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SMOOTH CANTOR FUNCTIONS

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Abstract. We characterise the set on which an infinitely differentiable function can be locally polynomial.

1. Introduction. Donoghue [1] has shown that there exists a smooth non-polynomial function $f:\mathbb{R}\to\mathbb{R}$ having the property that every interval contains a subinterval upon which f coincides with a polynomial. In this paper we characterise the sets where a smooth function can be locally polynomial in this manner. I have written this note so that it may be read independently of [1] but, as might be expected, the reader who consults that paper will find substantial overlaps. The reader must decide if the title of this paper is appropriate.

We make the following definitions.

Definition 1. A function $f:\mathbb{R}\to\mathbb{R}$ is said to be *real-analytic* at a point x if we can find a $\delta>0$ such that f has a power series expansion

$$f(x+h)=\sum_{r=0}^{\infty}a_r h^r$$

valid for $|h|<\delta$. We say that f is *locally polynomial* at x if, in addition, we can find an N such that

$$f(x+h)=\sum_{r=0}^{N}a_r h^r$$

for all $|h|<\delta$.

The following result goes back, effectively, to Du Bois-Reymond.

Theorem 2. *If E is closed, we can find an infinitely differentiable function $f:\mathbb{R}\to\mathbb{R}$ such that f is not real-analytic at each point E but is real-analytic at each point of its complement.*

Note that the set of points where a function is real-analytic must be open. There is a substantial literature dealing with this phenomenon. The paper [2] provides a particularly deep account.

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The object of this note is to prove the following result.

THEOREM 3. *Given a closed subset E of $\mathbb{R}$ with no isolated points, we can find an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ which is not real-analytic at each point of E but is locally polynomial at each point of its complement.*

The following observations explain why Theorem 3 takes the form it does.

LEMMA 4. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable. Let E be the set where f is not locally polynomial. Then:*

(i) *E is closed.*
(ii) *E contains no isolated points.*
(iii) *If x is a frontier point of E (that is to say, $x \in E \cap \operatorname{Cl}(\mathbb{R} \setminus E)$), then f is not real-analytic at x.*
(iv) *Suppose that E has empty interior. Then, if $x \in E$, we can find $x_j \in E$ and $n_j \to \infty$ such that $f^{(n_j)}(x_j) \neq 0$ and $x_j \to x$ as $j \to \infty$.*

Proof. (i) Direct from definition.

(ii) Write $U = \mathbb{R} \setminus E$. Suppose that $f(t) = P(t)$ for some polynomial P on an open interval I and $f(t) = Q(t)$ for some polynomial Q on an open interval J. If $I \cap J \neq \emptyset$ then, since $I \cap J$ is an open interval, $P = Q$ and $f(t) = P(t)$ on $I \cup J$. Thus, by standard arguments, if $f(t) = P(t)$ for some polynomial P on an open interval I and L is an open interval with $I \subseteq L \subseteq U$, we have $f(t) = P(t)$ on L.

Suppose that x does not lie in the closure of $E \setminus \{x\}$. Then we can find a $\delta > 0$ such that

$$(x - \delta, x), (x, x + \delta) \subseteq U$$

and polynomials P and Q such that $f(t) = P(t)$ for $t \in (x - \delta, x)$ and $f(t) = Q(t)$ for $t \in (x, x + \delta)$. Since f is infinitely differentiable, all its derivatives are continuous and

$$P^{(r)}(x) = f^{(r)}(x) = Q^{(r)}(x)$$

for all r. Thus $P = Q$ and $f(t) = P(t)$ for $t \in (x - \delta, x + \delta)$.

(iii) Suppose that $x \in \operatorname{Cl} U$ and there exists a $\delta > 0$ such that the power series

$$\sum_{r=0}^{\infty} a_r (t - x)^r$$

converges to $f(t)$ for all $|t - x| < \delta$. Choose $y \in U$ such that $|y - x| < \delta/2$. We can find an open interval J containing y and a polynomial P such that $f = P$ on J. By the uniqueness of power series, $f = P$ on $(x - \delta, x + \delta)$ so $x \in U$.

(iv) Suppose that x is such that we cannot find $x_j \in E$ and $n_j \to \infty$ with $f^{(n_j)}(x_j) \neq 0$ and $x_j \to x$ as $j \to \infty$. Then we can find a $\delta > 0$ and N such that, if $t \notin E$ and $|t-x| < \delta$, we have $f^{(n)}(t) = 0$ for all $n \geq N$. Since E has empty interior, it follows, by continuity, that $f^{(n)}(t) = 0$ for all $n \geq N$ and $|t-x| < \delta$. By repeated use of the mean value theorem, there is a polynomial P of degree at most $N-1$ such that $f(t) = P(t)$ for $|t-x| < \delta$ and so $x \notin E$. ∎

We shall also prove the following result.

THEOREM 5. *If U is a non-empty open subset of $\mathbb{R}$, we can find an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $x \notin U$ and a set $H \subset U$ with the following properties*:

(i) *$U \setminus H$ has Lebesgue measure zero.*
(ii) *If $x \in H$, then we can find an integer $N(x)$ with $f^{(n)}(x) = 0$ for all $n \geq N(x)$.*
(iii) *f is not locally polynomial at any point of U.*

This gives another proof of Theorem 2.

In order to make the proof of Theorem 5 as different as possible from the usual proof, we avoid the use of functions like $\exp(-1/x^2)$ and use instead a "stitching method" based on Lemma 7.

2. Main proof. In this section we prove the following version of Theorem 3. We use the notation $g|A$ to mean the restriction of the function g to a set A.

THEOREM 6. *Given a non-trivial closed subset E of $[0,1]$ with no isolated points and empty interior, we can find an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = 0$ for $x \notin [0,1]$, which is not real-analytic at each point of E but is locally polynomial at each point of its complement.*

The main point of difference between Theorem 3 and Theorem 6 is that, in Theorem 6, we suppose that E has empty interior. However, this is the interesting case and it should be fairly clear that there must be a number of *ad hoc* ways of getting from Theorem 6 to Theorem 3. We shall sketch one of them in the final section.

We need the following lemma which the reader may quite properly dismiss as trivial.

LEMMA 7.

(i) *Given an integer $n \geq 0$ and an interval $[a,b]$, we can find a constant K with the following property*: *Given α_j, $\beta_j \in \mathbb{R}$ with $|\alpha_j|$, $|\beta_j| \leq 1$, we can find a real-polynomial P of degree at most $2n+1$ such that*

$$P^{(j)}(a) = \alpha_j, \quad P^{(j)}(b) = \beta_j \quad \text{and} \quad |P^{(j)}(t)| \leq K$$

for all $t \in [a,b]$ and $0 \leq j \leq n$.

(ii) *Given an integer $n \geq 0$, $\eta > 0$, an interval $[a, b]$ and $\alpha, \beta \in \mathbb{R}$, we can find a real polynomial Q such that*

$$Q^{(n)}(a) = \alpha, \qquad Q^{(n)}(b) = \beta$$

but

$$Q^{(j)}(a) = Q^{(j)}(b) = 0 \quad \textit{and} \quad |Q^{(j)}(t)| \leq \eta$$

for all $t \in [a, b]$ and $0 \leq j \leq n - 1$.

(iii) *Given an integer $n \geq 0$, $\eta > 0$, and an interval $[a, b]$ we can find a real polynomial R of degree exactly $2n + 2$ such that*

$$R^{(j)}(a) = R^{(j)}(b) = 0 \quad \textit{and} \quad |R^{(j)}(t)| \leq \eta$$

for all $t \in [a, b]$ and $0 \leq j \leq n$.

Proof. By translation and rescaling we may take $a = 0$ and $b = 1$.

(i) It is sufficient to prove the result (with a different value of K) when $\beta_j = 0$ for all $0 \leq j \leq n$. Set $P_r(x) = x^r(1 - x)^{n+1}$ for $0 \leq r \leq n$ and observe that the matrix $(P_r^{(s)}(0))_{0 \leq r \leq n}^{0 \leq s \leq n}$ is triangular with non-zero diagonal elements. It follows that there exists a $\widetilde{K}$ such that, if $|\alpha_j| \leq 1$ for $0 \leq j \leq n$, we can find A_j with $|A_j| \leq \widetilde{K}$ and

$$\sum_{r=0}^{n} A_r P_r^{(s)}(0) = \alpha_s$$

for $0 \leq s \leq n$. Setting $P = \sum_{r=0}^{n} A_r P_r$ we see that

$$P^{(j)}(0) = \alpha_j, \qquad P^{(j)}(1) = 0$$

and

$$|P^{(j)}(t)| \leq (n + 1)\widetilde{K} \sup_{0 \leq r \leq n} \sup_{x \in [0,1]} |P_r^{(j)}(x)|$$

for all $t \in [0, 1]$ and all $0 \leq j \leq n$.

(ii) Let N be a large integer to be chosen later. Let $h(x) = \sin(Nx - n\pi/2)$ and set $g(x) = N^{-n}\alpha(1 - x)^{n+1}h(x)$. Then

$$g^{(n)}(0) = \alpha, \qquad g^{(j)}(1) = 0 \quad \text{for } 0 \leq j \leq n$$

and there is a constant A independent of N such that

$$|g^{(j)}(0)| \leq AN^{-1} \qquad \text{for } 0 \leq j \leq n - 1.$$

By considering the Taylor expansion of g we know that there is a polynomial G such that

$$|g^{(j)}(t) - G^{(j)}(t)| \leq AN^{-1}$$

for all $t \in [0, 1]$ and all $0 \leq j \leq n$.

Thus

$$\begin{aligned}|G^{(n)}(0)-\alpha| &\leq AN^{-1},\\ |G^{(j)}(0)| &\leq 2AN^{-1} \quad \text{for } 0\leq j\leq n-1,\\ |G^{(j)}(1)| &\leq AN^{-1} \quad \text{for } 0\leq j\leq n.\end{aligned}$$

By part (i) we can find a polynomial P with

$$\begin{aligned}Q^{(n)}(0) &= G^{(n)}(0)-\alpha,\\ P^{(j)}(0) &= G^{(j)}(0) \quad \text{for } 0\leq j\leq n-1,\\ Q^{(j)}(1) &= G^{(j)}(1) \quad \text{for } 0\leq j\leq n\end{aligned}$$

and

$$|P^{(j)}(t)| \leq 2KAN^{-1}$$

for all $t\in[0,1]$ and all $0\leq j\leq n$. If we set $Q=G-P$ and take N large enough, the required result follows.

(iii) Just set $R(t)=\varepsilon t^{n+1}(1-t)^{n+1}$ with ε sufficiently small but nonzero. ∎

Proof of Theorem 6. By rescaling, we may suppose $0,1\in E$. Standard results on topology show that $[0,1]\setminus E$ is the countable union $\mathcal{U}$ of disjoint open intervals $U_1, U_2, \dots$. Since E has no isolated points and empty interior, the U_r cannot share endpoints and cannot have 0 or 1 as endpoints. Thus

$$[0,1]\setminus\bigcup_{r=1}^{n}U_r=\bigcup_{r=0}^{n}J_{n,r}$$

where $J_{n,r}=[a_{n,r},b_{n,r}]$ and

$$0=a_{n,0}<b_{n,0}<a_{n,1}<b_{n,1}<a_{n,2}<\dots<b_{n,n-1}<a_{n,n}<b_{n,n}=1.$$

We take $f_0=0$ and $J_{0,0}=[0,1]$. We construct inductively functions $f_n:\mathbb{R}\to\mathbb{R}$ such that

(i)$_n$ $f_n|U_r$ is a polynomial for all $1\leq r\leq n$, $f_n|J_{n,r}$ is a polynomial for all $0\leq r\leq n$ and $f(x)=0$ for all $x\notin[0,1]$,
(ii)$_n$ f_n has a continuous nth derivative.

Suppose f_n has been constructed. Using Lemma 7 applied to the various intervals U_r and $J_{n+1,s}$ we can find a function $f_{n+1}:\mathbb{R}\to\mathbb{R}$ such that

(i)$_{n+1}$ $f_{n+1}|U_r$ is a polynomial for all $1\leq r\leq n+1$, $f|J_{n,r}$ is a polynomial for all $0\leq r\leq n+1$ and $f_{n+1}(x)=0$ for all $x\notin[0,1]$,
(ii)$_n$ f_{n+1} has a continuous $(n+1)$st derivative,

and, in addition,

(iii)$_{n+1}$ $f_{n+1}|U_r=f_n|U_r$ for $1\leq r\leq n$,
(iv)$_{n+1}$ $f_{n+1}|U_{n+1}$ is a polynomial of degree at least $n+1$,

whilst

$(\mathrm{v})_{n+1}$ $|f_{n+1}^{(r)}(x) - f_n^{(r)}(x)| \le 2^{-n}$ for all $x \in \mathbb{R}$, $0 \le r \le n$.

Now condition $(\mathrm{v})_n$ tells us that $f_n^{(r)}$ converges uniformly for each r and so f_n converges to an infinitely differentiable function f. Condition $(\mathrm{iii})_r$ combined with condition $(\mathrm{iii})_n$ tells us that $f|U_r$ is a polynomial of degree at least r.

If $x \in E$, then, since E has no interior and no isolated points, it follows that given any $\delta > 0$ and N we can find an $n \ge N$ and a $U_n \subseteq (x-\delta, x+\delta)$. Since $F|U_n$ is a polynomial of degree at least n, our standard arguments show that f cannot be real-analytic at x. ∎

3. Final remarks. We note an immediate consequence of Theorem 3.

LEMMA 8. *Given $a \in \mathbb{R}$, $N \ge 0$ and $\delta > 0$ we can find an infinitely differentiable function $g : \mathbb{R} \to \mathbb{R}$ and a set E of Lebesgue measure* 0 *with the following properties*:

(i) *$g(x) = 0$ whenever $|x-a| \ge \delta$.*
(ii) *If $x \notin E$, then there exists an $M(x)$ such that $g^{(m)}(x) = 0$ for all $m \ge M(x)$.*
(iii) *There exists an $m \ge N$ such that $g^{(m)}(a) \ne 0$.*

Proof. Choose a non-empty closed set $\widetilde{E}$ of Lebesgue measure zero (so, automatically, with empty interior) with no isolated points lying in $[0,1]$. By Theorem 6, we can find an infinitely differentiable function $\widetilde{g}$ with the following properties:

(i) $\widetilde{g}(x) = 0$ for $x \notin [0,1]$.
(ii) If $x \notin \widetilde{E}$ then $\widetilde{g}$ is locally polynomial at x and so in particular there exists an $M(x)$ such that $g^{(m)}(x) = 0$ for all $m \ge M(x)$.

Since $\widetilde{E}$ is non-empty, Lemma 4(iv) tells us that there exists a $b \in [0,1]$ and an $m \ge N$ such that $\widetilde{g}^{(m)}(b) \ne 0$. The required result follows by translation and dilation. ∎

We can now prove Theorem 5.

Proof of Theorem 5. Choose a countable dense subset $q_1, q_2, \ldots$ of U (without repeating points). Choose $\delta_j > 0$ so that $q_k \notin (q_j - 2\delta_j, q_j - 2\delta_j)$ for $1 \le k \le j-1$, $(q_j - 2\delta_j, q_j - 2\delta_j) \subseteq U$ and $\delta_j < 2^{-j}$. We now take $f_0 = 0$ and define f_j inductively as follows: If $f_{j-1}^{(m_j)}(q_j) \ne 0$ for some $m_j \ge j$ set $f_j = f_{j-1}$. If $f_{j-1}^{(m)}(q_j) = 0$ for all $m \ge j$ then, by Lemma 8, we can find a smooth function $g_j : \mathbb{R} \to \mathbb{R}$ and a set E_j of Lebesgue measure 0 such that:

(i) $g(x) = 0$ whenever $|x - q_j| \geq \delta_j$.
(ii) If $x \notin E$ then there exists an $M(x)$ such that $g^{(m)}(x) = 0$ for all $m \geq M(x)$.
(iii) There exists an $m_j \geq j$ such that $g^{(m_j)}(q_j) \neq 0$.

Now choose an $\varepsilon_j > 0$ with

$$\varepsilon_j |g_j^{(k)}(t)| \leq 2^{-j}$$

for all $t \in \mathbb{R}$ and all $0 \leq k \leq j$ and set $f_j = f_{j-1} + \varepsilon_j g_j$.

By the general principle of uniform convergence, all the derivatives of f_j converge uniformly and f_j converges uniformly to an infinitely differentiable function f. We note that $f_j(x) = 0$ and so $f(x) = 0$ for all $x \notin U$. Since

$$f_k^{(m_j)}(q_j) = f_j^{(m_j)}(q_j) \neq 0$$

for all $k \geq j$, we have $f^{(m_j)}(q_j) \neq 0$. Since the q_j are dense and $m_j \to \infty$, f cannot be locally polynomial at any point of U.

Suppose now that x is a point such that there does not exist an M such that $g^{(m)}(x) = 0$ for all $m \geq M$. If $x \notin \bigcup_{j=1}^{\infty} E_j = E$, say, then we know that for each j there exists an $N(j)$ such that $f_j^{(m)}(x) = 0$ for all $m \geq N(j)$. Thus $x \in \operatorname{supp}(f_j - f_{j-1})$ for infinitely many j and so

$$x \in \bigcap_{r=1}^{\infty} \bigcup_{s=r}^{\infty} [q_s - \delta_s, q_s + \delta_s] = F,$$

say. Elementary measure theory tells us that E and F have measure zero, so we are done. ∎

Theorem 3 can be proved in a similar manner:

Sketch proof of Theorem 3. If E is a closed set without isolated points we can write $E = E_0 \cup U$ where U is open and E_0 is a closed set without isolated points and with empty interior. (Note that E_0 may not be disjoint from U.) By Theorem 6 we can find an infinitely differentiable function f which is locally polynomial at each $x \notin E_0$ and is not locally polynomial at each $x \in E_0$. An inductive construction along the lines of the proof of Theorem 5 followed by a limiting argument produces a function f with the required properties. ∎

Our results generalise to higher dimensions though the proofs now seem to require the use of smooth partitions of unity.

LEMMA 9. *Suppose that E is a closed subset of $\mathbb{R}^m$ whose complement has connected open components $U_1, U_2, \ldots$ with the property that*

$$\mathrm{Cl}(U_j) \cap \mathrm{Cl}(U_k) = \emptyset$$

for $j \neq k$*. Then we can find an infinitely differentiable function* $f : \mathbb{R}^m \to \mathbb{R}$ *such that* $f|U_j = P_j|U_j$ *for some multinomial* P_j $[j \geq 1]$ *and* $P_j \neq P_k$ *when* $j \neq k$*.*

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