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GLOBAL ATTRACTOR FOR THE PERTURBED VISCOUS CAHN–HILLIARD EQUATION

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Abstract. We consider the initial-boundary value problem for the perturbed viscous Cahn–Hilliard equation in space dimension $n \le 3$. Applying semigroup theory, we formulate this problem as an abstract evolutionary equation with a sectorial operator in the main part. We show that the semigroup generated by this problem admits a global attractor in the phase space $(H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$ and characterize its structure.

1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded open set with the boundary $\partial\Omega$ of class C^4. In this paper we study the *perturbed viscous Cahn–Hilliard equation*

$$\varepsilon u_{tt} + u_t + \Delta(\Delta u + f(u) - \delta u_t) = 0, \quad x \in \Omega,\ t > 0, \tag{1}$$

where $\varepsilon, \delta \in (0,1]$, $n \le 3$, and the derivative of f grows like $|u|^q$, with $0 < q < 2$ if $n = 3$. This equation is considered with the initial-boundary conditions

$$u(0,x) = u_0(x), \qquad u_t(0,x) = v_0(x) \qquad \text{for } x \in \Omega, \tag{2}$$

$$u(t,x) = 0, \qquad \Delta u(t,x) = 0 \qquad \text{for } x \in \partial\Omega. \tag{3}$$

Equation (1) in one space dimension ($\Omega = (0,\pi)$) and with the polynomial nonlinear term $f(u) = -u^3 + u$ extending the classical Cahn–Hilliard parabolic equation ([10], [6]) has been introduced in [12]. The authors studied there the following four equations, named according to whether ε or δ vanishes or not:

- the nonviscous Cahn–Hilliard equation ($\varepsilon = \delta = 0$),
- the viscous Cahn–Hilliard equation ($\varepsilon = 0$, $\delta > 0$),
- the perturbed nonviscous Cahn–Hilliard equation ($\varepsilon > 0$, $\delta = 0$),
- the perturbed viscous Cahn–Hilliard equation ($\varepsilon > 0$, $\delta > 0$).

Zheng and Milani showed that the semigroup generated by the initial-boundary value problem for the perturbed (viscous and nonviscous) Cahn–Hilliard equation admits a global attractor in the phase space $H_0^1(0,\pi) \times$

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$H^{-1}(0,\pi)$ and that the family of such attractors (depending on $\varepsilon > 0$) is upper-semicontinuous with respect to the perturbation parameter as $\varepsilon \to 0^+$. In the case of the perturbed viscous Cahn–Hilliard equation, they also obtained the regularity of the attractor.

Our main goal here is to generalize part of results of [12] concerning the existence of the global attractor generated by problem (1)–(3) $(\varepsilon, \delta > 0)$. Considering this problem in higher space dimension $n \le 3$ and with a more general nonlinear term f, but with the initial conditions from a more regular phase space $(u_0, v_0) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times L^2(\Omega)$, we prove that the semigroup generated by this problem admits a global attractor $\mathcal{A}$ in $(H^2(\Omega) \cap H^1_0(\Omega)) \times L^2(\Omega)$. Moreover, we show that $\mathcal{A} = \mathcal{M}(\mathcal{N})$, where $\mathcal{M}(\mathcal{N})$ is an unstable manifold emanating from the set $\mathcal{N}$ of the equilibrium points for the semigroup $\{\mathcal{T}(t)\}$. We assume that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

(i) $f \in C^2(\mathbb{R}, \mathbb{R})$,
(ii) $\exists_{\overline{C}\in\mathbb{R}}\ \forall_{s\in\mathbb{R}}\ \overline{F}(s) := \int_0^s f(z)\,dz \le \overline{C}$,
(iii) $\exists_{\sigma \ge (2K_1^2+1)/(3\sqrt{\varepsilon})}\ \exists_{C_\sigma\in\mathbb{R}^+}\ \forall_{s\in\mathbb{R}}\ sf(s) - \frac{4}{3}\overline{F}(s) \le -\sigma s^2 + C_\sigma$, where K_1 is an embedding constant for $L^2(\Omega) \subset H^{-1}(\Omega)$ (see (9)),
(iv) $\exists_{\widehat{C}\in\mathbb{R}}\ \forall_{s\in\mathbb{R}}\ |f'(s)| \le \widehat{C}(1+|s|^q)$, where q is arbitrarily large if $n = 1, 2$, and $0 < q < 2$ if $n = 3$.

Notice that the function $f(u) = -u^3 + u$ used by Zheng and Milani satisfies the above assumptions for $n = 1, 2$.

Moreover, the technique used here is completely different. Precisely, working within semigroup theory, we consider problem (1)–(3) in the form of an abstract evolutionary equation; this approach makes our calculations easier than those in [12].

In this article all the Sobolev spaces H^k and C^k-type spaces are considered for functions defined on a fixed domain $\Omega \subset \mathbb{R}^n$, so we use the simplified notation $H^k = H^k(\Omega)$ and $C^m = C^m(\Omega)$ throughout. The norm in L^2 is denoted by $\|\cdot\|$ and the scalar product on this space by $(\cdot,\cdot)$. We reserve the letter K with suitable subscripts to denote constants such that the appropriate embedding estimate holds.

We denote by $-\Delta$ the Laplace operator with domain $D(-\Delta) = H^1_0$, and values in H^{-1}. We also consider the L^2-realization, $-\Delta_{L^2}$, of $-\Delta$ with the Dirichlet condition (see [1]), i.e. the linear operator in L^2 defined by

$$D(-\Delta_{L^2}) := \{u \in L^2 \cap D(-\Delta) : -\Delta u \in L^2\}, \qquad -\Delta_{L^2} u := -\Delta u.$$

We preserve the notation $-\Delta$ for this L^2-realization. Since $-\Delta$ is an unbounded, closed, positive self-adjoint linear operator with compact resolvent in L^2, we can define for $s \in \mathbb{R}$ the fractional powers $(-\Delta)^s$. The domain

$D((-\Delta)^s)$ of $(-\Delta)^s$ endowed with the scalar product and norm

$$(4)\qquad \begin{cases} (u,v)_{D((-\Delta)^s)} = ((-\Delta)^s u, (-\Delta)^s v), \\ \|u\|_{D((-\Delta)^s)} = ((u,u)_{D((-\Delta)^s)})^{1/2}, \end{cases}$$

is a Hilbert space for any $s > 0$. Let $D((-\Delta)^{-s})$ denote the dual space of $D((-\Delta)^s)$ $(s > 0)$. This Hilbert space can be endowed with the product and norm as above, where s is replaced by $-s$ (see [10, Section 2.1]). Moreover, we infer from [8, Section 1.4] that for $\alpha > 0$, $H^\alpha \supset D((-\Delta)^{\alpha/2})$ and the inner product on H^{-1} can be introduced as

$$(5)\qquad (\phi,\varphi)_{H^{-1}} = ((-\Delta)^{-1/2}\phi, (-\Delta)^{-1/2}\varphi), \qquad \varphi,\phi \in H^{-1}.$$

2. Operators A, B and their properties. Usually second order in time ("hyperbolic") equations are rewritten in the form of a first order system. Such a formulation and properties of operators appearing in it will now be discussed. Let A and B denote the operators $(-\Delta)^2$ and $(1/\sqrt{\varepsilon})(\delta(-\Delta)+I)$ with domains $D(A) = \{u \in H^3 : u_{|\partial\Omega} = \Delta u_{|\partial\Omega} = 0\}$ and $D(B) = H_0^1$ in the space H^{-1}, respectively. Making a suitable change of time variable, we can write (1) as an abstract equation in $H_0^1 \times H^{-1}$ in the following way:

$$(6)\qquad \frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A_B}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -\Delta(f(u)) \end{bmatrix}, \qquad t > 0,$$

where

$$(7)\qquad \mathbf{A_B} := \begin{bmatrix} 0 & I \\ -A & -B \end{bmatrix} : H_0^1 \times H^{-1} \supset (H^3 \cap H_0^2) \times H_0^1 \to H_0^1 \times H^{-1}.$$

We discuss the properties of A and B necessary to prove that $-\mathbf{A_B}$ is a sectorial, positive operator (i.e. $\operatorname{Re}\sigma(-\mathbf{A_B}) > 0$) and has compact resolvent. If we show that A and B are strictly positive definite self-adjoint operators on H^{-1}, the resolvent of A is compact and B is "comparable" with $A^{1/2}$, then $-\mathbf{A_B}$ will be sectorial and $\operatorname{Re}\sigma(-\mathbf{A_B}) > 0$ (see [2, Theorem 1.1]). Since C_0^∞ is dense in L^2 and L^2 is dense in H^{-1}, we deduce that A and B have dense domains.

LEMMA 2.1.

(i) *The operator* $B: H^{-1} \to H^{-1}$ *is strictly positive definite.*
(ii) *The operator* $A: H^{-1} \to H^{-1}$ *is strictly positive definite.*
(iii) *There exist two constants* ϱ_1 *and* ϱ_2, $0 < \varrho_1 < \varrho_2 < \infty$, *such that*

$$(8)\qquad \varrho_1(A^{1/2}\varphi,\varphi)_{H^{-1}} \le (B\varphi,\varphi)_{H^{-1}} \le \varrho_1(A^{1/2}\varphi,\varphi)_{H^{-1}}$$

for all $\varphi \in L^2$.

Proof. (i) For $\varphi \in H_0^1$, $\varphi \neq 0$, we have

$$(B\varphi, \varphi)_{H^{-1}} = \frac{\delta}{\sqrt{\varepsilon}} \|\varphi\|^2 + \frac{1}{\sqrt{\varepsilon}} \|\varphi\|_{H^{-1}}^2.$$

Thus from the embedding estimate

$$\|\varphi\|_{H^{-1}} \le K_1 \|\varphi\| \quad \text{for any } \varphi \in L^2 \tag{9}$$

we obtain

$$(B\varphi, \varphi)_{H^{-1}} \ge \left(\frac{\delta}{K_1^2\sqrt{\varepsilon}} + \frac{1}{\sqrt{\varepsilon}}\right) \|\varphi\|_{H^{-1}}^2 > 0.$$

(ii) Let $\varphi \in D(A)$ and $\varphi \neq 0$. Using the Poincaré inequality $\|\nabla\varphi\|^2 \ge \lambda_1 \|\varphi\|^2$, we obtain

$$(A\varphi, \varphi)_{H^{-1}} = \|(-\Delta)^{1/2}\varphi\|^2 \ge C\|\nabla\varphi\|^2 \ge C_1\|\varphi\|^2 \ge C_2\|\varphi\|_{H^{-1}}^2 > 0.$$

(iii) From the embedding estimate (9), for $\varphi \in L^2$, we obtain

$$(B\varphi, \varphi)_{H^{-1}} \le \frac{\delta + K_1^2}{\sqrt{\varepsilon}} \|\varphi\|^2 \quad \text{and} \quad (A^{1/2}\varphi, \varphi)_{H^{-1}} = (-\Delta\varphi, \varphi)_{H^{-1}} = \|\varphi\|^2,$$

so that inequality (8) holds with $\varrho_1 := \delta/\sqrt{\varepsilon}$ and $\varrho_2 := (\delta + K_1^2)/\sqrt{\varepsilon}$. ∎

Our next goal will be to show that A and B are self-adjoint. To this end, we introduce the differential operators $S_1 \colon H^{-1} \supset C^4 \cap C_0^2 \to H^{-1}$ and $S_2 \colon H^{-1} \supset C^2 \cap C_0 \to H^{-1}$, defined by

$$S_1\phi := (-\Delta)^2\phi, \quad \phi \in C^4 \cap C_0^2,$$

and

$$S_2\varphi := \frac{1}{\sqrt{\varepsilon}}(\delta(-\Delta) + I)\varphi, \quad \varphi \in C^2 \cap C_0.$$

It suffices to show that S_i is a symmetric operator in H^{-1}, strictly positive definite for $i = 1, 2$. Then there exists a unique, self-adjoint operator A_i such that $S_i \subset A_i$ (see [9, Section 8.10]). Since C_0^∞ is dense in L^2 and L^2 is dense in H^{-1}, we deduce that S_1 and S_2 have dense domains.

PROPOSITION 2.1. *The operators S_i, $i = 1, 2$, are symmetric and strictly positive definite.*

Proof. We just prove that S_i, $i = 1, 2$, are symmetric, because from Lemma 2.1 it follows that they are strictly positive definite. Integrating by parts, for $\phi, \varphi \in C^4 \cap C_0^2$ we obtain

$$\begin{aligned}(S_1\phi, \varphi)_{H^{-1}} &= (\Delta^2(-\Delta)^{-1/2}\phi, (-\Delta)^{-1/2}\varphi) \\ &= ((-\Delta)^{-1/2}\phi, \Delta^2(-\Delta)^{-1/2}\varphi) = (\phi, S_1\varphi)_{H^{-1}}.\end{aligned}$$

Using integration by parts again, for $\phi, \varphi \in C^2 \cap C_0$ we get

$$\begin{aligned}(S_2\phi, \varphi)_{H^{-1}} &= \frac{\delta}{\sqrt{\varepsilon}}((-\Delta)(-\Delta)^{-1/2}\phi, (-\Delta)^{-1/2}\varphi) + \frac{1}{\sqrt{\varepsilon}}(\phi, \varphi)_{H^{-1}} \\ &= \frac{\delta}{\sqrt{\varepsilon}}((-\Delta)^{-1/2}\phi, (-\Delta)(-\Delta)^{-1/2}\varphi) + \frac{1}{\sqrt{\varepsilon}}(\phi, \varphi)_{H^{-1}} \\ &= (\phi, S_2\varphi)_{H^{-1}}. \ \blacksquare\end{aligned}$$

We next show that the resolvent of $-\mathbf{A_B}$ is compact. Notice that for $u \in Y := \{\varphi \in H^{-1} : \varphi \in D(A),\ A\varphi \in D(B),\ B\varphi \in D(A)\}$ the operators A and B commute (i.e. $ABu = BAu$). It is easy to see that $Y \subset H^5$.

LEMMA 2.2. *If $AB = BA$ then for all $\lambda \in \varrho(-\mathbf{A_B})$ and sufficiently smooth functions we have*

(i) $(\lambda^2 I - \lambda B + A)^{-1}A = A(\lambda^2 I - \lambda B + A)^{-1}$,
(ii) $(\lambda^2 I - \lambda B + A)^{-1}(\lambda I - B) = (\lambda I - B)(\lambda^2 I - \lambda B + A)^{-1}$,
(iii) $A(\lambda I - B) = (\lambda I - B)A$.

Proof. If $\lambda = 0$ then the above equalities are obvious. Let $\lambda \neq 0$.

(i) We first show that

$$(\lambda^2 I - \lambda B + A)A = A(\lambda^2 I - \lambda B + A).$$

Indeed, from $AB = BA$ we obtain

$$\begin{aligned}(\lambda^2 I - \lambda B + A)A &= \lambda^2 A - \lambda BA + A^2 = \lambda^2 A - \lambda AB + A^2 \\ &= A(\lambda^2 I - \lambda B + A),\end{aligned}$$

hence

$$\begin{aligned}&(\lambda^2 I - \lambda B + A)^{-1}A \\ &\quad = (\lambda^2 I - \lambda B + A)^{-1}A(\lambda^2 I - \lambda B + A)(\lambda^2 I - \lambda B + A)^{-1} \\ &\quad = (\lambda^2 I - \lambda B + A)^{-1}(\lambda^2 I - \lambda B + A)A(\lambda^2 I - \lambda B + A)^{-1}.\end{aligned}$$

(ii) This property is a direct consequence of (i).

(iii) This is obvious. ∎

PROPOSITION 2.2. *The resolvent of $-\mathbf{A_B}$ is compact.*

Proof. From the properties of A and B we infer that for $\lambda \in \varrho(-\mathbf{A_B})$ the resolvent operator $(\lambda\mathbf{I} + \mathbf{A_B})^{-1}$ of $-\mathbf{A_B}$ is given by the formula

$$(\lambda\mathbf{I} + \mathbf{A_B})^{-1} = \begin{bmatrix} (\lambda I - B)(\lambda^2 I - \lambda B + A)^{-1} & -(\lambda^2 I - \lambda B + A)^{-1} \\ A(\lambda^2 I - \lambda B + A)^{-1} & \lambda(\lambda^2 I - \lambda B + A)^{-1} \end{bmatrix}.$$

For $(\phi,\varphi)^T \in H^1_0 \times H^{-1}$ we obtain

$$\begin{aligned}
&\|(\lambda \mathbf{I}+\mathbf{A_B})^{-1}[\phi,\varphi]^T\|_{H^3\times H^1}\\
&\qquad \le \frac{\delta}{\sqrt{\varepsilon}}\,\|(\lambda^2 I-\lambda B+A)^{-1}\phi\|_{H^5} + \left|\lambda-\frac{1}{\sqrt{\varepsilon}}\right| \|(\lambda^2 I-\lambda B+A)^{-1}\phi\|_{H^3}\\
&\qquad\quad + \|(\lambda^2 I-\lambda B+A)^{-1}\phi\|_{H^5}\\
&\qquad\quad + \|(\lambda^2 I-\lambda B+A)^{-1}\varphi\|_{H^3} + |\lambda|\,\|(\lambda^2 I-\lambda B+A)^{-1}\varphi\|_{H^1}\\
&\qquad \le \frac{\delta}{\sqrt{\varepsilon}}\,\|\phi\|_{H^1} + \left|\lambda-\frac{1}{\sqrt{\varepsilon}}\right| \|\phi\|_{H^{-1}} + \|\phi\|_{H^1} + \|\varphi\|_{H^{-1}} + |\lambda|\,\|\varphi\|_{H^{-3}}\\
&\qquad \le C\|(\phi,\varphi)^T\|_{H^1\times H^{-1}},
\end{aligned}$$

hence for any bounded subset $G \subset H^1_0 \times H^{-1}$ the set $(\lambda \mathbf{I}+\mathbf{A_B})^{-1}(G)$ is bounded in $H^3\times H^1$. Now, the compactness of the embedding $H^3\times H^1 \subset H^1_0\times H^{-1}$ implies that $-\mathbf{A_B}$ has compact resolvent. ∎

3. Local solutions and a priori estimates. Consider the semilinear Cauchy problem for the perturbed viscous Cahn–Hilliard equation

$$(10)\quad \begin{cases} u_{tt}+\dfrac{1}{\sqrt{\varepsilon}}\,u_t+\Delta\Big(\Delta u+f(u)-\dfrac{\delta}{\sqrt{\varepsilon}}\,u_t\Big)=0, & x\in\Omega,\ t>0,\\ u(0,x)=u_0(x),\quad u_t(0,x)=v_0(x), & x\in\Omega,\\ u(t,x)=0,\quad \Delta u(t,x)=0, & x\in\partial\Omega,\ t\ge 0,\end{cases}$$

where $\varepsilon,\delta\in(0,1]$, Ω is a nonempty, bounded, open subset of $\mathbb{R}^n$ for $n\le 3$, $\partial\Omega\in C^4$ and $f\in C^2(\mathbb{R},\mathbb{R})$. Then the problem (10) will be written in an abstract form in $X := H^1_0\times H^{-1}$ as

$$(11)\quad \frac{d}{dt}\begin{bmatrix} u\\ v\end{bmatrix} = \mathbf{A_B}\begin{bmatrix} u\\ v\end{bmatrix} + F(u,v),\qquad t>0,\qquad \begin{bmatrix} u\\ v\end{bmatrix}_{|t=0} = \begin{bmatrix} u_0\\ v_0\end{bmatrix},$$

where the operator $\mathbf{A_B}$ is given by formula (7) and the function $F\colon X^{1/2} := (H^2\cap H^1_0)\times L^2 \to X$ is defined as

$$(12)\qquad F(u,v)=\begin{bmatrix} 0\\ -\Delta(f(u))\end{bmatrix}.$$

Note that F is well defined. Indeed, taking $(u,v)^T\in X^{1/2}$, we have

$$(13)\quad \|F(u,v)\|_X = \|(-\Delta)f(u)\|_{H^{-1}} \le C_1\|\nabla f(u)\| = C_1\|f'(u)|\nabla u|\,\|.$$

Using the Hölder inequality and the embedding estimate

$$(14)\qquad \|u\|_{W^{1,6}} \le K_2\|u\|_{H^2},\qquad n\le 3,$$

we obtain

$$\|F(u,v)\|_X \le C_1\Big(\int_\Omega |f'(u)|^3\,dx\Big)^{1/3}\Big(\int_\Omega |\nabla u|^6\,dx\Big)^{1/6} \le C\|f'(u)\|_{L^\infty}\|u\|_{H^2}.$$

Thus, from the assumption that $f \in C^2(\mathbb{R},\mathbb{R})$ and the estimate

$$\|u\|_{L^\infty} \le K_3\|u\|_{H^2}, \quad n \le 4, \tag{15}$$

we deduce that the right-hand side of the last inequality is finite.

THEOREM 3.1. *Let* $(u_0, v_0) \in X^{1/2}$. *Then there exists a unique local solution* $(u,v)^T$ *of the problem* (11) *in* X, *defined on the maximal interval of existence* $(0,\tau_{\max})$ *and*

$$(u,v)^T \in C([0,\tau_{\max}), X^{1/2}) \cap C^1((0,\tau_{\max}), X) \cap C((0,\tau_{\max}), D(\mathbf{A_B})).$$

Proof. Since $-\mathbf{A_B}$ is a sectorial, positive operator, it suffices to show that $F\colon X^{1/2} \to X$ is Lipschitz continuous on bounded subsets of $X^{1/2}$ (see [7, Section 4.2]). Fix a bounded set $G \subset X^{1/2}$ and let $(u_1,v_1)^T, (u_2,v_2)^T \in G$. Then we have

$$\begin{aligned}\|F(u_1,v_1) - F(u_2,v_2)\|_X &= \|(-\Delta)(f(u_1) - f(u_2))\|_{H^{-1}}\\ &\le C_1(\|f'(u_1)|\nabla(u_1-u_2)|\,\| + \|(f'(u_1) - f'(u_2))|\nabla u_2|\,\|).\end{aligned}$$

Using the Hölder inequality, continuity of f' and the fact that for any $(u,v) \in G$, thanks to (15), there is a constant m such that $\|u\|_{L^\infty} \le m$, we have

$$\begin{aligned}\|F(u_1,v_1) - F(u_2,v_2)\|_X &\le C_1\Big(\int_\Omega |f'(u_1)|^2|\nabla(u_1-u_2)|^2\,dx\Big)^{1/2}\\ &\quad + C_1\Big(\int_\Omega |f''(\zeta)|^3|u_1-u_2|^3\,dx\Big)^{1/3}\|u_2\|_{W^{1,6}}\\ &\le \sup_{|s|\le m}|f'(s)|\,\|u_1-u_2\|_{H_0^1} + \sup_{|s|\le m}|f''(s)|\,\|u_1-u_2\|_{L^3}\|u_2\|_{W^{1,6}}.\end{aligned}$$

Consequently, from (14) and the assumption that $f \in C^2(\mathbb{R},\mathbb{R})$, we deduce

$$\|F(u_1,v_1) - F(u_2,v_2)\|_X \le C(G)\|u_1-u_2\|_{H^2}. \ \blacksquare$$

Throughout the remainder of this section we need a condition on the nonlinear term f weaker than (iv), that is,

$$|f(s)| \le \widetilde{C}(1 + |s|^{q+1}), \qquad s \in \mathbb{R}, \tag{16}$$

where $q > 0$ can be arbitrarily large. Moreover, assume from now on the dissipativity conditions

$$\exists_{\sigma \ge (2K_1^2+1)/(3\sqrt{\varepsilon})}\ \exists_{C_\sigma \in \mathbb{R}^+}\ \forall_{s\in\mathbb{R}} \qquad sf(s) - \frac{4}{3}\,\overline{F}(s) \le -\sigma s^2 + C_\sigma, \tag{17}$$

where K_1 was introduced in (9), and

$$\exists_{\overline{C}\in\mathbb{R}}\ \forall_{s\in\mathbb{R}}\quad \overline{F}(s) := \int_0^s f(z)\,dz \le \overline{C}. \tag{18}$$

Denote by $\langle\cdot,\cdot\rangle_{H^{-1}\times H_0^1}$ the duality pairing between H^{-1} and H_0^1, and for $u, v \in H^{-1}$ set

$$[u,v] := \langle v, (-\Delta)^{-1}u\rangle_{H^{-1}\times H_0^1}. \tag{19}$$

Our next goal will be to investigate the behavior of the Lyapunov type functional $\Phi_0\colon X^{1/2}\to\mathbb{R}$ connected with (10) and defined by

$$\Phi_0(u,v) = E_0(u,v) + \frac{\delta}{2\sqrt{\varepsilon}}\,\|u\|^2 - 2\int_\Omega \overline{F}(u)\,dx, \tag{20}$$

where

$$E_0(u,v) = \|v\|^2_{H^{-1}} + [u,v] + \frac{1}{2\sqrt{\varepsilon}}\,\|u\|^2_{H^{-1}} + \|u\|^2_{H_0^1}, \tag{21}$$

and derive uniform in time estimates of local solutions to (11) in X. Notice that the square root of E_0 defines an equivalent norm on X. We infer from (16) that the functional Φ_0 is well defined. It is easy to check that it is bounded from below. Indeed, by (18) and (20), we obtain

$$\Phi_0(u,v) \ge -2\int_\Omega \overline{F}(u)\,dx \ge -2\overline{C}|\Omega| =: -M_0. \tag{22}$$

Now we estimate Φ_0 from above.

Lemma 3.1. *Under the assumptions* (17) *and as long as a local solution* $(u,v)^T$ *to* (11) *exists, we have*

$$\Phi_0(u(t),v(t)) \le \Big(\Phi_0(u_0,v_0) - \frac{3}{2}\,M_1\Big)e^{-2t/3} + \frac{3}{2}\,M_1, \tag{23}$$

where M_1 is a positive constant.

Proof. Consider the equation formally obtained by applying $(-\Delta)^{-1}$ to (10), i.e.

$$(-\Delta)^{-1}u_{tt} + \frac{1}{\sqrt{\varepsilon}}\,(-\Delta)^{-1}u_t + (-\Delta)u - f(u) + \frac{\delta}{\sqrt{\varepsilon}}\,u_t = 0. \tag{24}$$

Multiplying (24) in L^2 first by $2u_t$, then by u we obtain

$$\frac{d}{dt}\Big(\|u_t\|^2_{H^{-1}} + \|u\|^2_{H_0^1} - 2\int_\Omega \overline{F}(u)\,dx\Big) + \frac{2}{\sqrt{\varepsilon}}\,\|u_t\|^2_{H^{-1}} + \frac{2\delta}{\sqrt{\varepsilon}}\,\|u_t\|^2 = 0 \tag{25}$$

and

$$\frac{d}{dt}\Big([u,u_t] + \frac{1}{2\sqrt{\varepsilon}}\,\|u\|^2_{H^{-1}} + \frac{\delta}{2\sqrt{\varepsilon}}\,\|u\|^2\Big) - \|u_t\|^2_{H^{-1}} + \|u\|^2_{H_0^1} - \int_\Omega f(u)u\,dx = 0.$$

Adding these identities and recalling (20), we get

$$\frac{d}{dt}\Phi_0(u,u_t) + \frac{2-\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\,\|u_t\|^2_{H^{-1}} + \frac{2\delta}{\sqrt{\varepsilon}}\,\|u_t\|^2 + \|u\|^2_{H^1_0} - \int_\Omega f(u)u\,dx = 0,$$

but since $\varepsilon \le 1$ and $\delta > 0$, we have

$$\frac{d}{dt}\Phi_0(u,u_t) \le -\|u_t\|^2_{H^{-1}} - \|u\|^2_{H^1_0} + \int_\Omega f(u)u\,dx. \tag{26}$$

Further, we deduce from (20), (21) and $[u,u_t] \le \frac{1}{2}\|u\|^2_{H^{-1}} + \frac{1}{2}\|u_t\|^2_{H^{-1}}$ that

$$\frac{2}{3}\,\Phi_0(u,u_t) \le \|u_t\|^2_{H^{-1}} + \frac{1+\sqrt{\varepsilon}}{3\sqrt{\varepsilon}}\,\|u\|^2_{H^{-1}} + \frac{2}{3}\,\|u\|^2_{H^1_0} + \frac{1}{3\sqrt{\varepsilon}}\,\|u\|^2 - \frac{4}{3}\int_\Omega \overline{F}(u)\,dx. \tag{27}$$

Adding (26) and (27), we get

$$\frac{d}{dt}\Phi_0(u,u_t) + \frac{2}{3}\,\Phi_0(u,u_t) \le \frac{2K_1^2+1}{3\sqrt{\varepsilon}}\,\|u\|^2 + \int_\Omega f(u)u\,dx - \frac{4}{3}\int_\Omega \overline{F}(u)\,dx.$$

From the dissipativity condition (17) it follows that

$$\frac{d}{dt}\Phi_0(u,u_t) + \frac{2}{3}\,\Phi_0(u,u_t) \le C_\sigma|\Omega| =: M_1. \tag{28}$$

Integrating the last inequality over $[0,t]$, we obtain (23). ■

COROLLARY 3.1. *Under the assumptions* (16)–(18) *and as long as a local solution* $(u,v)^T$ *to* (11) *exists, we have*

$$\|(u,v)^T\|_X \le c(\|(u_0,v_0)^T\|_{X^{1/2}}),$$

where $c\colon [0,\infty) \to [0,\infty)$ *is a locally bounded function.*

Proof. From Lemma 3.1 we obtain

$$E_0(u(t),v(t)) \le \left(\Phi_0(u_0,v_0) - \frac{3}{2}\,M_1\right)e^{-2t/3} + \frac{3}{2}\,M_1 + M_0. \tag{29}$$

Since $u \in H^2$ conditions (15) and (16) give

$$\left|\int_\Omega uf(u)\,dx\right| \le \widetilde{C}(\|u\|_{L^1} + \|u\|^{q+2}_{L^{q+2}}),$$

hence from (17), recalling that $\sigma > 0$, we have

$$-2\int_\Omega \overline{F}(u)\,dx \le \frac{3}{2}\,(\widetilde{C}\|u\|_{L^1} + \widetilde{C}\|u\|^{q+2}_{L^{q+2}} + M_1),$$

so that

$$\Phi_0(u,v) \le E_0(u,v) + \frac{\delta}{2\sqrt{\varepsilon}}\,\|u\|^2 + \frac{3}{2}\,(\widetilde{C}\|u\|_{L^1} + \widetilde{C}\|u\|^{q+2}_{L^{q+2}} + M_1).$$

From (15), (29) and the last inequality we deduce that

$$(30)\qquad \begin{aligned} &E_0(u(t),v(t))\\ &\quad\le \Big(E_0(u_0,v_0)+\frac{\delta}{2\sqrt{\varepsilon}}\,\|u_0\|^2+\frac{3}{2}\,\widetilde{C}(\|u_0\|_{L^1}+\|u_0\|_{L^{q+2}}^{q+2})\Big)e^{-2t/3}\\ &\qquad+\frac{3}{2}\,M_1+M_0\\ &\quad\le C_1(\|(u_0,v_0)^T\|_{H^2\times L^2}^2+\|u_0\|_{H^2}^2+\|u_0\|_{H^2}+\|u_0\|_{H^2}^{q+2})e^{-2t/3}\\ &\qquad+\frac{3}{2}\,M_1+M_0, \end{aligned}$$

since the square root of E_0 defines an equivalent norm on X. ∎

4. Global solutions. Under an additional growth restriction on the derivative of f local solutions will now be extended to global ones.

THEOREM 4.1. *Under assumptions* (17), (18) *and the growth restriction*

$$(31)\qquad |f'(s)|\le \widehat{C}(1+|s|^q),\qquad s\in\mathbb{R},$$

where q can be arbitrarily large if $n=1,2$, and $0<q<2$ if $n=3$, a local solution to (11) *exists globally in time.*

Proof. Note that for every $s\ge 1/2q$ and $r\ge 1$ if $n=1,2$, and for every $s\in[1/2q,3/q]$ and $r\in[1,3)$ if $n=3$, we have

$$(32)\qquad \|u\|_{L^{2sq}}\le K_4\|u\|_{H_0^1}\quad \text{for } u\in H_0^1,$$

and

$$(33)\qquad \|u\|_{W^{1,2r}}\le \check{C}\|u\|_{H^2}^{\eta}\|u\|_{H_0^1}^{1-\eta}\quad \text{for } u\in H^2\cap H_0^1,$$

with some $\eta\in[0,1)$. By (13), (31) we get

$$\|F(u,v)\|_X\le C_1\Big[\Big(\int_\Omega |\nabla u|^2\,dx\Big)^{1/2}+\Big(\int_\Omega |u|^{2q}|\nabla u|^2\,dx\Big)^{1/2}\Big].$$

Using the Hölder inequality with $s>\max\{1/2q,1\}$ if $n=1,2$, and $s=3/q$ if $n=3$ ($r=s/(s-1)$), we obtain

$$\|F(u,v)\|_X\le C_1(\|u\|_{H_0^1}+\|u\|_{L^{2sq}}^q\|u\|_{W^{1,2r}}).$$

Consequently, from (32) and (33),

$$\begin{aligned} \|F(u,v)\|_X&\le C\max\{\|u\|_{H_0^1},\|u\|_{H_0^1}^{q+1-\eta}\}(1+\|u\|_{H^2}^{\eta})\\ &\le g(\|(u,v)^T\|_X)(1+\|(u,v)^T\|_{X^{1/2}}^{\eta}), \end{aligned}$$

where $g\colon[0,\infty)\to[0,\infty)$ is some nondecreasing function, so that any local solution to (11) exists globally in time (see [3, Theorem 3.1.1]). ∎

Denote by $\{\mathcal{T}(t)\}$ the C^0 semigroup of global solutions to (11), which is defined on $X^{1/2} = (H^2 \cap H_0^1) \times L^2$ by the relation

$$\mathcal{T}(t)(u_0, v_0) = (u(t), v(t)), \quad t \geq 0.$$

THEOREM 4.2. *The semigroup $\{\mathcal{T}(t)\}$ has a global attractor $\mathcal{A}$ in $X^{1/2}$.*

Proof. Since the resolvent of $\mathbf{A_B}$ is compact, we know (see [3, Theorem 3.3.1]) that the semigroup is compact. If we show that $\{\mathcal{T}(t)\}$ is point dissipative, then $\{\mathcal{T}(t)\}$ will have a global attractor in $X^{1/2}$ (see [3, Corollary 1.1.6]). To this end, it suffices to prove (see [3, Corollary 4.1.4]) that for all $(u_0, v_0) \in X^{1/2}$,

$$\limsup_{t\to\infty} \|(u,v)\|_X \leq \frac{3}{2} M_1 + M_0,$$

where M_0 and M_1 are the constants from (22) and (28), respectively. Note that this inequality follows directly from (30). ∎

4.1. *Geometric structure of the global attractor.* Following [4, Section 1.6] we now study the structure of the global attractor for the semigroup $\{\mathcal{T}(t)\}$. To this end, we discuss the properties of the Lyapunov type functional $\Phi_1 \colon X^{1/2} \to \mathbb{R}$ defined as

$$\Phi_1(u,v) = \|v\|_{H^{-1}}^2 + \|u\|_{H_0^1}^2 - 2\int_\Omega \overline{F}(u)\,dx. \tag{34}$$

PROPOSITION 4.1.

(i) *Φ_1 is bounded from below.*
(ii) *Φ_1 is continuous.*
(iii) *For each $(u_0, v_0) \in X^{1/2}$ the function $0 < t \mapsto \Phi_1(\mathcal{T}(t)(u_0, v_0))$ is nonincreasing.*
(iv) *If $\Phi_1(\mathcal{T}(t)(u_0, v_0)) = \Phi_1(u_0, v_0)$ for all $t > 0$ and some $(u_0, v_0) \in X^{1/2}$ then $\mathcal{T}(t)(u_0, v_0) = (u_0, v_0)$ for all $t > 0$.*

Proof. (i) We show that Φ_1, like Φ_0, is bounded from below by $-M_0$. Indeed, by (18), (34) and the definition of M_0 (see (22)) we obtain

$$\Phi_1(u,v) \geq -2\int_\Omega \overline{F}(u)\,dx \geq -M_0.$$

(ii) Let $(u,v), (u_n, v_n) \in X^{1/2}$ be such that $\|(u_n - u, v_n - v)\|_{X^{1/2}} \to 0$ as $n \to \infty$, hence we may assume that $\|u_n\|_{L^\infty}, \|u\|_{L^\infty} \leq M$. Since

$$|\Phi_1(u_n, v_n) - \Phi_1(u,v)| \leq \|v_n - v\|_{H^{-1}}(\|v_n\|_{H^{-1}} + \|v\|_{H^{-1}}) + \|u_n - u\|_{H_0^1}(\|u_n\|_{H_0^1} + \|u\|_{H_0^1}) + 2\int_\Omega |\overline{F}(u_n) - \overline{F}(u)|\,dx,$$

it suffices to show that $\int_\Omega |\overline{F}(u_n) - \overline{F}(u)|\,dx \to 0$ as $n \to \infty$. From (16) we have

$$\int_\Omega |\overline{F}(u_n(x)) - \overline{F}(u(x))|\,dx \le \int_\Omega \Big| \int_0^{u_n(x)} f(s)\,ds - \int_0^{u(x)} f(s)\,ds \Big|\,dx$$
$$\le \int_\Omega \Big| \int_{u(x)}^{u_n(x)} |f(s)|\,ds \Big|\,dx \le \int_\Omega \Big| \int_{u(x)}^{u_n(x)} (1+|s|^q)\,ds \Big|\,dx$$
$$\le |\Omega| \sup_{|s|\le M} (1+|s|^q)\|u_n - u\|_{L^\infty}.$$

(iii) For $(u_0, v_0) \in X^{1/2}$ from (25) and the definition of the semigroup $\{\mathcal{T}(t)\}$, we deduce that

$$\frac{d}{dt}\Phi_1(u(t), u_t(t)) = -\frac{2}{\sqrt{\varepsilon}}\,\|u_t\|^2_{H^{-1}} - \frac{2\delta}{\sqrt{\varepsilon}}\,\|u_t\|^2 \le 0.$$

(iv) Let $(u_0, v_0) \in X^{1/2}$ be such that $\Phi_1(\mathcal{T}(t)(u_0, v_0)) = \Phi_1(u_0, v_0)$ for $t > 0$. Then from (25) we obtain

$$0 = \frac{d}{dt}\Phi_1(\mathcal{T}(t)(u_0, v_0)) = -\frac{2}{\sqrt{\varepsilon}}\,\|u_t\|^2_{H^{-1}} - \frac{2\delta}{\sqrt{\varepsilon}}\,\|u_t\|^2,$$

but the left hand side is independent of t, hence $\|u_t\|_{H^{-1}} = \|u_t\| = 0$, so that $u_t(t,x) = 0$ a.e. for $t > 0$. ∎

Let $\mathcal{N}$ be the set of equilibrium points for the semigroup $\{\mathcal{T}(t)\}$, i.e.

$$\mathcal{N} = \{(\varphi, \phi) \in X^{1/2} : \mathcal{T}(t)(\varphi, \phi) = (\varphi, \phi) \text{ for } t \ge 0\}.$$

We define the unstable manifold $\mathcal{M}(\mathcal{N})$ emanating from the set $\mathcal{N}$ as the set of all $(u_0, v_0) \in X^{1/2}$ such that there exists a full trajectory $\gamma = \{(u(t), v(t)) \colon t \in \mathbb{R}\}$ with the properties

$$(u(0), v(0)) = (u_0, v_0) \quad \text{and} \quad \lim_{t\to-\infty} \operatorname{dist}_{X^{1/2}}((u(t), v(t)), \mathcal{N}) = 0.$$

PROPOSITION 4.2. *We have $\mathcal{A} = \mathcal{M}(\mathcal{N})$. Moreover, the global attractor consists of full trajectories $\gamma = \{(u(t), v(t)) \colon t \in \mathbb{R}\}$ such that*

$$\lim_{t\to\infty} \operatorname{dist}_{X^{1/2}}((u(t), v(t)), \mathcal{N}) = 0 \quad \textit{and} \quad \lim_{t\to-\infty} \operatorname{dist}_{X^{1/2}}((u(t), v(t)), \mathcal{N}) = 0.$$

Proof. This follows directly from [4, Theorem 6.1] and Proposition 4.1. ∎

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