathcal{S}$-dense in G_R.*

COROLLARY 4.11. *Let R be a residually finite Dedekind domain which has only a finite number of units such that the set $\widetilde{\mathcal{P}_1}$ is infinite. Assume $\mathcal{S}$ contains infinitely many prime ideals* (p)*, where $p \in \widetilde{\mathcal{P}_1}$. Then R satisfies the $\mathcal{S}$-Dirichlet condition for $\mathcal{P}_1$ if and only if $\widetilde{\mathcal{P}_1}$ is $\mathcal{S}$-dense in G_R.*

COROLLARY 4.12. *Let R be a residually finite Dedekind domain which has only a finite number of units such that the set $\widetilde{\mathcal{P}_1}$ is infinite. Assume $\mathcal{S}$ contains infinitely many prime ideals* (p)*, where $p \in \widetilde{\mathcal{P}_1}$. Let $\mathcal{P}_s$, for $s \in \mathbb{N}$, denote the set of products of s nonassociated primes of R. Then $\widetilde{\mathcal{P}_s}$ satisfies* *the $\mathcal{S}$-Dirichlet condition for $\widetilde{\mathcal{P}_s}$ for every $s \in \mathbb{N}$ if and only if $\widetilde{\mathcal{P}}_1$ is $\mathcal{S}$-dense in G_R.*

The following statement is Corollary 25.2 of [KP] (see [KP, p. 146] for references and discussion):

Let $(a, b) = 1$ with $0 \le b < a$. Then $ax + b$ assumes for $x = 0, 1, 2, \dots$ infinitely many prime values if and only if it assumes at least one prime value.

The second proof of this statement on p. 146 of [KP] does not prove this corollary but only a weaker statement as follows.

PROPOSITION 4.13. *Assume that every arithmetic progression $ax + b$, $x = 0, 1, 2, \dots,$ such that $(a, b) = 1$ and $0 \le b < a$ assumes at least one prime value. Then every such $ax + b$ assumes infinitely many prime values.*

Proof. Proceed as on p. 146 of [KP] to conclude that the set P of all primes is strongly dense in the quotient topology $\mathfrak{D}^*$ relative to the topology Δ^* and a cross-section mapping $\rho : G_R \to R^0$ (consult [KP] for more details) and apply Corollary 4.6. ■

5. Closure of primes in the S-coprime topology for $\mathbb{Z}$. The purpose of this section is to generalize some results of [Br] and determine the cluster points of arithmetic progressions in certain coprime topologies.

In this section we assume that $R = \mathbb{Z}$, and identify $\mathcal{P}$ with the set $P \subset \mathbb{N}$ of prime numbers, and a subset $\mathcal{S} \subset \mathcal{P}$ with a subset $S \subset P$. The next proposition is a generalization of Theorem 4.2 of [Br].

PROPOSITION 5.1. *The closure $\overline{P}$ of the set P in the topology τ_S is $P \cup \{\pm \prod_{p_i \in S} p_i^{n_i} : n_i \ge 0\}$.*

Proof. Let a be a composite number divisible by a prime $p \notin S$ and suppose $p^l > |a|$. Then the coset $a + (p^l)$ is S-coprime and does not intersect P because any number $a + p^l z = p(a/p + p^{l-1} z)$ is composite.

On the other hand, if $a = \pm \prod_{p_i \in S} p_i^{n_i}$ and the coset $a + (x)$ is S-coprime, then x is coprime to a. By the Dirichlet theorem about primes in arithmetic progressions, the set $a + (x)$ contains infinitely many primes. This implies that $a \in \overline{P}$. ■

REMARK 5.2. Let Δ_S^* be the quotient topology on $\mathbb{N}$ induced by τ_S on $\mathbb{Z}$. Then the closure of P with respect to Δ_S^* is $\overline{P} = P \cup \{\prod_{p_i \in S} p_i^{n_i} : n_i \ge 0\}$.

Now we turn our attention to infinite sets $S \subset P$ related to arithmetic progressions.

PROPOSITION 5.3. *Let S be a set of primes in an arithmetic progression $\{b + cn\}_{n=1}^{\infty}$, where $(b, c) = 1$ and $c \ge 1$. Then the cluster points of S in the* *topology τ_S are precisely the numbers $a = \pm\prod_{p_i\in S} p_i^{n_i}$ for $n_i \ge 0$ satisfying the following property:*

(∗) *If a prime power p^t, $t \ge 1$, divides c but p does not divide a, then $a \equiv b \bmod p^t$.*

Proof. Assume $a = \pm\prod_{p_i\in S} p_i^{n_i}$ does not satisfy (∗). Then there is p^t that divides c, p does not divide a and $a \not\equiv b \pmod{p^t}$. The coset $a + (p^t)$ is invertible (hence S-coprime), but its intersection with the arithmetic progression $\{b + cn\}_{n=1}^{\infty}$ is empty since $b + cn \equiv b \pmod{p^t} \not\equiv a \pmod{p^t}$. Thus a is not a cluster point of S.

Assume now that $a = \pm\prod_{p_i\in S} p_i^{n_i}$ satisfies (∗) and $a + (x)$, where $x \ge 1$, is S-coprime. Then x is coprime to a and $a \equiv b \pmod A$, where

$$A = \prod_{\substack{q_j^{t_j} \| c \\ q_j \nmid a}} q_j^{t_j}$$

and q_j are primes. Since x is coprime to a, we find that (c, x) divides A. Then $(c/A, x) = 1$ and $(c/A, A) = 1$, which implies $(c/A, xA) = 1$.

The cosets $a + (xA)$ and $b + (c/A)$ are invertible, and their intersection is nonempty and of the form $d + (xc)$ by the Chinese remainder theorem. By Lemma 3.2, $d + (xc)$ is an invertible coset.

The Dirichlet theorem implies that there are infinitely many primes in the arithmetic progression $\{d+n(xc)\}_{n=1}^{\infty}$. Since $d+(xc) \subset a+(A) = b+(A)$, $d + (xc) \subset b + (c/A)$ and $(c/A, A) = 1$, we infer that $d + (xc) \subset b + (c)$. Since $\{d + n(xc)\}_{n=1}^{\infty}$ is contained in $a + (x)$, and it contains infinitely many elements of S, we conclude that a is a cluster point of S.

Finally, if a is divisible by a prime $p \notin S$, then $a + (p)$ is S-coprime and does not intersect S, showing that a is not a cluster point of S. ∎

Proposition 5.4. *Let S be a set of primes in an arithmetic progression $\{b + cn\}_{n=1}^{\infty}$, where $(b, c) = 1$ and $c \ge 1$. Then the cluster points of $P \setminus S$ in the topology $\tau_{P\setminus S}$ are precisely the numbers $a = \pm\prod_{p_i\in P\setminus S} p_i^{n_i}$ for $n_i \ge 0$ such that either $a \not\equiv b \pmod A$ or $c > 2A$, where $A = \prod_{q_j^{t_j}\|c,\, q_j\nmid a} q_j^{t_j}$ and the q_j are primes.*

Proof. If a is divisible by a prime $p \in S$, then $a + (p)$ is $(P \setminus S)$-coprime and does not intersect $P \setminus S$, showing that a is not a cluster point of $P \setminus S$.

Let $a = \pm\prod_{p_i\in P\setminus S} p_i^{n_i}$ and assume first that a does not satisfy (∗), that is, there is a prime p that does not divide a, p^t divides c for some $t \ge 1$ and $a \not\equiv b \pmod{p^t}$. If $a + (x)$, where $x \ge 1$, is $(P \setminus S)$-coprime, then $(x, a) = 1$, hence $a + (x)$ is invertible. The coset $a + (xp^t)$ is invertible (hence $(P \setminus S)$-coprime) and it contains infinitely many primes of the arithmetic progression $\{a + xp^t n\}_{n=1}^{\infty}$ by the Dirichlet theorem. Since $b + cn \equiv b$

(mod p^t) $\not\equiv a \equiv a + xp^tn$ (mod p^t), none of these primes belongs to S. Therefore $a + (xp^t) \subset a + (x)$ contains infinitely many elements from $P \setminus S$, and so a is a cluster point of $P \setminus S$.

Next, let $a = \pm \prod_{p_i \in P\setminus S} p_i^{n_i}$ and assume that $a \equiv b$ (mod A).

If $c = A$, then $(a, c) = 1$ and $a \equiv b$ (mod c). Therefore the coset $a + (c)$ is invertible (hence $(P \setminus S)$-coprime) and $a + (c)$ contains only finitely many elements of $P \setminus S$. There is a large enough x such that $(x, a) = 1$ and the invertible coset $a + (xc)$ contains no elements of $P \setminus S$ except for a itself (if $a \in P \setminus S$). Therefore such an element a is not a cluster point of $P \setminus S$.

If $c = 2A$, then $(a, c) = 2$ and $a \equiv b$ (mod $c/2$). Every prime (possibly except 2) in the invertible (hence $(P \setminus S)$-coprime) coset $a + (c/2)$ belongs to $a + c/2 + (c)$. Since $(b, c) = 1$, we must have $a + c/2 \equiv b$ (mod c). Therefore there are only finitely many elements of $P \setminus S$ in $a + (c/2)$. Taking x suitably large such that $(x, a) = 1$ we obtain an invertible coset $a + (cx/2)$ that contains no elements of $P \setminus S$ except for a itself. Thus a is not a cluster point of $P \setminus S$.

Assume now that $c/A = D > 2$ and $a + (x)$, where $x \geq 1$, is $(P \setminus S)$-coprime. Then x is coprime to a, $a \equiv b$ (mod A) and $A = c/D$. Since x is coprime to a, we find that (c, x) divides A. Then $(c/A, x) = 1$ and $(c/A, A) = 1$, which implies $(c/A, xA) = 1$.

The cosets $a + (xA)$ and $b + (D)$ are invertible, and their intersection is a nonempty invertible coset $d + (xc)$ by the Chinese remainder theorem and Lemma 3.2. By the Chebotarev density theorem, the density of primes in the sequence $\{a + xAn\}_{n=1}^{\infty}$ equals $1/\phi(xA)$, where ϕ denotes the Euler function. If a prime from S belongs to $a + (xA)$, then it belongs to $d + (xc)$. But the density of primes in the sequence $\{d + xcn\}_{n=1}^{\infty}$ equals $1/\phi(xc)$, which is smaller than $1/\phi(xA)$ because $c/A = D > 2$. Therefore there are infinitely many primes from $P \setminus S$ in the sequence $\{a + xAn\}_{n=1}^{\infty}$ and we have $a + (xA) \subset a + (x)$. Hence a is a cluster point of $P \setminus S$. ■

6. Infinitude of primes

6.1. A topological property. The nucleus of Furstenberg's topological proof [Fu] of the infinitude of rational primes in $\mathbb{Z}$ is the observation (cf. Remark 2.2) that the set of units is not open in the linear topology. As observed in the text preceding Theorem 14 in [KP], the set of units cannot be open in the invertible cosets topology if we assume that every arithmetic progression $a + b\mathbb{Z}$ with $(a, b) = 1$ contains at least one rational prime.

The next theorem unifies the above mentioned Theorem 14 of [KP] addressing the invertible cosets topology and the known modification of Furstenberg's linear topology for rings R in terms of the $\mathcal{F}$-coprime topology.

Theorem 6.1. *If a subset $\mathcal{M}_{\mathcal{S}}$ of maximal ideals in $\mathcal{S}$ of a ring R is such that*

$$\bigcup_{\mathfrak{M}\in\mathcal{M}_{\mathcal{S}}} (\mathfrak{M}\setminus\{0\})$$

is not closed in $\tau_{\mathcal{F}}$ on R, then $\mathcal{M}_{\mathcal{S}}$ is infinite.

Proof. If $\mathfrak{M}\in\mathcal{M}_{\mathcal{S}}$, then $W=\bigcup_{x\in R\setminus\mathfrak{M}}(x+\mathfrak{M})$ is a union of $\mathcal{S}$-coprime cosets, and thus open in R. Hence $\mathfrak{M}\setminus\{0\}$ is closed for each $\mathfrak{M}\in\mathcal{M}_{\mathcal{S}}$. Since $\bigcup_{\mathfrak{M}\in\mathcal{M}_{\mathcal{S}}}(\mathfrak{M}\setminus\{0\})$ is not closed, it cannot be a finite union of the closed sets $\mathfrak{M}\setminus\{0\}$. Consequently, $\mathcal{M}_{\mathcal{S}}$ must be infinite. ∎

Corollary 6.2. *If there exists a set $\mathcal{M}_{\mathcal{S}}$ satisfying the assumptions of Theorem* 6.1, *then the set of prime ideals in $\mathcal{S}$ is infinite.*

Since the set of units in a commutative ring R with identity is the complement of the union of its maximal ideals, we have:

Corollary 6.3. *If the set U of units is not open in the invertible cosets topology or the linear topology on R, then the set of maximal ideals in R is infinite.*

The proof of the next theorem is analogous to the last one and is omitted.

Theorem 6.4. *Let $\mathcal{M}^{\mathcal{S}}$ be the set of all maximal ideals of R that contain an ideal of $\mathcal{S}$. If the set*

$$\bigcup_{\mathfrak{M}\in\mathcal{M}^{\mathcal{S}}} (\mathfrak{M}\setminus\{0\})$$

is not closed in $\tau_{\mathcal{F}}$ on R^0, then $\mathcal{M}^{\mathcal{S}}$ is infinite.

Analogues of the last two theorems remain true for the induced topology $\tau^*_{\mathcal{F}}$ on $R^0=R\setminus\{0\}$.

6.2. Infinitude of primes in number fields. In this subsection we assume that K is a number field and $R=R_K$ is the ring of algebraic numbers of K, $\mathcal{I}=\mathcal{I}_K$ is the set of all proper ideals of R, and $\mathcal{P}=\mathcal{P}_K$ is the set of all prime ideals of R.

As was observed in Remark 2.2, every open set in R_K must be infinite, and this simple observation was used to extend Furstenberg's result to derive the infinitude of prime ideals in number fields with a finite set of units. However, the assumption of the finiteness of the group U is not necessary in the case of number fields.

Proposition 6.5. *For any number field K, the set U_K of units is not open in the invertible cosets topology on R_K. Consequently, the set of prime ideals in R_K is infinite.*

Proof. If U_K is open in the invertible cosets topology on R_K, then it contains a coset $1+\mathfrak{A}$ for some ideal $\mathfrak{A}\neq R_K$ of R_K. For $a\in\mathfrak{A}$, if

$a \neq 0$, then $A = \mathrm{Norm}_{K/\mathbb{Q}}(a)$ is an integer different from 0, -1 and 1. Then $1 + 2A \in 1 + \mathfrak{A}$ is an integer different from 1 and -1, hence is not a unit, which is a contradiction. ■

Acknowledgments. The second author was supported by the Grant Agency of the Czech Republic, Grant # P201/12/2351 and by the strategic development financing RVO 67985807.

REFERENCES

[Au] K. E. Aubert, *On the ideal theory of commutative semi-groups*, Math. Scand. 1 (1953), 39–54.

[Br] K. A. Broughan, *Adic topologies for the rational numbers*, Canad. J. Math. 55 (2003), 711–723.

[Co] R. C. Courter, *Rings all of whose factor rings are semi-prime*, Canad. Math. Bull. 12 (1969), 417–426.

[Fu] H. Furstenberg, *On the infinitude of primes*, Amer. Math. Monthly 62 (1955), 353.

[Gi1] R. W. Gilmer, Jr., *Integral domains which are almost Dedekind*, Proc. Amer. Math. Soc. 15 (1964), 813–818.

[Gi2] R. W. Gilmer, *Multiplicative Ideal Theory*, Queen's Univ., Kingston, Ont., 1968.

[Gi3] R. Gilmer, *An existence theorem for non-noetherian rings*, Amer. Math. Monthly 77 (1970), 621–623.

[Go1] S. W. Golomb, *A connected topology for the integers*, Amer. Math. Monthly 66 (1959), 663–665.

[Go2] S. W. Golomb, *Arithmetica topologica*, in: General Topology and its Relations to Modern Analysis and Algebra (Praha, 1961), J. Novák (ed.), Publ. House Czechoslovak Acad. Sci., Praha, 1962, 179–186.

[Ha] S. Hanly, *Solution of E 1218: A proposition equivalent to Dirichlet's theorem*, Amer. Math. Monthly 64 (1957), 742.

[JKL] Y. Ch. Jeon, N. K. Kim and Y. Lee, *On fully idempotent rings*, Bull. Korean Math. Soc. 47 (2010), 715–726.

[KP] J. Knopfmacher and Š. Porubský, *Topologies related to arithmetical properties of integral domains*, Expo. Math. 15 (1997), 131–148.

[MP] F. Marko and Š. Porubský, *Density and the Dirichlet condition*, Int. J. Number Theory 8 (2012), 823–830.

[Po] Š. Porubský, *Arithmetically related ideal topologies and the infinitude of primes*, Quaest. Math. 24 (2001), 373–391.

[Sc] I. Schur, *Über die Existenz unendlich vieler Primzahlen in einigen speziellen arithmetischen Progressionen*, S.-B. Berlin Math. Ges. 11 (1912), 40–50.

[Si1] W. Sierpiński, *Theory of Numbers*, 3rd ed., Monografie Mat. 19, Warszawa, 1950 (in Polish).

[Si2] W. Sierpiński, *Elementary Theory of Numbers*, PWN–Polish Sci. Publ., Warszawa, 1964.

[Si3] W. Sierpiński, *Elementary Theory of Numbers* (ed. by A. Schinzel), North-Holland Math. Library 31, North-Holland and PWN–Polish Sci. Publ., Amsterdam–Warszawa, 1988.

[Sp] R. Spira, *Problem E 1218*, Amer. Math. Monthly 63 (1956), 342.

[Wo] J. Wójcik, *A refinement of a theorem of Schur on primes in arithmetic progressions*, Acta Arith. 11 (1966), 433–436.

[Ze] D. Zeitlin, *Solution of E 1218: A proposition equivalent to Dirichlet's theorem*, Amer. Math. Monthly 64 (1957), 46.

František Marko
Pennsylvania State University
76 University Drive
Hazleton, PA 18202, U.S.A.
E-mail: fxm13@psu.edu

Štefan Porubský
Institute of Computer Science
Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 2
182 07 Praha 8, Czech Republic
E-mail: sporubsky@hotmail.com

Received 7 November 2013;
revised 21 January 2015 (6067)