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GALOIS COVERINGS AND THE CLEBSCH–GORDAN PROBLEM FOR QUIVER REPRESENTATIONS

BY

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Abstract. We study the Clebsch–Gordan problem for quiver representations, i.e. the problem of decomposing the point-wise tensor product of any two representations of a quiver into its indecomposable direct summands. For this purpose we develop results describing the behaviour of the point-wise tensor product under Galois coverings. These are applied to solve the Clebsch–Gordan problem for the double loop quivers with relations $\alpha\beta = \beta\alpha = \alpha^n = \beta^n = 0$. These quivers were originally studied by I. M. Gelfand and V. A. Ponomarev in their investigation of representations of the Lorentz group. We also solve the Clebsch–Gordan problem for all quivers of type $\widetilde{\mathbb{A}}_n$.

1. Introduction. Given any Krull–Schmidt category equipped with a tensor product, one can pose the Clebsch–Gordan problem, i.e. the problem of decomposing the tensor product of any two objects into a direct sum of indecomposables. This problem originates from representation theory of groups. Here we consider it for quiver representations where the tensor product is defined point-wise and arrow-wise.

In this form it arises naturally in the investigation of lattices over curve singularities [3]. For the loop quiver $\widetilde{\mathbb{A}}_0$ it has been studied by Huppert [11] and independently by Martsinkovsky and Vlassov [12]. Previous results by the author deal with the Kronecker quiver [8] and extended Dynkin quivers of type $\widetilde{\mathbb{A}}_n$ (see [10]).

One of the most fundamental problems in representation theory is the classification problem for the indecomposable objects of a Krull–Schmidt category. By solving it we mean finding a list of indecomposable objects such that each isomorphism class of indecomposables is represented exactly once. Assuming that the classification problem is solved one can present a solution to the Clebsch–Gordan problem in the following way: for any pair of elements from the classifying list provide a formula for their decomposition into a direct sum of indecomposables from the classifying list.

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The concept of coverings comes from topology and was introduced in representation theory by P. Gabriel [5], [1]. In some cases it can be used as a tool to solve the classification problem (cf. e.g. [4]).

In the present article we investigate the relationship between Galois coverings and the tensor product of quiver representations. Our main results (Theorem 2 and Corollary 2) allow the reduction of parts of the Clebsch–Gordan problem for the base quiver to the Clebsch–Gordan problem for the covering quiver, provided that a classification of indecomposables is given in terms of the covering.

We apply these results to solve the Clebsch–Gordan problem for the double loop quivers with relations $\alpha\beta = \beta\alpha = \alpha^n = \beta^n = 0$ and quivers of type $\widetilde{\mathbb{A}}_n$.

2. Preliminaries. We recall a few of the basic notions associated with linear categories and quivers, some of which can be found in [6]. Let $\Bbbk$ be a field. A category $\mathcal{C}$ is called *linear* if all its morphism sets are endowed with a $\Bbbk$-linear structure and all its composition maps are $\Bbbk$-bilinear. For linear categories $\mathcal{A}$ and $\mathcal{B}$ a functor $F : \mathcal{A} \to \mathcal{B}$ is called *linear* if the maps $\mathcal{A}(x, y) \to \mathcal{B}(F(x), F(y))$, $\alpha \mapsto F(\alpha)$, are $\Bbbk$-linear.

An *ideal* $\mathcal{I}$ of a linear category $\mathcal{C}$ is a family of subspaces $\mathcal{I}(x, y) \subset \mathcal{C}(x, y)$ such that $\beta\mathcal{I}(x, y)\alpha \subset \mathcal{I}(w, z)$ for all $\beta \in \mathcal{C}(y, z)$ and $\alpha \in \mathcal{C}(w, x)$. For an ideal $\mathcal{I}$ of a category $\mathcal{C}$ we define the quotient category $\mathcal{C}/\mathcal{I}$ by $\mathrm{Ob}(\mathcal{C}/\mathcal{I}) = \mathrm{Ob}\,\mathcal{C}$ and $(\mathcal{C}/\mathcal{I})(x, y) = \mathcal{C}(x, y)/\mathcal{I}(x, y)$. The composition of morphisms in $\mathcal{C}/\mathcal{I}$ is the residue class of the composition of chosen representatives in $\mathcal{C}$.

A *quiver* Q is a quadruple (Q_0, Q_1, t, h), where Q_0 is the set of vertices and Q_1 the set of arrows. The maps $t, h : Q_1 \to Q_0$ map an arrow α to its tail $t\alpha$ and head $h\alpha$ respectively. We write $x \xrightarrow{\alpha} y$ to state that $t\alpha = x$ and $h\alpha = y$. A *path* from $x \in Q_0$ to $y \in Q_0$ of length $d \geq 1$ is a sequence of arrows $\alpha_d \ldots \alpha_1$ such that $t\alpha_1 = x$, $h\alpha_i = t\alpha_{i+1}$ for all $i = 1, \ldots, d-1$ and $h\alpha_d = y$. For each vertex $x \in Q_0$ there is moreover a path e_x of length zero from x to x. With each quiver Q we associate its *path category* $\widehat{Q}$ whose set of objects is Q_0 and whose morphism sets $\widehat{Q}(x, y)$ consist of all paths from x to y. Composition of paths is given by concatenation. We also consider the linearized path category $\Bbbk Q$, which has the same objects as $\widehat{Q}$ and whose morphism sets $\Bbbk Q(x, y)$ are the vector spaces having $\widehat{Q}(x, y)$ as basis. The composition maps in this category are the bilinear extensions of the composition maps in $\widehat{Q}$.

A *subquiver* of a quiver Q is a quiver $Q' = (Q'_0, Q'_1, t', h')$ such that $Q'_0 \subset Q_0$, $Q'_1 \subset Q_1$ and $t'(\alpha) = t(\alpha)$, $h'(\alpha) = h(\alpha)$ for all $\alpha \in Q'_1$. Let Q' and Q'' be subquivers of Q. Their union $Q' \cup Q''$ and intersection $Q' \cap Q''$

respectively are the subquivers of Q determined by

$$(Q' \cup Q'')_i = Q'_i \cup Q''_i \quad \text{for } i \in \{0,1\},$$
$$(Q' \cap Q'')_i = Q'_i \cap Q''_i \quad \text{for } i \in \{0,1\}.$$

We say that Q' and Q'' are *disjoint* if $(Q' \cap Q'')_0$ is empty. In that case we write $Q' \dot{\cup} Q''$ for the union of Q' and Q''.

Let Q be a quiver. An ideal $\mathcal{I}$ of $\Bbbk Q$ is called *semimonomial* if it is generated by elements of the form α or $\alpha - \beta$, where $\alpha, \beta \in \widehat{Q}(x,y)$.

Let Γ be a small linear category. A *Γ-module* is a linear functor

$$m : \Gamma \to \operatorname{Mod} \Bbbk$$

where $\operatorname{Mod} \Bbbk$ denotes the category of all $\Bbbk$-linear spaces. A morphism from a Γ-module m to a Γ-module n is defined to be a natural transformation

$$\phi : m \to n.$$

We denote by $\operatorname{Mod} \Gamma$ the category of all Γ-modules and by $\operatorname{mod} \Gamma$ the full subcategory formed by all finite-dimensional modules, i.e. modules m such that $\bigoplus_{x \in \Gamma} m(x)$ is finite-dimensional.

If $\Gamma = \Bbbk Q$ for some quiver Q, then a Γ-module m is uniquely determined by the choice of vector spaces $m(x)$ for all $x \in Q_0$ and linear maps $m(\alpha)$ for all $\alpha \in Q_1$. The collection of vector spaces $m(x)$ and linear maps $m(\alpha)$ is called a *representation* of Q. If $\mathcal{I}$ is an ideal of $\Bbbk Q$, then the category $\operatorname{mod}(\Bbbk Q/\mathcal{I})$ is identified with the full subcategory of $\operatorname{mod} \Bbbk Q$ formed by all modules m satisfying $m(\alpha) = 0$ for each $\alpha \in \mathcal{I}$.

For any two modules $m, n \in \operatorname{Mod} \Gamma$ we define their direct sum $m \oplus n$ by

$$(m \oplus n)(x) = m(x) \oplus n(x) \quad \text{for each } x \in \operatorname{Ob} \Gamma,$$
$$(m \oplus n)(\alpha) = m(\alpha) \oplus n(\alpha) \quad \text{for each } \alpha \in \Gamma(x,y).$$

A module $m \in \operatorname{Mod} \Gamma$ is called *indecomposable* if $m \xrightarrow{\sim} m' \oplus m''$ implies $m' = 0$ or $m'' = 0$ but not both. The full subcategories of $\operatorname{Mod} \Gamma$ and $\operatorname{mod} \Gamma$ formed by all indecomposable modules are denoted by $\operatorname{Ind} \Gamma$ and $\operatorname{ind} \Gamma$ respectively.

For any linear functor $F : \Gamma \to \Lambda$ of small linear categories, we define the associated pull-up functor

$$F^* : \operatorname{Mod} \Lambda \to \operatorname{Mod} \Gamma$$

by $F^* m = m \circ F$ for each Λ-module m and $(F^*(\phi))_x = \phi_{F(x)}$ for each morphism ϕ of Λ-modules.

For $\Gamma = \Bbbk Q/\mathcal{I}$, where Q is a quiver and $\mathcal{I}$ a semimonomial ideal, we define the tensor product $m \otimes n$ of Γ-modules by

$$(m \otimes n)(x) = m(x) \otimes n(x) \quad \text{for each } x \in \operatorname{Ob} \Gamma,$$
$$(m \otimes n)(\alpha) = m(\alpha) \otimes n(\alpha) \quad \text{for each } \alpha \in Q_1.$$

Since the tensor product of linear maps respects compositions we see that $(m \otimes n)(\alpha) = m(\alpha) \otimes n(\alpha)$ for every path α in Q. Moreover, the tensor product respects the zero morphism in the sense that $(m \otimes n)(0) = 0 = m(0) \otimes n(0)$. It follows that if α, α' are paths in Q or zero morphisms such that $m(\alpha) = m(\alpha')$ and $n(\alpha) = n(\alpha')$, then $(m \otimes n)(\alpha) = (m \otimes n)(\alpha')$. Since the ideal $\mathcal{I}$ is semimonomial we deduce that $m \otimes n$ is a well-defined Γ-module. The canonical isomorphism $m(x) \otimes n(x) \xrightarrow{\sim} n(x) \otimes m(x)$ defines an isomorphism of Γ-modules $m \otimes n \xrightarrow{\sim} n \otimes m$.

The *Clebsch–Gordan problem* for $\operatorname{mod} \Gamma$ is the problem of decomposing $m \otimes n$ into a direct sum of indecomposable modules, for all $m, n \in \operatorname{mod} \Gamma$. Since the tensor product commutes with direct sums, we may assume without loss of generality that $m, n \in \operatorname{ind} \Gamma$.

We recall from [9] the notion of characteristic representations. Let Q' be a subquiver of a quiver Q. The *characteristic representation* associated with Q' is the $\Bbbk Q$-module $\chi_{Q'}$ defined by

$$\chi_{Q'}(x) = \begin{cases} \Bbbk & \text{if } x \in Q'_0, \\ 0 & \text{if } x \notin Q'_0, \end{cases} \qquad \chi_{Q'}(\alpha) = \begin{cases} 1_{\Bbbk} & \text{if } \alpha \in Q'_1, \\ 0 & \text{if } \alpha \notin Q'_1. \end{cases}$$

The canonical vector space isomorphism $\Bbbk \otimes \Bbbk \xrightarrow{\sim} \Bbbk$ gives rise to the isomorphism of representations

$$\chi_{Q'} \otimes \chi_{Q''} \xrightarrow{\sim} \chi_{Q' \cap Q''}. \tag{1}$$

3. Galois coverings

3.1. *Generalities.* Let us briefly recall some basic facts about the concept of Galois coverings, as presented in [5] and [1]. A linear functor $F : \Gamma \to \Lambda$ between linear categories is called a *covering functor* if the induced linear maps

$$\bigoplus_{y' \in F^{-1}(b)} \Gamma(x, y') \to \Lambda(a, b) \quad \text{and} \quad \bigoplus_{x' \in F^{-1}(a)} \Gamma(x', y) \to \Lambda(a, b)$$

are bijective for all $a, b \in \operatorname{Ob} \Lambda$ and $x \in F^{-1}(a)$, $y \in F^{-1}(b)$.

Let G be a group and Γ a small linear category. A *G-action* on Γ is a group morphism $G \to \operatorname{Aut} \Gamma$, $g \mapsto F_g$, such that all F_g are linear. It defines a G-action on $\operatorname{Ob} \Gamma$ by $gx = F_g(x)$ for all $x \in \operatorname{Ob} \Gamma$. It is called *free* if the stabilizer G_x is trivial for all $x \in \operatorname{Ob} \Gamma$, and *locally bounded* if for all $x, y \in \operatorname{Ob} \Gamma$ the identities $\Gamma(gx, y) = \Gamma(x, gy) = 0$ hold for all but finitely many $g \in G$. For any $m \in \operatorname{mod} \Gamma$ and $g \in G$ we denote by ${}^g m$ the translated module $m \circ F_{g^{-1}}$. To simplify notation we identify F_g with g. If Λ is a linear subcategory of Γ, then $g\Lambda$ is the subcategory of Γ defined by $\operatorname{Ob}(g\Lambda) = g(\operatorname{Ob} \Lambda)$ and $(g\Lambda)(x, y) = g(\Lambda(x, y))$ for all $g \in G$.

Following [6, p. 9], a *spectroid* is a small linear category Γ with the following properties: all endomorphism algebras are local, different objects are non-isomorphic and all morphism spaces are finite-dimensional.

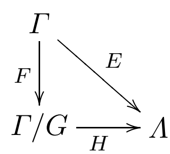
Let G be a group acting on a spectroid Γ. We assume that this action is free and locally bounded. Then the quotient category Γ/G is defined as follows. The objects of Γ/G are the G-orbits of objects of Γ. A morphism $\alpha \in (\Gamma/G)(a,b)$ is a double sequence $\alpha = (\alpha_{yx}) \in \prod_{x\in a, y\in b} \Gamma(x,y)$ such that $g(\alpha_{yx}) = \alpha_{gy,gx}$ for all $g \in G$, $x \in a$ and $y \in b$. If $\alpha \in (\Gamma/G)(a,b)$ and $\beta \in (\Gamma/G)(b,c)$, then the composition $\beta\alpha$ is defined by $(\beta\alpha)_{zx} = \sum_{y\in b} \beta_{zy}\alpha_{yx}$. All but finitely many terms in the sum are zero since the G-action is locally bounded. The linear projection functor

$$F : \Gamma \to \Gamma/G$$

sends an object x to its orbit and a morphism $\alpha \in \Gamma(x,y)$ to the double sequence $F(\alpha)$ defined by

$$F(\alpha)_{hy,gx} = \begin{cases} g\alpha & \text{if } g = h, \\ 0 & \text{if } g \neq h. \end{cases}$$

It is shown in [5] that F is a covering functor such that $Fg = F$ for all $g \in G$. Moreover, it has the universal property that if Λ is a spectroid and a linear functor $E : \Gamma \to \Lambda$ satisfies $Eg = E$ for all $g \in G$, then there is a unique linear functor $H : \Gamma/G \to \Lambda$ such that the diagram



commutes. If in addition E is a covering functor, surjective on the objects of Λ and such that G acts transitively on $E^{-1}(x)$ for all $x \in \operatorname{Ob}\Lambda$, then H is an isomorphism. In this case E is called a *Galois covering*.

If a group G acts on a small linear category Γ we say that an ideal $\mathcal{I}$ of Γ is *G-invariant* if $g\mathcal{I}(x,y) \subset \mathcal{I}(gx,gy)$ for all $g \in G$ and all $x, y \in \operatorname{Ob}\Gamma$. In this case we get an induced G-action on $\Gamma/\mathcal{I}$ defined by $g(\alpha + \mathcal{I}(x,y)) = g\alpha + \mathcal{I}(gx,gy)$. We proceed by investigating the case $\Gamma = \Bbbk Q/\mathcal{I}$ in more detail. Our goal is to find a canonical Galois covering $\Gamma \to \Lambda$ where Λ is the linear path category of a quiver modulo some ideal.

We say that a group G *acts on a quiver* Q if it acts on Q_0 and on Q_1 in such a way that $t(g\alpha) = gt(\alpha)$ and $h(g\alpha) = gh(\alpha)$ for all $g \in G$ and $\alpha \in Q_1$. If Q' is a subquiver of Q, then gQ' denotes the subquiver determined by $(gQ)_i = g(Q_i)$ for $i \in \{0,1\}$. The orbit quiver Q/G is defined by $(Q/G)_0 = Q_0/G$, $(Q/G)_1 = Q_1/G$, $t(G\alpha) = G(t\alpha)$ and $h(G\alpha) = G(h\alpha)$.

Let P be the linear functor

$$P : \Bbbk Q \to \Bbbk(Q/G)$$

which sends vertices and arrows to their respective orbits. For any ideal $\mathcal{I}$ of $\Bbbk Q$ we define the ideal $\mathcal{I}/G$ of $\Bbbk(Q/G)$ by

$$(\mathcal{I}/G)(X,Y) = \sum_{(x,y)\in X\times Y} P(\mathcal{I}(x,y)).$$

Let $\overline{P}$ be the functor

$$\overline{P} : \Bbbk Q/\mathcal{I} \to \Bbbk(Q/G)/(\mathcal{I}/G)$$

induced by P. If $\mathcal{I}$ is semimonomial then so is $\mathcal{I}/G$.

If a group G acts on a quiver Q, then it induces a G-action on $\Bbbk Q$ by $g(\beta\alpha) = (g\beta)(g\alpha)$ for all paths α, β. We observe that $Pg = P$ since $(Pg)(x) = Gx = P(x)$ for each vertex $x \in Q_0$, and $(Pg)(\alpha) = G\alpha = P(\alpha)$ for each arrow $\alpha \in Q_1$. If $\mathcal{I}$ is a G-invariant ideal of $\Bbbk Q$, then $\overline{P}g = \overline{P}$. We proceed to show that, under suitable assumptions, $\overline{P}$ is a covering functor.

LEMMA 1. *Let Q be a quiver and G a group acting on Q. For all $x, y \in Q_0$ and $\xi \in \widehat{(Q/G)}(Gx,Gy)$, there are $g \in G$ and $\alpha \in \widehat{Q}(x,gy)$ such that $P(\alpha) = \xi$.*

Proof. The proof proceeds by induction on d, the length of ξ. If $d = 0$ then $\xi = e_{Gx}$ and $Gx = Gy$. Choose $g \in G$ such that $gy = x$ and $\alpha = e_x \in \widehat{Q}(x,x) = \widehat{Q}(x,gy)$. Assume that $d > 0$. Then $\xi = G\beta\xi'$ for some arrow $z \xrightarrow{\beta} g_1y$ in Q and some path $\xi' \in \widehat{(Q/G)}(Gx,Gz)$. By induction hypothesis there are $g_2 \in G$ and $\alpha' \in \widehat{Q}(x,g_2z)$ such that $P(\alpha') = \xi'$. Choose $g = g_2g_1$ and $\alpha = (g_2\beta)\alpha'$. Then $P(\alpha) = G\beta P(\alpha') = \xi$. ∎

LEMMA 2. *Let Q be a quiver and G a group acting freely on Q. Let $x, y \in Q_0$ and $g \in G$. Then $P(\alpha) = P(\beta)$ implies $\alpha = \beta$ for all $\alpha \in \widehat{Q}(x,y)$, $\beta \in \widehat{Q}(x,gy)$.*

Proof. Since the functor P sends arrows to arrows, it sends paths of length d to paths of length d for all $d \in \mathbb{N}$. We show that if $P(\alpha) = P(\beta)$ then $\alpha = \beta$ by induction on d, the length of α, which coincides with the length of β. If $d = 0$ then $\alpha = e_x = \beta$. Assume that $d > 0$. Then $\alpha = \alpha'\alpha_1$ for some arrow α_1 from x to z and some path $\alpha' \in \widehat{Q}(z,y)$. Similarly, $\beta = \beta'\beta_1$ for some arrow β_1 from x to z' and some path $\beta' \in \widehat{Q}(z',gy)$. Since $P(\alpha) = P(\beta)$ we have $P(\alpha_1) = P(\beta_1)$ and $P(\alpha') = P(\beta')$. Hence there is $h \in G$ such that $h\alpha_1 = \beta_1$ and thus $hx = x$. Since the G-action is free, $h = 1$ and $\alpha_1 = \beta_1$. It follows that $z = z'$, and by induction that $\alpha' = \beta'$. Hence $\alpha = \alpha'\alpha_1 = \beta'\beta_1 = \beta$. ∎

THEOREM 1. *Let Q be a quiver and G a group acting freely on Q. Let $\mathcal{I}$ be a G-invariant ideal of $\Bbbk Q$. Then*

$$\overline{P} : \Bbbk Q/\mathcal{I} \to \Bbbk(Q/G)/(\mathcal{I}/G)$$

is a covering functor.

Proof. Let $X, Y \in (Q/G)_0$ and $x \in X$, $y \in Y$. Then $\overline{P}^{-1}(X) = Gx$ and $\overline{P}^{-1}(Y) = Gy$. Since the action of G is free we obtain

$$\bigoplus_{y' \in \overline{P}^{-1}(Y)} \Bbbk Q(x, y') = \bigoplus_{g \in G} \Bbbk Q(x, gy), \qquad \bigoplus_{x' \in \overline{P}^{-1}(X)} \Bbbk Q(x', y) = \bigoplus_{g \in G} \Bbbk Q(gx, y).$$

Our aim is to show that the linear maps

$$\overline{P}_{xY} : \bigoplus_{g \in G} (\Bbbk Q/\mathcal{I})(x, gy) \to (\Bbbk(Q/G)/(\mathcal{I}/G))(X, Y)$$

and

$$\overline{P}_{Xy} : \bigoplus_{g \in G} (\Bbbk Q/\mathcal{I})(gx, y) \to (\Bbbk(Q/G)/(\mathcal{I}/G))(X, Y)$$

induced by $\overline{P}$ are bijective.

The functor P induces a map

$$\bigcup_{g \in G} \widehat{Q}(x, gy) \to (\widehat{Q/G})(X, Y),$$

which according to Lemmas 1 and 2 is a bijection. Since $\widehat{Q}(x, gy)$ and $(\widehat{Q/G})(X, Y)$ are bases of $\Bbbk Q(x, gy)$ and $\Bbbk(Q/G)(X, Y)$ respectively, the linear map

$$P_{xY} : \bigoplus_{g \in G} \Bbbk Q(x, gy) \to \Bbbk(Q/G)(X, Y)$$

defined by $P_{xY}(\alpha) = P(\alpha)$ for all $\alpha \in \Bbbk Q(x, gy)$ is bijective. Using the fact that $\mathcal{I}$ is G-invariant we obtain

$$\begin{aligned}(\mathcal{I}/G)(X, Y) &= \sum_{g,h \in G} P(\mathcal{I}(gx, hy)) = \sum_{g,h \in G} Pg(\mathcal{I}(x, g^{-1}hy)) \\ &= \sum_{g,h \in G} P(\mathcal{I}(x, g^{-1}hy)) = \sum_{g \in G} P(\mathcal{I}(x, gy)).\end{aligned}$$

Hence P_{xY} induces an isomorphism

$$\widetilde{P}_{xY} : \bigoplus_{g \in G} \mathcal{I}(x, gy) \to (\mathcal{I}/G)(X, Y).$$

Consider the following commutative diagram of linear maps; note that the columns are short exact sequences:

$$\begin{array}{ccc}
\bigoplus_{g\in G}(\Bbbk Q/\mathcal{I})(x,gy) & \xrightarrow{\overline{P}_{xY}} & (\Bbbk(Q/G)/(\mathcal{I}/G))(X,Y) \\
\uparrow & & \uparrow \\
\bigoplus_{g\in G}\Bbbk Q(x,gy) & \xrightarrow{P_{xY}} & \Bbbk(Q/G)(X,Y) \\
\uparrow & & \uparrow \\
\bigoplus_{g\in G}\mathcal{I}(x,gy) & \xrightarrow{\widetilde{P}_{xY}} & (\mathcal{I}/G)(X,Y)
\end{array}$$

Since both P_{xY} and $\widetilde{P}_{xY}$ are bijective so is $\overline{P}_{xY}$.

Define the linear map

$$\phi : \bigoplus_{g\in G}(\Bbbk Q/\mathcal{I})(gx,y) \to \bigoplus_{g\in G}(\Bbbk Q/\mathcal{I})(x,gy)$$

by $\phi(\alpha) = g^{-1}\alpha$ for all $\alpha \in (\Bbbk Q/\mathcal{I})(gx,y)$. It is an isomorphism. The composition

$$\overline{P}_{xY}\phi : \bigoplus_{g\in G}(\Bbbk Q/\mathcal{I})(gx,y) \to (\Bbbk(Q/G)/(\mathcal{I}/G))(a,b)$$

sends α to $\overline{P}g^{-1}\alpha = \overline{P}\alpha$ for all $\alpha \in (\Bbbk Q/\mathcal{I})(gx,y)$. Therefore it coincides with $\overline{P}_{Xy}$, which is therefore bijective. ■

COROLLARY 1. *If in addition to the assumptions of Theorem* 1, $\Bbbk Q/\mathcal{I}$ *is a spectroid and the* G*-action on* $\Bbbk Q/\mathcal{I}$ *is locally bounded*, *then* $\overline{P}$ *is a Galois covering.*

Proof. We have already seen that $\overline{P}g = \overline{P}$ for all $g \in G$. Observe that each $a \in \mathrm{Ob}(\Bbbk(Q/G)/(\mathcal{I}/G))$ is of the form $a = Gx$. Therefore $\overline{P}^{-1}(a) = Gx \neq \emptyset$. So $\overline{P}$ is surjective on the objects. Since G acts transitively on Gx we conclude that $\overline{P}$ is a Galois covering. ■

From now on we write P instead of $\overline{P}$ to simplify the notation.

Throughout the remainder of this section we make the following assumptions. Let Q be a quiver and G a group acting freely on Q. Let $\mathcal{I}$ be a G-invariant semimonomial ideal of $\Bbbk Q$. Set $\Gamma = \Bbbk Q/\mathcal{I}$, $\Lambda = \Bbbk(Q/G)/(\mathcal{I}/G)$ and let

$$P : \Gamma \to \Lambda$$

be the covering functor defined above. Identifying $\mathrm{mod}\,\Gamma$ with a full subcategory of $\mathrm{mod}\,\Bbbk Q$ and $\mathrm{mod}\,\Lambda$ with a full subcategory of $\mathrm{mod}\,\Bbbk(Q/G)$, as explained in Section 2, for all $m \in \mathrm{mod}\,\Gamma$ and $n \in \mathrm{mod}\,\Lambda$ we write

$$m(\alpha) = m(\alpha + \mathcal{I}(x,y)), \qquad n(G\alpha) = n(G\alpha + (\mathcal{I}/G)(Gx,Gy))$$

whenever $x \xrightarrow{\alpha} y$ is an arrow in Q.

Denote by

$$P_* : \operatorname{mod} \Gamma \to \operatorname{mod} \Lambda$$

the push-down functor induced by P, i.e. the left adjoint of the pull-up functor $P^* : \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma$ associated with P.

Since P is a covering functor we have, according to [1],

$$(P_* m)(Gx) = \bigoplus_{x' \in P^{-1}(Gx)} m(x') = \bigoplus_{g \in G} m(gx). \tag{2}$$

Furthermore, for each arrow $x \xrightarrow{\alpha} y$ in Q and each $h \in G$ the diagram

$$\begin{array}{ccc} m(hx) & \xrightarrow{m(h\alpha)} & m(hy) \\ \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle \mathrm{incl}} \\ \bigoplus_{g \in G} m(gx) & \xrightarrow{(P_* m)(G\alpha)} & \bigoplus_{g \in G} m(gy) \end{array}$$

commutes. Hence

$$(P_* m)(G\alpha) = \bigoplus_{g \in G} m(g\alpha) : \bigoplus_{g \in G} m(gx) \to \bigoplus_{g \in G} m(gy). \tag{3}$$

For the pull-up functor we have

$$(P^* n)(x) = n(Gx), \qquad (P^* n)(\alpha) = n(G\alpha). \tag{4}$$

So we see that

$$(P^* P_* m)(x) = \bigoplus_{g \in G} m(gx), \qquad (P^* P_* m)(\alpha) = \bigoplus_{g \in G} m(g\alpha),$$

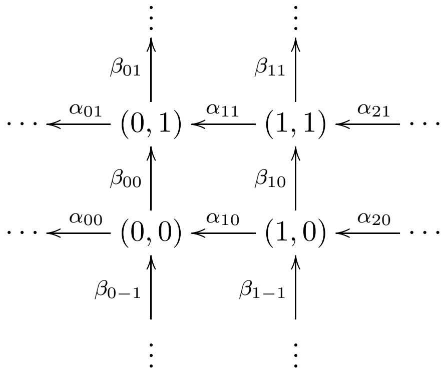
that is,

$$P^* P_* m = \bigoplus_{g \in G} {}^{g^{-1}} m = \bigoplus_{g \in G} {}^{g} m.$$

This latter result can be found as a lemma in [5].

A Λ-module n is said to be *of the first kind* if $n \xrightarrow{\sim} P_* m$ for some m in $\operatorname{mod} \Gamma$. It is said to be *of the second kind* if it contains no direct summand of the first kind. We denote by $\operatorname{mod}_1 \Lambda$ and $\operatorname{mod}_2 \Lambda$ the full subcategories of $\operatorname{mod} \Lambda$ formed by all modules of the first and second kind respectively. Further we denote by $\operatorname{ind}_1 \Lambda$ and $\operatorname{ind}_2 \Lambda$ the full subcategories of $\operatorname{mod}_1 \Lambda$ and $\operatorname{mod}_2 \Lambda$ respectively formed by all indecomposable modules.

3.2. *Example.* We illustrate the concept of Galois coverings with a concrete example, which can be found in [4]. Let Q be the quiver



i.e. $Q_0 = \mathbb{Z}^2$ and $Q_1 = \{\alpha_z, \beta_z \mid z \in \mathbb{Z}^2\}$. The group $G = \mathbb{Z}^2$ acts freely on Q by translation. Let $n \geq 2$ and $\mathcal{I}_n$ be the ideal of $\Bbbk Q$ generated by all morphisms $\beta_{ij}\alpha_{i+1,j}$, $\alpha_{i,j+1}\beta_{ij}$, $\alpha_{i+1,j} \ldots \alpha_{i+n,j}$ and $\beta_{i,j+n} \ldots \beta_{i,j+1}$. It is a G-invariant ideal and hence Theorem 1 yields the covering functor $P : \Gamma \to \Lambda$, where $\Gamma = \Bbbk Q/\mathcal{I}_n$ and $\Lambda = \Bbbk(Q/G)/(\mathcal{I}_n/G)$. Furthermore Q/G is the quiver

where $a = G(0,0)$, $\alpha = G\alpha_{00}$ and $\beta = G\beta_{00}$. The ideal $\mathcal{I}_n/G$ is generated by the morphisms $\beta\alpha$, $\alpha\beta$, α^n and β^n. This quiver with relations appears in [7], where the authors investigate representations of the Lorentz group.

A *line* of Γ is a subquiver of Q of type $\mathbb{A}_\infty$, $\mathbb{A}_\infty^\infty$ or $\mathbb{A}_m$ for some m such that $\Bbbk L$ forms a subcategory of Γ. According to [4] the category $\operatorname{ind}\Gamma$ is classified up to isomorphism by the characteristic representations χ_L, where L runs through all finite lines of Γ. Hence every indecomposable Λ-module of the first kind is isomorphic to $P_*(\chi_L)$ for some finite line L.

3.3. *Coverings and tensor product.* In this section we investigate the relationship between coverings and the tensor product. The following result provides a means of computing the tensor product of a Λ-module of the first kind and any other Λ-module.

THEOREM 2. *For all $m \in \operatorname{mod}\Gamma$ and $n \in \operatorname{mod}\Lambda$ there is an isomorphism*

$$(P_*m) \otimes n \xrightarrow{\sim} P_*(m \otimes (P^*n)).$$

Proof. We compute the right hand side at $Gx \in \operatorname{Ob}\Lambda$ and $G\alpha \in (Q/G)_1$ using the identities (2), (3) and (4):

$$P_*(m \otimes (P^*n))(Gx) = \bigoplus_{g\in G}(m(gx) \otimes (P^*n)(gx)) = \bigoplus_{g\in G}(m(gx) \otimes n(Gx)),$$
$$P_*(m \otimes (P^*n))(G\alpha) = \bigoplus_{g\in G}(m(g\alpha) \otimes (P^*n)(g\alpha)) = \bigoplus_{g\in G}(m(g\alpha) \otimes n(G\alpha)).$$

On the other hand, the equalities (2) and (3) give

$$((P_*m)\otimes n)(Gx) = \Big(\bigoplus_{g\in G} m(gx)\Big)\otimes n(Gx),$$
$$((P_*m)\otimes n)(G\alpha) = \Big(\bigoplus_{g\in G} m(g\alpha)\Big)\otimes n(G\alpha).$$

Now the identification

$$\Big(\bigoplus_{g\in G} m(gx)\Big)\otimes n(Gx) \xrightarrow{\sim} \bigoplus_{g\in G}(m(gx)\otimes n(Gx))$$

constitutes the claimed isomorphism. ∎

COROLLARY 2. *For all $m,n\in \operatorname{mod}(\Gamma)$ there is an isomorphism*

$$(P_*m)\otimes(P_*n) \xrightarrow{\sim} \bigoplus_{g\in G} P_*(m\otimes {}^g n)).$$

Proof. We have seen that

$$P^*P_*n = \bigoplus_{g\in G} {}^g n.$$

According to Theorem 2 we obtain

$$(P_*m)\otimes(P_*n) \xrightarrow{\sim} P_*\Big(m\otimes \bigoplus_{g\in G} {}^g n\Big) \xrightarrow{\sim} \bigoplus_{g\in G} P_*(m\otimes {}^g n),$$

since P_* commutes with direct sums. ∎

If Q' and Q'' are subquivers of Q, then combining Corollary 2 with formula (1) yields

$$(P_*\chi_{Q'})\otimes(P_*\chi_{Q''}) \xrightarrow{\sim} \bigoplus_{g\in G} P_*(\chi_{Q'\cap gQ''}), \tag{5}$$

upon noting that ${}^g\chi_{Q''} = \chi_{gQ''}$.

It has been shown in [5] that if Γ is a spectroid, the G-action on Γ is locally bounded and the G-action on $\operatorname{ind}\Gamma/\xrightarrow{\sim}$ is free, then P_* preserves indecomposability. In this case Corollary 2 yields the Clebsch–Gordan formulae for Λ-modules of the first kind, provided that the Clebsch–Gordan problem is solved for $\operatorname{mod}\Gamma$.

3.4. *Example revisited.* To illustrate the usefulness of the results from the previous section we return to the example of Section 3.2 and present a solution the Clebsch–Gordan problem in that case. Let Γ and Λ be as in Section 3.2.

We already have a description of the indecomposable Λ-modules of the first kind as $P_*(\chi_L)$, where L runs through all finite lines of Γ. The following proposition provides the Clebsch–Gordan formula for these modules.

PROPOSITION 1. *Let L and L' be finite lines of Γ and $L \cap gL' = \dot{\bigcup}_{i \in I_g} L^i$ a decomposition of $L \cap gL'$ into finite lines for all $g \in G$. Then*

$$(P_*\chi_L) \otimes (P_*\chi_{L'}) \xrightarrow{\sim} \bigoplus_{g \in G} \bigoplus_{i \in I_g} P_*(\chi_{L^i}).$$

Proof. Formula (5) gives

$$(P_*\chi_L) \otimes (P_*\chi_{L'}) \xrightarrow{\sim} \bigoplus_{g \in G} P_*(\chi_{L \cap gL'}).$$

Since P_* commutes with direct sums the proposition follows from the fact that

$$\chi_{L \cap gL'} = \chi_{\dot{\bigcup}_{i \in I_g} L^i} = \bigoplus_{i \in I_g} \chi_{L^i}. \blacksquare$$

Proposition 1 reduces the Clebsch–Gordan problem for $\mathrm{mod}_1 \Lambda$ to the simple combinatorial task of determining the decomposition $L \cap gL' = \bigcup_{i \in I_g} L^i$ for all finite lines L and L', and $g \in G$.

We proceed to describe the modules of the second kind, based on the description in [4], but adapted to our setting. The original classification however is due to [7]. See also [2].

Let L be a *G-periodic* line in Γ, i.e. a line with non-trivial stabilizer G_L, and such that $(0,0) \in L_0$. Then G_L acts as a group of automorphisms on L. Since G_L is non-trivial we obtain $G_L \xrightarrow{\sim} \mathbb{Z}$ as L is of type $\mathbb{A}_\infty^\infty$. For all $z \in L_0$ set $\bar{z} = z + G_L \in G/G_L$.

For any indecomposable linear automorphism $\phi : V \to V$ of a finite-dimensional $\Bbbk$-linear space V let $B_\phi(L)$ be the Λ-module defined as follows. Let U_L be the $\Bbbk$-linear space having

$$\{u_{\overline{z}} \mid \overline{z} \in L_0/G_L\}$$

as basis. Set

$$B_\phi(L)(a) = U_L \otimes V.$$

The linear maps $A = (B_\phi(L))(\alpha)$ and $B = (B_\phi(L))(\beta)$ are determined by

$$A(u_{\overline{z}} \otimes v) = \begin{cases} u_{\overline{z-(1,0)}} \otimes v & \text{if } \overline{z} \neq \overline{(1,0)} \text{ and } \alpha_z \in L_1, \\ u_{\overline{z-(1,0)}} \otimes \phi^{-1} v & \text{if } \overline{z} = \overline{(1,0)} \text{ and } \alpha_z \in L_1, \\ 0 & \text{otherwise,} \end{cases}$$

$$B(u_{\overline{z}} \otimes v) = \begin{cases} u_{\overline{z+(0,1)}} \otimes v & \text{if } \overline{z} \neq \overline{(0,0)} \text{ and } \beta_z \in L_1, \\ u_{\overline{z+(0,1)}} \otimes \phi v & \text{if } \overline{z} = \overline{(0,0)} \text{ and } \beta_z \in L_1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Bbbk L$ is a subcategory of Γ, $B_\phi(L)$ is well-defined. The Λ-modules $B_\phi(L)$ are called *band modules*. Moreover, every indecomposable Λ-module of the second kind is isomorphic to $B_\phi(L)$ for some ϕ and L. Two band modules

$B_\phi(L)$ and $B_\psi(L')$ are isomorphic precisely when ϕ and ψ are conjugate, and $L = gL'$ for some $g \in G$ (cf. [4] and [2]).

In order to apply Theorem 2 we must determine $P^*(B_\phi(L))$. For this purpose let $C_\phi(L)$ be the Γ-module defined by

$$(C_\phi(L))(z) = (\chi_L(z)) \otimes_{\Bbbk} V$$

and

$$(C_\phi(L))(\alpha_z) = \begin{cases} \chi_L(\alpha_z) \otimes \mathbb{I}_V & \text{if } \overline{z} \neq \overline{(1,0)}, \\ \chi_L(\alpha_z) \otimes \phi^{-1} & \text{if } \overline{z} = \overline{(1,0)}, \end{cases}$$

$$(C_\phi(L))(\beta_z) = \begin{cases} \chi_L(\beta_z) \otimes \mathbb{I}_V & \text{if } \overline{z} \neq \overline{(0,0)}, \\ \chi_L(\beta_z) \otimes \phi & \text{if } \overline{z} = \overline{(0,0)}. \end{cases}$$

It follows from this definition that ${}^gC_\phi(L) = C_\phi(L)$ for all $g \in G_L$. For each $h \in G/G_L$ set ${}^hC_\phi(L) = {}^gC_\phi(L)$, where $g \in G$ is a representative of h.

Lemma 3. *For all G-periodic lines L of Γ containing $(0,0)$ and all linear automorphisms $\phi : V \to V$ there is an isomorphism*

$$P^*(B_\phi(L)) \xrightarrow{\sim} \bigoplus_{h\in G/G_L} {}^hC_\phi(L)$$

of Γ-modules.

Proof. We construct the claimed isomorphism

$$\psi : P^*(B_\phi(L)) \to \bigoplus_{h\in G/G_L} {}^hC_\phi(L).$$

Let x be a point in Q. Observe that

$$(P^*(B_\phi(L)))(x) = (B_\phi(L))(a) = U_L \otimes_{\Bbbk} V.$$

Let $z \in L_0$, $g_0 = x - z$ and $h_0 = g_0 + G_L$. Then

$${}^{h_0}C_\phi(L)(x) = (\chi_{g_0L}(x)) \otimes_{\Bbbk} V = (\chi_L(z)) \otimes_{\Bbbk} V = \Bbbk \otimes_{\Bbbk} V$$

since $z \in L_0$.

We let

$$\iota : {}^{h_0}C_\phi(L)(x) \to \bigoplus_{h\in G/G_L} {}^hC_\phi(L)(x)$$

be the inclusion and set

$$\psi_x(u_{\overline{z}} \otimes v) = \iota(1 \otimes v).$$

Let $h \in G/G_L$ be represented by some $g \in G$ and such that ${}^hC_\phi(L)(x) \neq 0$. Then $x \in gL_0$ and there is $z \in L_0$ such that $g + z = x$.

Hence $\psi_x(u_{\overline{z}} \otimes V) = {}^hC_\phi(L)(x)$. Moreover, $\overline{z}$ is uniquely determined by $x - z + G_L = h$ and thus ψ_x is a bijection.

We proceed to show that $\psi = (\psi_x)_{x\in Q_0}$ is a Γ-module morphism, and hence an isomorphism. Let $x \xrightarrow{\mu} y$ be an arrow in Q. Moreover, let $u_{\overline{z}} \in U_L$ and $h = g + G_L \in G/G_L$ be such that $x = g + z$. Then $t(g^{-1}\mu) = z$, and $g^{-1}\mu = \alpha_z$ or $g^{-1}\mu = \beta_z$. Assume that $g^{-1}\mu = \alpha_z$. Then

$$(P^*(B_\phi(L))(\mu))(u_{\overline{z}} \otimes v) = A(u_{\overline{z}} \otimes v) \in U_L \otimes V = P^*(B_\phi(L))(y)$$

and

$$^hC_\phi(L)(\mu) = \begin{cases} \chi_L(\alpha_z) \otimes \mathbb{I}_V & \text{if } z \neq (1,0), \\ \chi_L(\alpha_z) \otimes \phi^{-1} & \text{if } z = (1,0). \end{cases}$$

If $\alpha_z \notin L_1$, then

$$\psi_y(A(u_{\overline{z}} \otimes v)) = 0 = (^hC_\phi(L)(\mu))(\psi_x(u_{\overline{z}} \otimes v)).$$

Now assume that $\alpha_z \in L_1$. If $\overline{z} \neq \overline{(1,0)}$, then

$$\psi_y(A(u_{\overline{z}} \otimes v)) = \psi_y(u_{\overline{z-(1,0)}} \otimes v) = (^hC_\phi(L)(\mu))(\psi_x(u_{\overline{z}} \otimes v)).$$

If $\overline{z} = \overline{(1,0)}$, then

$$\psi_y(A(u_{\overline{z}} \otimes v)) = \psi_y(u_{\overline{z-(1,0)}} \otimes \phi^{-1}v) = (^hC_\phi(L)(\mu))(\psi_x(u_{\overline{z}} \otimes v)).$$

The case $g^{-1}\mu = \beta_z$ is treated analogously. ■

For any G-periodic line L of Γ and $h = g + G_L \in G/G_L$, set $hL = gL$.

PROPOSITION 2. *Let L and L' be lines in Γ such that L is G-periodic and contains $(0,0)$, and L' is finite. Moreover, let $L' \cap hL = \dot{\bigcup}_{i\in I_h} L^i$ be a decomposition of $L' \cap hL$ into finite lines for all $h \in G/G_L$. Then there is an isomorphism*

$$(P_*\chi_{L'}) \otimes (B_\phi(L)) \xrightarrow{\sim} \dim V \bigoplus_{h\in G/G_L} \bigoplus_{i\in I_h} P_*\chi_{L^i}.$$

Proof. Theorem 2 yields

$$(P_*\chi_{L'}) \otimes (B_\phi(L)) \xrightarrow{\sim} P_*(\chi_{L'} \otimes (P^*(B_\phi(L)))) \xrightarrow{\sim} P_*\Big(\bigoplus_{h\in G/G_L} \chi_{L'} \otimes {}^hC_\phi(L)\Big)$$

by Lemma 3. Observe that $(\chi_{L'} \otimes {}^hC_\phi(L))(z) \neq 0$ if and only if $z \in S = \dot{\bigcup}_{i\in I_h} L^i$. Furthermore, $\dim(\chi_{L'} \otimes {}^hC_\phi(L))(z) = \dim V$ for all $z \in S$, and all arrows in S act as isomorphisms in $\chi_{L'} \otimes {}^hC_\phi(L)$. Due to the classification of all indecomposable Γ-modules the only possible decomposition of $\chi_{L'} \otimes {}^hC_\phi(L)$ is

$$\chi_{L'} \otimes {}^hC_\phi(L) \xrightarrow{\sim} \dim V \bigoplus_{i\in I_h} \chi_{L^i}.$$

Hence

$$(P_*\chi_{L'}) \otimes (B_\phi(L)) \xrightarrow{\sim} P_*\Big(\dim V \bigoplus_{h\in G/G_L} \bigoplus_{i\in I_h} \chi_{L^i}\Big) \xrightarrow{\sim} \dim V \bigoplus_{h\in G/G_L} \bigoplus_{i\in I_h} P_*\chi_{L^i}.$$

The last step is valid since all but finitely many summands are zero. ■

To complete the solution of the Clebsch–Gordan problem for Λ it remains to find a formula for the decomposition of $(B_\phi(L))\otimes(B_\psi(L'))$. In this situation we cannot apply our results on coverings. Instead we will use elementary methods to obtain the desired result.

Let L, L' be G-periodic lines in Γ containing the point $(0,0)$. Let X be a cross-section of $G/(G_L+G_{L'})$ such that $0\in X$.

Let $L\cap gL' = \dot{\bigcup}_{i\in J_g} L^i$ be a decomposition of $L\cap gL'$ into lines for all $g\in G$. Choose $I_g\subset J_g$ such that $\{L^i \mid i\in I_g\}$ forms a cross-section for the $G_L\cap G_{L'}$-action on $\{L^i\mid i\in J_g\}$.

Define the linear map

$$T:\bigoplus_{x\in X}\bigoplus_{i\in I_x}\Bbbk L_0^i\to U_L\otimes U_{L'}$$

by $T(z)=u_{\overline{z}}\otimes u_{\overline{z-x}}$ for all $z\in L_0^i$, $i\in I_x$ and $x\in X$. Here $\Bbbk L_0^i$ denotes the vector space having L_0^i as basis. To see that T is well-defined note that if $z\in L_0^i$ then $z\in L_0$ and $z\in xL_0'$. Hence $z-x\in L_0'$ and $u_{\overline{z}}\otimes u_{\overline{z-x}}\in U_L\otimes U_{L'}$.

Since $B_\phi(L)\xrightarrow{\sim}B_\phi(gL)$ for all $g\in G$ we can cover all interesting cases by only considering the cases $L\neq gL'$ for all $g\in G$, and $L=L'$.

LEMMA 4. *If $L\neq gL'$ for all $g\in G$, then T is an isomorphism. If $L=L'$, then T induces a linear isomorphism*

$$\widetilde{T}:\bigoplus_{x\in X\setminus\{0\}}\bigoplus_{i\in I_x}\Bbbk L_0^i\to U$$

where $U\subset U_L\otimes U_L$ is spanned by

$$\{u_{\overline{z}}\otimes u_{\overline{z'}}\mid \overline{z}\neq\overline{z'}\}.$$

Proof. Note that $\widetilde{T}$ is well-defined in case $L=L'$, since if $0\neq x\in X$, then $\overline{z}\neq\overline{z-x}$.

Let $z\in L_0$ and $z'\in L_0'$. Write $z'=z+f-x+h$ for some $f\in G_L$, $x\in X$ and $h\in G_{L'}$. Then $z+f-x\in L_0'$ and $z+f\in xL_0'$. Hence $z+f\in L_0\cap xL_0'=\dot{\bigcup}_{i\in J_x}L^i$. Let $y\in G_L\cap G_{L'}$ be such that $z_0=z+f+y\in L_0^i$ for some $i\in I_x$. Then

$$u_{\overline{z}}\otimes u_{\overline{z'}}=u_{\overline{z+y}}\otimes u_{\overline{z'+y}}=u_{\overline{z+f+y}}\otimes u_{\overline{z+f-x+y}}=u_{\overline{z_0}}\otimes u_{\overline{z_0-x}}=T(z_0).$$

Hence T is an epimorphism. If $L=L'$ and $\overline{z}\neq\overline{z'}$ then $x\neq 0$ and thus $\widetilde{T}$ is also an epimorphism.

Assume that $u_{\overline{z}}\otimes u_{\overline{z-x}}=u_{\overline{z'}}\otimes u_{\overline{z'-x'}}$ for some $z\in L_0^i$, $i\in I_x$, $z'\in L_0^{i'}$, $i'\in I_{x'}$. Then

$$z\equiv z \bmod G_L,\qquad z-x\equiv z'-x' \bmod G_{L'}.$$

In particular $x\equiv x' \bmod G_L+G_{L'}$ and thus $x=x'$. We obtain

$$z\equiv z' \bmod G_L\cap G_{L'}.$$

Hence there is some $g \in G_L \cap G_{L'}$ such that $z' = z + g$. In particular

$$z' \in gL_0^i \cap L_0^{i'}$$

and thus $L^{i'} = gL^i$. We obtain $i' = i$, since $\{L^i \mid i \in I_g\}$ forms a cross-section for the $G_L \cap G_{L'}$-action on $\{L^i \mid i \in J_g\}$. Hence

$$L^i = gL^i.$$

If $g \neq 0$, then $L^i \subset L \cap xL'$ is G-periodic and $L = L^i = xL'$. Thus, if $L \notin GL'$, then $g = 0$ and $z = z'$. Hence T is a monomorphism and thus an isomorphism in that case. If $L = L'$ and $g \neq 0$, then $x \in G_L$ and thus $x = 0$. This is a contradiction if $u_{\overline{z}} \otimes u_{\overline{z-x}} \in U$. Again $g = 0$ and $\widetilde{T}$ is an isomorphism. ∎

We now present the Clebsch–Gordan formula for band modules and thus complete our solution to the Clebsch–Gordan problem for Λ.

THEOREM 3. *Let L, L' be periodic lines in Q containing the point* $(0,0)$ *and $\phi : V \to V$, $\psi : W \to W$ be linear automorphisms. Let X be a cross-section of $G/(G_L + G_{L'})$ such that $0 \in X$. Let $L \cap gL' = \dot{\bigcup}_{i\in J_g} L^i$ be a decomposition of $L \cap gL'$ into lines for all $g \in G$. Let $I_g \subset J_g$ be such that $\{L^i \mid i \in I_g\}$ forms a cross-section for the $G_L \cap G_{L'}$-action on $\{L^i \mid i \in J_g\}$.*

If $L \neq gL'$ for all $g \in G$, then

$$B_\phi(L) \otimes B_\psi(L') \xrightarrow{\sim} \dim V \dim W \bigoplus_{x\in X} \bigoplus_{i\in I_x} P_* \chi_{L^i}.$$

If $L = L'$, then

$$B_\phi(L) \otimes B_\psi(L') \xrightarrow{\sim} \Big(\dim V \dim W \bigoplus_{x\in X\setminus\{0\}} \bigoplus_{i\in I_x} P_* \chi_{L^i}\Big) \oplus \Big(\bigoplus_j B_{\phi_j}(L)\Big),$$

where

$$\phi \otimes \psi \xrightarrow{\sim} \bigoplus_j \phi_j$$

is a decomposition of $\phi \otimes \psi$ into indecomposable automorphisms.

The case $B_\phi(L) \otimes B_\psi(L')$ is now reduced to the simple task of determining the set $\{L^i \mid i \in I_g\}$ given L and L', and the more complicated problem of finding

$$\phi \otimes \psi \xrightarrow{\sim} \bigoplus_j \phi_j$$

for all linear automorphisms ϕ and ψ. This is equivalent to solving the Clebsch–Gordan problem for the loop quiver $\widetilde{\mathbb{A}}_0$. In case the ground field $\Bbbk$ is algebraically closed and of characteristic 0, it has been solved by Huppert [11] and independently by Martsinkovsky and Vlassov [12]. In other cases the solution is still unknown.

Proof of Theorem 3. Assume that $L \neq gL'$ for all $g \in G$. From Lemma 4 we obtain a linear isomorphism

$$S : (B_\phi(L) \otimes B_\psi(L))(a) = U_L \otimes V \otimes U_{L'} \otimes W \xrightarrow{\sim} \bigoplus_{x\in X} \bigoplus_{i\in I_x} (\Bbbk L_0^i \otimes V \otimes W)$$

defined by

$$S(u_{\overline{z}} \otimes v \otimes u_{\overline{z'}} \otimes w) = T^{-1}(u_{\overline{z}} \otimes u_{\overline{z'}}) \otimes v \otimes w.$$

We define a Λ-module structure on

$$\bigoplus_{x\in X} \bigoplus_{i\in I_x} (\Bbbk L_0^i \otimes V \otimes W)$$

via S, and denote this Λ-module by M.

Let $z \in L_0^i$ for some $i \in I_x$, $v \in V$ and $w \in W$. Then

$$S(u_{\overline{z}} \otimes v \otimes u_{\overline{z-x}} \otimes w) = z \otimes v \otimes w.$$

Let $A = (B_\phi(L) \otimes B_\psi(L))(\alpha)$. If $\alpha_z \in L_1$ and $\alpha_{z-x} \in L_1'$, then

$$A(u_{\overline{z}} \otimes v \otimes u_{\overline{z-x}} \otimes w) = u_{\overline{z-(1,0)}} \otimes \phi^{m_z}(v) \otimes u_{\overline{z-x-(1,0)}} \otimes \psi^{n_z}(w)$$

for some integers m_z and n_z. Otherwise

$$A(u_{\overline{z}} \otimes u_{\overline{z-x}}) = 0.$$

If $\alpha_z \in L_1$ and $\alpha_z \in xL_1'$, then $z - (1,0) \in L_0^i$ and

$$S(u_{\overline{z-(1,0)}} \otimes \phi^{m_z}(v) \otimes u_{\overline{z-(1,0)-x}} \otimes \psi^{n_z}(w)) = (z-(1,0)) \otimes \phi^{m_z}(v) \otimes \psi^{n_z}(w).$$

Hence

$$M(\alpha)(z \otimes v \otimes w) = \begin{cases} (z-(1,0)) \otimes \phi^{m_z}(v) \otimes \psi^{n_z}(w) & \text{if } \alpha_z \in L_1,\ \alpha_z \in xL_1', \\ 0 & \text{otherwise.} \end{cases}$$

A similar calculation shows that

$$M(\beta)(z\otimes v\otimes w) = \begin{cases} (z+(0,1)) \otimes \phi^{m_z'}(v) \otimes \psi^{n_z'}(w) & \text{if } \beta_z \in L_1,\ \ \beta_z \in xL_1', \\ 0 & \text{otherwise,} \end{cases}$$

for some integers m_z' and n_z'.

As has been noted earlier, $\alpha_z \in L_1$ and $\alpha_z \in xL_1'$ implies $z-(1,0) \in L_0^i$. Similarly, $\beta_z \in L_1$ and $\beta_z \in xL_1'$ implies $z+(0,1) \in L_0^i$. Hence

$$M = \bigoplus_{x\in X} \bigoplus_{i\in I_x} M_i,$$

where M_i is the submodule of M corresponding to $\Bbbk L_0^i \otimes V \otimes W$.

For each $x \in X$ and $i \in I_x$ we define the Γ-module N_i by

$$N_i(z) = \chi_{L^i}(z) \otimes V \otimes W,$$

and

$$N_i(\alpha_z) = \begin{cases} 1 \otimes \phi^{m_z} \otimes \psi^{n_z} & \text{if } \alpha_z \in L_1^i, \\ 0 & \text{otherwise}, \end{cases}$$
$$N_i(\beta_z) = \begin{cases} 1 \otimes \phi^{m'_z} \otimes \psi^{n'_z} & \text{if } \beta_z \in L_1^i, \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$M_i \xrightarrow{\sim} P_* N_i.$$

On the other hand, since $N_i(\mu)$ is an isomorphism for each $\mu \in L_1^i$,

$$N_i \xrightarrow{\sim} \dim V \dim W (P_* \chi_{L^i}).$$

Hence

$$B_\phi(L) \otimes B_\psi(L') \xrightarrow{\sim} M \xrightarrow{\sim} \dim V \dim W \bigoplus_{x\in X} \bigoplus_{i \in I_x} P_* \chi_{L^i}.$$

Now assume $L = L'$. Then $(B_\phi(L) \otimes B_\psi(L'))(a) = U_L \otimes V \otimes U_L \otimes W$ and from Lemma 4 we obtain a linear isomorphism

$$\widetilde{S} : (B_\phi(L) \otimes B_\psi(L'))(a) \xrightarrow{\sim} \Big(\Big(\bigoplus_{x\in X\setminus\{0\}} \bigoplus_{i\in I_x} \Bbbk L_0^i\Big) \oplus D\Big) \otimes V \otimes W,$$

where $D \subset U_L \otimes U_L$ is the subspace spanned by all vectors $u_{\overline{z}} \otimes u_{\overline{z}}$, as in the previous case. We define a Λ-module structure on

$$\Big(\Big(\bigoplus_{x\in X\setminus\{0\}} \bigoplus_{i\in I_x} \Bbbk L_0^i\Big) \oplus D\Big) \otimes V \otimes W.$$

via $\widetilde{S}$, and denote this Λ-module by M.

Let $A = (B_\phi(L) \otimes B_\psi(L))(\alpha)$ and $B = (B_\phi(L) \otimes B_\psi(L))(\beta)$. Let $z \in L_0$. If $\alpha_z \in L_1$, then

$$A(u_{\overline{z}} \otimes v \otimes u_{\overline{z}} \otimes w) = \begin{cases} u_{\overline{z-(1,0)}} \otimes v \otimes u_{\overline{z-(1,0)}} \otimes w & \text{if } \overline{z} \neq \overline{(1,0)}, \\ u_{\overline{z-(1,0)}} \otimes \phi^{-1} v \otimes u_{\overline{z-(1,0)}} \otimes \psi^{-1} v & \text{if } \overline{z} = \overline{(1,0)}. \end{cases}$$

Otherwise

$$A(u_{\overline{z}} \otimes v \otimes u_{\overline{z}} \otimes w) = 0.$$

If $\beta_z \in L_1$, then

$$B(u_{\overline{z}} \otimes v \otimes u_{\overline{z}} \otimes w) = \begin{cases} u_{\overline{z+(0,1)}} \otimes v \otimes u_{\overline{z+(0,1)}} \otimes w & \text{if } \overline{z} \neq \overline{(1,0)}, \\ u_{\overline{z+(0,1)}} \otimes \phi v \otimes u_{\overline{z+(0,1)}} \otimes \psi v & \text{if } \overline{z} = \overline{(1,0)}. \end{cases}$$

Otherwise

$$B(u_{\overline{z}} \otimes v \otimes u_{\overline{z}} \otimes w) = 0.$$

We obtain a submodule N of M determined by

$$N(a) = D \otimes V \otimes W.$$

Furthermore, we get the isomorphism

$$B_{\phi\otimes\psi}(L) \xrightarrow{\sim} N$$

determined by $u_{\overline{z}} \otimes v \otimes w \mapsto u_{\overline{z}} \otimes u_{\overline{z}} \otimes v \otimes w$.

Let $\tau : V \otimes W \xrightarrow{\sim} V \otimes W$ be a linear automorphism such that

$$\tau(\phi \otimes \psi)\tau^{-1} = \bigoplus_j \phi_j.$$

Then τ yields the isomorphism

$$\theta : B_{\phi\otimes\psi}(L) \xrightarrow{\sim} B_{\bigoplus_j \phi_j}(L) = \bigoplus_j B_{\phi_j}(L)$$

determined by

$$\theta_a : (B_{\phi\otimes\psi}(L))(a) \xrightarrow{\sim} (B_{\bigoplus_j \phi_j}(L))(a), \qquad u_{\overline{z}} \otimes v \otimes w \mapsto u_{\overline{z}} \otimes \tau(v \otimes w).$$

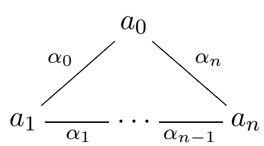
By arguments analogous to those in the previous case one shows that $\widetilde{S}$ induces a Λ-module structure on $(\bigoplus_{x\in X\setminus\{0\}} \bigoplus_{i\in I_x} \Bbbk L_0^i) \otimes V \otimes W$ which is isomorphic to

$$\dim V \dim W \bigoplus_{x\in X\setminus\{0\}} \bigoplus_{i\in I_x} P_* \chi_{L^i}.$$

Hence

$$B_\phi(L) \otimes B_\psi(L') \xrightarrow{\sim} \Big(\dim V \dim W \bigoplus_{x\in X\setminus\{0\}} \bigoplus_{i\in I_x} P_* \chi_{L^i}\Big) \oplus \Big(\bigoplus_j B_{\phi_j}(L)\Big). \blacksquare$$

4. Quivers of type $\widetilde{\mathbb{A}}_n$. In this section we revisit the Clebsch–Gordan problem for quivers of type $\widetilde{\mathbb{A}}_n$, i.e. quivers whose underlying graph is



for some $n \in \mathbb{N}$. We assume that $\Bbbk$ is algebraically closed. This problem has originally been solved in [10], by means of explicit computations. Here we present a more streamlined approach, using the results on coverings and characteristic representations developed above. For the reader's convenience we include those computations from [10] which are indispensable even in the present approach (cf. proof of Theorem 5(iii)).

4.1. *Indecomposable modules.* Let $n \in \mathbb{N}$ and Q be a quiver of type $\mathbb{A}_\infty^\infty$, i.e. a quiver with underlying graph

$$\cdots \overset{\alpha_{-1}}{\text{———}} a_0 \overset{\alpha_0}{\text{———}} a_1 \overset{\alpha_1}{\text{———}} \cdots.$$

Assume that the orientation of Q is periodic in the sense that $a_i \xrightarrow{\alpha_i} a_{i+1}$ implies $a_{i+n+1} \xrightarrow{\alpha_{i+n+1}} a_{i+n+2}$ and $a_i \xleftarrow{\alpha_i} a_{i+1}$ implies $a_{i+n+1} \xleftarrow{\alpha_{i+n+1}} a_{i+n+2}$ for all $i \in \mathbb{Z}$. Then $\mathbb{Z}$ acts freely on Q by

$$ka_i = a_{i+k(n+1)}, \qquad k\alpha_i = \alpha_{i+k(n+1)}$$

for all $k \in \mathbb{Z}$. The quotient quiver $Q/\mathbb{Z}$ is of type $\widetilde{\mathbb{A}}_n$. Moreover, every quiver of type $\widetilde{\mathbb{A}}_n$ arises in this way. Theorem 1 yields a covering functor

$$P : \Bbbk Q \to \Bbbk(Q/\mathbb{Z})$$

together with the associated push-down functor

$$P_* : \operatorname{mod} \Bbbk Q \to \operatorname{mod} \Bbbk(Q/\mathbb{Z}).$$

We interpret the classification of $\operatorname{ind} \Bbbk Q$ found in [6] in terms of coverings. For all integers i, j such that $i \le j$ let $X_{ij} = \chi_{Q^{ij}} \in \operatorname{mod} \Bbbk Q$, where Q^{ij} is the subquiver of Q with underlying graph

$$a_i \overset{\alpha_i}{\textrm{———}} \cdots \overset{\alpha_{j-1}}{\textrm{———}} a_j.$$

Set

$$S(i, j) = P_*(X_{ij}).$$

The modules $S(i, j)$ are modules of the first kind and are called *strings*.

For each positive integer m and scalar $\lambda \in \Bbbk \setminus \{0\}$ let $B_\lambda(m)$ be the $\Bbbk(Q/\mathbb{Z})$-module defined by

$$B_\lambda(m)(a_i) = \Bbbk^m, \qquad B_\lambda(m)(\alpha_i) = \begin{cases} \mathbb{I}_m & \text{if } i \neq n, \\ J_\lambda(m) & \text{if } i = n, \end{cases}$$

where $\mathbb{I}_m$ is the identity matrix of size m and $J_\lambda(m)$ is the Jordan block of size m with eigenvalue λ. The modules $B_\lambda(m)$ are called bands and are modules of the second kind.

Theorem 4 ([6, p. 121]). *The set*

$$\{S(i, j) \mid 0 \le i \le n, i \le j\} \cup \{B_\lambda(m) \mid \lambda \in \Bbbk \setminus \{0\},\, m \in \mathbb{N} \setminus \{0\}\}$$

classifies $\operatorname{ind} \Bbbk(Q/\mathbb{Z})$, *up to isomorphism.*

4.2. *Clebsch–Gordan formulae.* Let $i \wedge j = \min\{i, j\}$ and $i \vee j = \max\{i, j\}$ for all integers i, j. The following result provides the Clebsch–Gordan formulae for $\widetilde{\mathbb{A}}_n$ in terms of strings and bands.

Theorem 5. *Assume that* $\operatorname{char}(\Bbbk) = 0$. *For all integers* i, i', j, j' *such that* $0 \le i \le i' \le n$, $i \le j$ *and* $i' \le j'$, *scalars* $\lambda, \mu \in \Bbbk \setminus \{0\}$ *and* $l, m \in \mathbb{N} \setminus \{0\}$ *the following formulae hold*:

(i) $S(i,j) \otimes S(i',j') \xrightarrow{\sim} \bigoplus_{k=0}^{[(j'-i)/(n+1)]} S(i, j \wedge (j'-k(n+1)))$
$$\oplus \bigoplus_{k=1}^{[(j-i')/(n+1)]} S(i', j' \wedge (j-k(n+1))),$$

(ii) $S(i,j) \otimes B_\mu(m) \xrightarrow{\sim} mS(i,j)$,

(iii) $B_\lambda(l) \otimes B_\mu(m) \xrightarrow{\sim} \bigoplus_{k=0}^{(l\wedge m)-1} B_{\lambda\mu}(l+m-2k-1)$.

Here $[x]$ denotes the integer part of x for all $x \in \mathbb{Q}$. The restriction $i \le i'$ does not affect the generality of formula (i), as the tensor product is commutative.

Proof. (i) We extend the notation by letting $S(i,j)$ and X_{ij} be zero whenever $i > j$. Formula (5) yields

$$\begin{aligned} S(i,j) \otimes S(i',j') &= (P_* X_{ij}) \otimes (P_* X_{i'j'}) \\ &\xrightarrow{\sim} \bigoplus_{k\in\mathbb{Z}} P_*(\chi_{Q^{ij} \cap kQ^{i'j'}}) = \bigoplus_{k\in\mathbb{Z}} P_*(X_{i\vee(i'+k(n+1)), j\wedge(j'+k(n+1))}). \end{aligned}$$

From the inequality $i \ge i'$ we obtain

$$\begin{aligned} &S(i,j) \otimes S(i',j') \\ &\quad\xrightarrow{\sim} \bigoplus_{k\le 0} P_*(X_{i, j\wedge(j'+k(n+1))}) \oplus \bigoplus_{k>0} P_*(X_{i'+k(n+1), j\wedge(j'+k(n+1))}) \\ &\quad= \bigoplus_{k\ge 0} S(i, j \wedge (j'-k(n+1))) \oplus \bigoplus_{k>0} S(i', (j-k(n+1)) \wedge j') \end{aligned}$$

using the equality $P_*({}^kX) = P_*(X)$ for all $k \in \mathbb{Z}$ and $X \in \operatorname{mod} \Bbbk Q$. The limits $k \le [(j'-i)/(n+1)]$ and $k \le [(j-i')/(n+1)]$ arise from the fact that $S(i, j \wedge (j'-k(n+1)))$ and $S(i', (j-k(n+1)) \wedge j')$ are zero when $(n+1)k > j'-i$ and $(n+1)k > j-i'$ respectively.

(ii) From Theorem 2 we obtain

$$S(i,j) \otimes B_\lambda(m) = (P_* X_{ij}) \otimes B_\lambda(m) \xrightarrow{\sim} P_*(X_{ij} \otimes (P^* B_\lambda(m))).$$

Since $(X_{ij} \otimes (P^* B_\lambda(m)))(a_k)$ is of dimension m for all $i \le k \le j$ and zero otherwise, and $(X_{ij} \otimes (P^* B_\lambda(m)))(\alpha_k)$ is an isomorphism for all $i \le k < j$, it follows that

$$X_{ij} \otimes (P^* B_\lambda(m)) \xrightarrow{\sim} mX_{ij}.$$

Hence

$$S(i,j) \otimes B_\lambda(m) \xrightarrow{\sim} P_*(mX_{ij}) \xrightarrow{\sim} mP_*(X_{ij}) = mS(i,j)$$

since P_* commutes with direct sums.

(iii) Let $l, m \in \mathbb{N} \setminus \{0\}$ and $\lambda, \mu \in \Bbbk \setminus \{0\}$. Set $A = B_\lambda(l)$, $B = B_\mu(m)$ and $T = J_\lambda(l) \otimes J_\mu(m)$, the Kronecker product of the Jordan blocks. By definition we have

$$(A \otimes B)(a_k) = \Bbbk^l \otimes \Bbbk^m \xrightarrow{\sim} \Bbbk^{lm}.$$

In the standard basis $(e_i \otimes e_j)_{(i,j)\in \underline{l}\times\underline{m}}$ the linear map $(A \otimes B)(\alpha_k)$ is given by the identity matrix $\mathbb{I}_{lm}$ if $k \neq n$ whereas $(A \otimes B)(\alpha_n)$ is given by T.

Any $C \in \mathrm{GL}_{lm}(\Bbbk)$ determines a new representation $(A \otimes B)^C$ given by $(A\otimes B)^C(a_k) = \Bbbk^{lm}$ for all $k \in \{0, \dots, n\}$, $(A\otimes B)^C(\alpha_k) = \mathbb{I}_{lm}$ if $k < n$ and $(A \otimes B)^C(\alpha_n) = CTC^{-1}$, together with an isomorphism

$$C? : A \otimes B \to (A \otimes B)^C$$

given by $(A\otimes B)(a_k) \to (A\otimes B)^C(a_k)$, $x \mapsto Cx$, for all $k \in \{0, \dots, n\}$. Since $\mathrm{char}(\Bbbk) = 0$ we know from [11, p. 51] that there exists a $C \in \mathrm{GL}_{lm}(\Bbbk)$ such that $CTC^{-1} = \bigoplus_{k=0}^{(l\vee m)-1} J_{\lambda\mu}(l + m - 2k - 1)$. Accordingly, $(A \otimes B)^C = \bigoplus_{k=0}^{(l\vee m)-1} B_{\lambda\mu}(l + m - 2k - 1)$. We conclude that

$$A \otimes B \xrightarrow{\sim} \bigoplus_{k=0}^{(l\vee m)-1} B_{\lambda\mu}(l + m - 2k - 1). \ \blacksquare$$

Note that the assumption $\mathrm{char}(\Bbbk) = 0$ only enters in the proof of part (iii), namely in order to ensure that the matrix T has the Jordan decomposition $\bigoplus_{k=0}^{(l\vee m)-1} J_{\lambda\mu}(l+m-2k-1)$. The case $\mathrm{char}(\Bbbk) = p$ can be treated similarly as soon as the Jordan decomposition of $J_\lambda(l) \otimes J_\mu(m)$ is provided. However, at present I do not know any general formula for this decomposition.

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