COLLOQUIUM MATHEMATICUM

VOL. 109 2007 NO. 2

A CONVOLUTION PROPERTY OF SOME MEASURES WITH SELF-SIMILAR FRACTAL SUPPORT

BY

DENISE SZECSEI (Daytona Beach, FL)

Abstract. We define a class of measures having the following properties:

(1) the measures are supported on self-similar fractal subsets of the unit cube $I^M = [0,1)^M$, with 0 and 1 identified as necessary;
(2) the measures are singular with respect to normalized Lebesgue measure m on I^M;
(3) the measures have the convolution property that $\mu * L^p \subseteq L^{p+\varepsilon}$ for some $\varepsilon = \varepsilon(p) > 0$ and all $p \in (1,\infty)$.

We will show that if $(1/p,1/q)$ lies in the triangle with vertices $(0,0)$, $(1,1)$ and $(1/2,1/3)$, then $\mu * L^p \subseteq L^q$ for any measure μ in our class.

1. Introduction. Let T denote the circle group $\mathbb{R}/\mathbb{Z}$ and, for $1 \le p < \infty$, let L^p denote the usual Lebesgue space formed with respect to normalized Lebesgue measure m on T. While every complex Borel measure μ on T acts as a convolution operator on any L^p-space: $\mu * L^p \subseteq L^p$, there are also probability measures μ on T which are singular with respect to m and have the property that for each $p \in (1,\infty)$, $\mu * L^p \subseteq L^{p+\varepsilon}$ for some $\varepsilon = \varepsilon(p) > 0$. An example of such a measure, as well as a discussion of this phenomenon, can be found in [4]. The Cantor–Lebesgue measure is a singular measure on the circle group $\mathbb{R}/\mathbb{Z}$, and its support is the Cantor set, which is a self-similar fractal subset of $\mathbb{R}$. Oberlin [3] showed that for each $p \in (1,\infty)$ there is an $\varepsilon > 0$ for which the Cantor–Lebesgue measure has the convolution property that $\|\lambda * f\|_{L^{p+\varepsilon}} \le \|f\|_{L^p}$. We will generalize this result by defining a class of measures having the following properties:

(1) the measures are supported on self-similar fractal subsets of the unit cube $I^M = [0,1)^M$, with 0 and 1 identified as necessary;
(2) the measures are singular with respect to normalized Lebesgue measure m on I^M;
(3) the measures have the convolution property that $\mu * L^p \subseteq L^{p+\varepsilon}$ for some $\varepsilon = \varepsilon(p) > 0$ and all $p \in (1,\infty)$.

2000 *Mathematics Subject Classification*: Primary 43A05; Secondary 43A15.
Key words and phrases: singular measures, convolution.

We will show that if $(1/p, 1/q)$ lies in the triangle with vertices $(0, 0)$, $(1, 1)$ and $(1/2, 1/3)$, then $\mu * L^p \subseteq L^q$ for any measure μ in our class.

This paper is organized as follows: §2 introduces our class of sets and measures, while §3 is concerned with their convolution properties.

2. The class $\Im$ of self-similar fractal sets. Let I^M denote the unit cube in $\mathbb{R}^M$ viewed as an abelian group with binary operation componentwise addition modulo 1. Fix $0 < r < 1$ and distinct $x_0, x_1, \ldots, x_n \in I^M$, where $\{x_0, x_1, \ldots, x_n\}$ forms a subgroup of I^M. Denote this subgroup G_1. We will be dealing with certain iterated function systems $(f_0, f_1, \ldots, f_n)$ on I^M where f_i will have the form $f_i = rx + x_i$. A discussion of iterated function systems can be found in [1] and [2]. This type of iterated function system realizes the ratio list $(r, \ldots, r)$. Because of the identification of the edges of the M-dimensional torus, there may be some confusion regarding the interpretation of "+". If we consider I^M as a subset of $\mathbb{R}^M$, where "+" denotes addition inherited from $\mathbb{R}^M$, then each f_i is a similarity, and we can obtain the invariant set for these iterated function systems ([1], [2]). When we generate the invariant set using the sets G_1 and S_1, as described below, we will identify the edges of the M-dimensional torus, and "+" will denote addition modulo 1, so that we remain in the group G_n.

Let $S_1 = \{x_1, \ldots, x_n\}$ and consider the iterated function system $(f_1, \ldots, f_n)$ realizing the ratio list $(r, \ldots, r)$. Write S for the invariant set of this iterated function system. We will define two sequences of sets, $\{S_N\}$ and $\{G_N\}$, in similar fashions. Let

$$S_N = \bigcup_{k=1}^{n} f_k(S_{N-1}) \doteq \bigcup_{k=1}^{n} (rS_{N-1} + x_k) = \bigcup_{k=1}^{n} (S_{N-1} + r^{N-1}x_k),$$

$$G_N = \bigcup_{k=0}^{n} f_k(G_{N-1}) \doteq \bigcup_{k=0}^{n} (rG_{N-1} + x_k) = \bigcup_{k=0}^{n} (G_{N-1} + r^{N-1}x_k)$$

for $N \geq 2$. Since S_1 and G_1 are compact sets, the invariant sets S and G for their respective iterated function systems can be generated from S_1 and G_1.

We will say that $S \in \Im$ if the following three conditions hold:

- $0 \in S_1$.
- There exists a non-empty bounded open set V in I^M such that $f_i(V) \cap f_j(V) = \emptyset$ for $i \neq j$, and $V \supset \bigcup_{i=0}^{n} f_i(V)$. This condition is referred to as the *open set condition.*
- G_N is the subgroup of I^M generated by S_N, $|S_N| = |S_1|^N$ and $|G_N| = |G_1|^N$.

Examples of fractal sets belonging to $\Im$ include the triadic Cantor set, the Sierpiński gasket and the Sierpiński carpet [2]. For the triadic Cantor set,

$$S_1 = \{0, 2/3\}, \qquad G_1 = \{0, 1/3, 2/3\}.$$

For the Sierpiński gasket,

$$S_1 = \{(0,0), (1/4, 1/2), (3/4, 1/2)\},$$
$$G_1 = \{(0,0), (1/2, 0), (1/4, 1/2), (3/4, 1/2)\}.$$

For the Sierpiński carpet,

$$S_1 = \{(0,0), (1/3,0), (2/3,0), (0,1/3), (2/3,1/3), (0,2/3), (1/3,2/3), (2/3,2/3)\},$$
$$G_1 = \{(0,0), (1/3,0), (2/3,0), (0,1/3), (1/3,1/3), (2/3,1/3), (0,2/3), (1/3,2/3), (2/3,2/3)\}.$$

In general, and roughly, to construct self-similar fractal sets in I^M belonging to $\Im$, begin with a geometric subset of I^M, such as a square, triangle, cube, etc. Divide it evenly into n congruent pieces, each of which has the same geometric shape as the original, and remove one of the pieces. Construct the sets S_1 and G_1 from the vertices of the divided geometric shape. Determine the ratio list from the geometry of the setting, and define the iterated function system using the set S_1 and the ratio list.

The open set condition ensures that the components $f_i(S)$ of S do not overlap "too much". Because $0 \in S_1$, we have $S_1 \subset S_2 \subset \cdots$. The third condition ensures that $\{G_N\}$ is a nested sequence of subgroups of I^M, from which it follows that G is a subgroup of I^M.

Since S_1 is compact, $\{S_N\}$ converges to S in the Hausdorff metric, and hence $\bigcup_{N=1}^{\infty} S_N$ is dense in S. Thus the invariant set S for the iterated function system $(f_1, \ldots, f_n)$ satisfies $S = \overline{\bigcup_{N=1}^{\infty} S_N}$. Similarly, $\bigcup_{N=1}^{\infty} G_N$ is dense in G, and the invariant set G for the iterated function system $(f_0, f_1, \ldots, f_n)$ satisfies $G = \overline{\bigcup_{N=1}^{\infty} G_N}$.

Let $L^p(G_N)$ denote the Lebesgue space formed with respect to normalized counting measure (denoted m_N) on G_N, and let m denote the Haar measure on G. Then m is the weak* limit of the probability measures m_N. The norm in $L^p(G_N)$ will be written as $\|\cdot\|_{p,N}$. Denote by $C(G)$ the space of continuous functions on G. Denote by μ_N the normalized counting measure on S_N, i.e. the probability measure uniformly distributed on S_N. Then $\{\mu_N\}$ is a weak*-Cauchy sequence of measures; we will denote its weak* limit by μ.

3. Convolution properties. Suppose S and G are self-similar fractal sets constructed as above, with $S \in \Im$. We will prove the following convolution theorem:

THEOREM 1. *Let μ be the measure on S as defined above. For each $p \in (1, \infty)$ there is an $\varepsilon > 0$ such that $\|\mu * f\|_{L^{p+\varepsilon}(G)} \le \|f\|_{L^p(G)}$ for all $f \in L^p(G)$.*

The proof of this theorem requires two lemmas, the first of which is stated in a more general setting. Suppose G_1 and G_2 are abelian groups satisfying $G_1 \subset G_2$, $|G_1| = n^J$, $|G_2| = n^{J+1}$, and $G_2 = \bigcup_{j=1}^{n}(x_j + G_1)$. Let S_1 and S_2 be subsets of G_1 and G_2 respectively, satisfying $|S_1| = (n-1)^J$, $|S_2| = (n-1)^{J+1}$, and $S_2 = \bigcup_{j=1}^{n-1}(x_j + S_1)$. Let μ_i denote the normalized counting measure on S_i, and $\|g\|_{p,i}$ denote the L^p norm with respect to the normalized counting measure on G_i.

Lemma 1. *Suppose that the n-point inequality*

$$\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n-1}\sum_{j\neq i}a_j\right)^q\right)^{1/q} \le \left(\frac{1}{n}\sum_{i=1}^{n}a_i^p\right)^{1/p} \tag{1}$$

holds for all positive real numbers $\{a_i\}_{i=1}^n$. If the inequality

$$\left(\frac{1}{n^J}\sum_{x\in G_1}\left|\frac{1}{(n-1)^J}\sum_{t\in S_1}h(x-t)\right|^q\right)^{1/q} \le \left(\frac{1}{n^J}\sum_{x\in G_1}|h(x)|^p\right)^{1/p} \tag{2}$$

holds for all functions $h \in L^p(G_1)$, then the inequality

$$\left(\frac{1}{n^{J+1}}\sum_{x\in G_2}\left|\frac{1}{(n-1)^{J+1}}\sum_{t\in S_2}g(x-t)\right|^q\right)^{1/q} \le \left(\frac{1}{n^{J+1}}\sum_{x\in G_2}|g(x)|^p\right)^{1/p} \tag{3}$$

holds for all functions $g \in L^p(G_2)$.

Lemma 2. *Inequality* (1) *is valid for $q = 3$ and $p = 2$.*

We observe that (2) is just $\|\mu_1 * h\|_{q,1} \le \|h\|_{p,1}$, and (3) is just $\|\mu_2 * g\|_{q,2} \le \|g\|_{p,2}$. Once the two lemmas are proven, an inductive argument will show that $\|\mu_N * ft\|_{L^3(G_N)} \le \|f\|_{L^2(G_N)}$ for all $f \in L^p(G_N)$ and all N. Then if $f \in C(G)$, it follows that $|\mu_N * f| \to |\mu * f|$ uniformly on G, and we have

$$\int |\mu_N * f|^3\, dm_N \to \int |\mu * f|^3\, dm.$$

Since

$$\left[\int |\mu_N * f|^3\, dm_N\right]^{2/3} \le \int |f|^2\, dm_N \quad \text{and} \quad \int |f|^2\, dm_N \to \int |f|^2\, dm,$$

we see that

$$\|\mu * f\|_{L^3(G)} \le \|f\|_{L^2(G)}$$

for all non-negative continuous functions f on G. In addition, we know that

$$\|\mu * f\|_{L^1(G)} \le \|f\|_{L^1(G)}$$

for $f \in L^1(G)$ and

$$\|\mu * f\|_{L^\infty(G)} \le \|f\|_{L^\infty(G)},$$

so application of the Riesz–Thorin theorem will complete the proof of Theorem 1.

Proof of Lemma 1. We begin by using a coset expansion of S_2 and G_2 in terms of S_1 and G_1 to show that

$$\left(\frac{1}{n^{J+1}}\sum_{x\in G_2}\left|\frac{1}{(n-1)^{J+1}}\sum_{t\in S_2} g(x-t)\right|^q\right)^{1/q} = \left(\frac{1}{n}\sum_{i=1}^{n}\left\|\frac{1}{n-1}\sum_{j=1}^{n-1}\mu_i * g(x+(x_i-x_j))\right\|_{q,1,x}^q\right)^{1/q}.$$

We calculate

$$\begin{aligned}
&\left(\frac{1}{n^{J+1}}\sum_{x\in G_2}\left|\frac{1}{(n-1)^{J+1}}\sum_{t\in S_2} g(x-t)\right|^q\right)^{1/q}\\
&= \left(\frac{1}{n^{J+1}}\sum_{x\in G_2}\left|\frac{1}{(n-1)^{J+1}}\sum_{j=1}^{n-1}\sum_{t\in x_j+S_1} g(x-t)\right|^q\right)^{1/q}\\
&= \left(\frac{1}{n^{J+1}}\sum_{i=1}^{n}\sum_{x\in x_i+G_1}\left|\frac{1}{(n-1)^{J+1}}\sum_{j=1}^{n-1}\sum_{t\in S_1} g(x-x_j-t)\right|^q\right)^{1/q}\\
&= \left(\frac{1}{n^{J+1}}\sum_{i=1}^{n}\sum_{x\in G_1}\left|\frac{1}{(n-1)^{J+1}}\sum_{j=1}^{n-1}\sum_{t\in S_1} g(x+x_i-x_j-t)\right|^q\right)^{1/q}\\
&= \left(\frac{1}{n^{J+1}}\sum_{i=1}^{n}\sum_{x\in G_1}\left|\frac{1}{(n-1)^{1}}\sum_{j=1}^{n-1}\frac{1}{(n-1)^{J}}\sum_{t\in S_1} g(x-t+(x_i-x_j))\right|^q\right)^{1/q}\\
&= \left(\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{n^J}\sum_{x\in G_1}\left|\frac{1}{n-1}\sum_{j=1}^{n-1}\mu_1 * g(x+(x_i-x_j))\right|^q\right]\right)^{\{1/q}\\
&= \left[\frac{1}{n}\sum_{i=1}^{n}\left\|\frac{1}{n-1}\sum_{j=1}^{n-1}\mu_1 * g(x+(x_i-x_j))\right\|_{q,1,x}^q\right]^{1/q}.
\end{aligned}$$

Using the triangle inequality and the inductive hypothesis $\|\mu_1 * g\|_{q,1} \leq \|g\|_{p,1}$, we see that

$$\begin{aligned}
&\left[\frac{1}{n}\sum_{i=1}^{n}\left\|\frac{1}{n-1}\sum_{j=1}^{n-1}\mu_1 * g(x+(x_i-x_j))\right\|_{q,1,x}^q\right]^{1/q}\\
&\qquad \leq \left[\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{n-1}\sum_{j=1}^{n-1}\|\mu_1 * g(x+(x_i-x_j))\|_{q,1,x}\right]^q\right]^{1/q}\\
&\qquad = \left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{n-1}\Big[\sum_{j=1}^{n-1}\|g\|_{p,1,(x_i-x_j)+G_1}\Big]^q\right]^{1/q}.
\end{aligned}$$

Now, for fixed i, $\{(x_i - x_j) + G_1\}_{j=1}^{n-1}$ spans all of the cosets of G_1 in G_2 except $(x_i - x_n) + G_1$. And, for fixed k, $\{(x_i - x_k) + G_1\}_{i=1}^{n}$ spans all of the cosets of G_1 in G_2, so by (1),

$$\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{n-1}\left[\sum_{j=1}^{n-1}\|g\|_{p,1,(x_i-x_j)+G_1}\right]^q\right]^{1/q}$$
$$\leq \left[\frac{1}{n}\sum_{i=1}^{n}\|g\|_{p,1,x_i+G_1}^p\right]^{1/p} = \left[\frac{1}{n}\sum_{i=1}^{n} n\|g\|_{p,2,x_i+G_1}^p\right]^{1/p}$$
$$= \left[\sum_{i=1}^{n}\|g\|_{p,2,x_i+G_1}^p\right]^{1/p} = [\|g\|_{p,2}^p]^{1/p} = \|g\|_{p,2}.$$

Proof of Lemma 2. Cubing both sides of

$$\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n-1}\sum_{j\neq i}a_j\right)^3\right)^{1/3} \leq \left(\frac{1}{n}\sum_{i=1}^{n}a_i^2\right)^{1/2}$$

yields

$$\sum_{i=1}^{n}\Big(\sum_{j\neq i}a_j\Big)^3 \leq (n-1)^3 n^{-1/2}\Big(\sum_{i=1}^{n}a_i^2\Big)^{3/2}.$$

Since both sides are homogeneous of degree 3, it is enough to show that the maximum of $\sum_{i=1}^{n}(\sum_{j\neq i}a_j)^3$ subject to the constraint $\sum_{i=1}^{n}a_i^2 = 1$ is $(n-1)^3n^{-1/2}$. By Lagrange's method, the maximum of $\sum_{i=1}^{n}(\sum_{j\neq i}a_j)^3$ subject to the constraint $\sum_{i=1}^{n}(\sum_{j\neq i}a_j)^3 = 1$ occurs when the a_i's satisfy the system of equations

$$\frac{\partial}{\partial a_k}\Big(\sum_{i=1}^{n}\Big(\sum_{j\neq i}a_j\Big)^3\Big) = 2\lambda a_k \quad \text{for } 1 \leq k \leq n. \tag{4}$$

Expanding the left-hand side of (4) yields the following system of equations:

$$\left[a_k^2 + 2\sum_{\substack{j=1\\ j\neq k}}^{n}a_ja_k + (n-2)\sum_{j=1}^{n}a_j^2 + 2(n-3)\sum_{i=1}^{n}\sum_{j>i}^{n}a_ia_j\right] = -2\lambda a_k \tag{5}$$

for $1 \leq k \leq n$, $n \geq 3$. This system of equations is satisfied only when $a_i = a_j$ for $1 \leq i, j \leq n$. We can therefore write $a = a_i$, and given that $\sum_{i=1}^{n}a_i^2 = 1$, we have

$$\sum_{i=1}^{n}\Big(\sum_{\substack{j=1\\ j\neq i}}^{n}a\Big)^3 = (n-1)^3n^{-1/2}.$$

REFERENCES

[1] G. Edgar, *Measure, Topology, and Fractal Geometry*, Springer, New York, 1990.
[2] K. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, Wiley, New York, 1990.
[3] D. Oberlin, *A convolution property of the Cantor–Lebesgue measure*, Colloq. Math. 47 (1982), 113–117.
[4] E. Stein, *Harmonic analysis on* $\mathbb{R}^N$, in: Studies in Harmonic Analysis, MAA Stud. Math. 13, Math. Assoc. Amer., Washington, DC, 1976, 97–135.

Department of Mathematics
Stetson University
P.O. Box 7532
Daytona Beach, FL 32116, U.S.A.
E-mail: szecsei@math.fsu.edu

Received 27 June 2005 (4633)