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ON THE RADIUS OF CONVEXITY FOR A CLASS OF CONFORMAL MAPS

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Abstract. Let $\mathcal{A}$ denote the class of all analytic functions f in the open unit disc $\mathbb{D}$ in the complex plane satisfying $f(0)=0, f'(0)=1$. Let $U(\lambda)$ $(0<\lambda\le 1)$ denote the class of functions $f\in\mathcal{A}$ for which

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<\lambda \quad \text{for } z\in\mathbb{D}.$$

The behaviour of functions in this class has been extensively studied in the literature. In this paper, we shall prove that no member of $U_0(\lambda)=\{f\in U(\lambda): f''(0)=0\}$ is convex in $\mathbb{D}$ for any λ and obtain a lower bound for the radius of convexity for the family $U_0(\lambda)$. These results settle a conjecture proposed in the literature negatively. We also improve the existing lower bound for the radius of convexity of the family $U_0(\lambda)$.

1. Introduction. Let $\mathcal{A}$ denote the class of all analytic functions f in the open unit disc $\mathbb{D}$ in the complex plane satisfying $f(0)=0$ and $f'(0)=1$. Let $U(\lambda)$ $(0<\lambda\le 1)$ denote the class of functions $f\in\mathcal{A}$ for which

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<\lambda \quad \text{for } z\in\mathbb{D},$$

and let $U_0(\lambda)$ be the class of $f\in U(\lambda)$ with $f''(0)=0$. The properties of functions in $U(\lambda)$ and $U_0(\lambda)$ have been studied in detail in the literature (see [2]–[5]). Recently, Ponnusamy and Vasundhra [5] proposed the conjecture that $f\in U_0(\lambda)$ is convex at least when $0<\lambda\le 3-2\sqrt{2}$. In [6], Vasundhra also obtained a lower bound for the radius of convexity of the families $U(\lambda)$ and $U_0(\lambda)$.

The aim of the present paper is to show that the above conjecture is not valid. We shall also improve the lower bound for the radius of convexity of $U_0(\lambda)$. Further we shall obtain a lower bound for $\operatorname{Re}(zf'(z)/f(z))$ on each $|z|=r<1$, thereby giving an alternative proof for the order of starlikeness of the family $U_0(\lambda)$.

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2. Main results

THEOREM 2.1. *If $f \in U_0(\lambda)$ then $|z| \le r < 1$ we have*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \begin{cases} \dfrac{1-\lambda r^2}{1+\lambda r^2} & \text{if } r^2 \le 1/2\lambda, \\ \dfrac{1-2\lambda^2 r^4}{2(1-\lambda^2 r^4)} & \text{if } r^2 \ge 1/2\lambda. \end{cases}$$

Proof. From [6, p. 22] (replacing w by $-w$) we have

$$\frac{zf'(z)}{f(z)} = \frac{1-\lambda w}{1+\lambda w_1}$$

where $w(z)$ is analytic in $|z|<1$ with $|w(z)| \le |z|^2 < 1$ and

$$w_1(z) = \int_0^1 \frac{w(tz)}{t^2}\, dt. \tag{1}$$

By a simple computation we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \frac{(1-\lambda a)(1+\lambda a_1) - \lambda^2 b b_1}{1+\lambda^2(a_1^2+b_1^2)+2\lambda a_1}$$

with $w(z) = a+ib$ and $w_1(z) = a_1+ib_1$ (so that $a^2+b^2 \le r^4$ and $a_1^2+b_1^2 \le r^4$).

Put $\lambda|a| = y, \lambda a_1 = x$ so that $0 \le y \le \lambda r^2$ and $-\lambda r^2 \le x \le \lambda r^2$. Using

$$a \le |a|, \qquad \lambda|b| \le \sqrt{\lambda^2 r^4 - y^2}, \qquad \lambda|b_1| \le \sqrt{\lambda^2 r^4 - x^2}$$

we get

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \frac{(1-y)(1+x) - \sqrt{\lambda^2 r^4 - x^2}\sqrt{\lambda^2 r^4 - y^2}}{1+\lambda^2 r^4 + 2x}.$$

We now fix x with $|x| \le \lambda r^2$. We observe that the function

$$F(y) = (1-y)(1+x) - \sqrt{\lambda^2 r^4 - x^2}\sqrt{\lambda^2 r^4 - y^2}$$

for $0 \le y \le \lambda r^2$ attains its absolute minimum at

$$y = \frac{\lambda r^2(1+x)}{\sqrt{1+\lambda^2 r^4 + 2x}} = y_0 \quad \text{with} \quad F(y_0) = (1+x) - \lambda r^2\sqrt{1+\lambda^2 r^4 + 2x}.$$

Hence

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge H(x) = \frac{1+x}{1+\lambda^2 r^4 + 2x} - \frac{\lambda r^2}{\sqrt{1+\lambda^2 r^4 + 2x}}.$$

We now observe that

$$H'(x) = -\frac{1-\lambda^2 r^4}{(1+\lambda^2 r^4 + 2x)^2} + \frac{\lambda r^2}{(1+\lambda^2 r^4 + 2x)^{3/2}}$$

so that

$$H'(x) < 0 \quad \text{for } x < x_0 = \frac{1-3\lambda^2 r^4}{2\lambda^2 r^4},$$

$H'(x_0) = 0$ and $H'(x) \geq 0$ for $x \geq x_0$. However, x_0 is an admissible value for x if and only if

$$-\lambda r^2 \leq x_0 \leq \lambda r^2.$$

We can verify that $-\lambda r^2 \leq x_0$ holds trivially and that $x_0 \leq \lambda r^2$ if and only if $(1+\lambda r^2)(2\lambda^2 r^4 + \lambda r^2 - 1) \geq 0$, that is, $r^2 \geq 1/2\lambda$.

Thus $H(x) \geq H(x_0)$ for $r^2 \geq 1/2\lambda$. On the other hand, if $0 < r^2 \leq 1/2\lambda$ then $-\lambda r^2 \leq x \leq \lambda r^2 < x_0$ and hence $H(x)$ is a decreasing function of x. Thus in this case $H(x) \geq H(\lambda r^2)$. By a simple computation we have

$$H(\lambda r^2) = \frac{1-\lambda r^2}{1+\lambda r^2} \quad \text{and} \quad H(x_0) = \frac{1-2\lambda^2 r^4}{2(1-\lambda^2 r^4)}.$$

This completes the proof of our theorem.

Corollary 2.2. *Let $f \in U_0(\lambda)$ for $0 < \lambda \leq 1$. Then $f(z)$ is starlike of order δ with $0 < \delta < 1$ where $\delta = \delta(\lambda)$ is given by*

$$\delta(\lambda) = \begin{cases} \dfrac{1-\lambda}{1+\lambda} & \text{if } \lambda \leq 1/2, \\[2ex] \dfrac{1-2\lambda^2}{2(1-\lambda^2)} & \text{if } \lambda \geq 1/2. \end{cases}$$

In particular f is starlike of order 0 for $0 < \lambda \leq 1/\sqrt{2}$, and of order $1/2$ for $0 < \lambda \leq 1/3$.

Theorem 2.3. *For each fixed λ with $0 < \lambda \leq 1$ we have*

$$\lim_{r\to 1-} \operatorname{Re}\left(1 + \frac{r f_r''(r)}{f_r'(r)}\right) = -\infty$$

where for each α in $\mathbb{D}$, $f_\alpha \in U_0(\lambda)$ is defined by

$$\frac{z f_\alpha'(z)}{f_\alpha(z)} = \frac{1+\lambda w_\alpha(z)}{1-\lambda w_{\alpha 1}(z)}$$

with $w_\alpha(z) = -z^2 g_\alpha(z) = -z^2 \frac{z-\alpha}{1-z\bar{\alpha}}$ and $w_{\alpha 1} = (w_\alpha)_1$ (see (1)).

Proof. From [6, p. 22] we have, for $f \in U_0(\lambda)$,

$$\frac{z f'(z)}{f(z)} = \frac{1+\lambda w}{1-\lambda w_1} \tag{2}$$

where $w(z)$ is analytic in $|z| < 1$ with $|w(z)| \leq |z|^2 < 1$ and w_1 is given by (1).

Note that, conversely, all functions f defined by (2) are members of $U_0(\lambda)$. Let $\mathcal{F}$ be the subfamily of $U_0(\lambda)$ consisting of functions f_α with

$$\frac{z f_\alpha'(z)}{f_\alpha(z)} = \frac{1+\lambda w_\alpha(z)}{1-\lambda w_{\alpha 1}(z)}$$

for $w_\alpha(z)$ and $w_{\alpha 1}(z)$ as in the statement.

Using (2) we have, for each $f \in U_0(\lambda)$,

$$1 + \frac{zf''(z)}{f'(z)} = 2\,\frac{zf'(z)}{f(z)} - 1 + \frac{\lambda z w'(z)}{1+\lambda w(z)}$$
$$= 2\,\frac{1+\lambda w}{1-\lambda w_1} - 1 + \frac{\lambda z w'(z)}{1+\lambda w(z)}.$$

For $f_r \in \mathcal{F}$ and $z = r$ we also have

$$1 + \frac{r f_r''(r)}{f_r'(r)} = 2\,\frac{1+\lambda w_r(r)}{1-\lambda w_{r1}(r)} - 1 + \frac{\lambda r w_r'(r)}{1+\lambda w_r(r)}$$

with $w_{r1}(r) = r + \frac{1-r^2}{r}\log(1-r^2)$. Now

$$2\,\frac{1+\lambda w_r(r)}{1-\lambda w_{r1}(r)} - 1 + \frac{\lambda r w_r'(r)}{1+\lambda w_r(r)}$$
$$= \frac{2}{1-\lambda\big[r + \frac{1-r^2}{r}\log(1-r^2)\big]} - 1 - \frac{\lambda r^3}{1-r^2}.$$

Hence

$$\lim_{r\to 1} \operatorname{Re}\left(1 + \frac{r f_r''(r)}{f_r'(r)}\right) = \lim_{r\to 1} \operatorname{Re}\left(\frac{2}{1-\lambda\big[r + \frac{1-r^2}{r}\log(1-r^2)\big]} - 1 - \frac{\lambda r^3}{1-r^2}\right)$$
$$= -\infty,$$

proving the theorem.

THEOREM 2.4. *Let $f \in U_0(\lambda)$. Then*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq 0 \qquad \text{for } |z| \leq R$$

where

$$R^2 = \begin{cases} \dfrac{1}{7\lambda}, & \dfrac{8}{35} \leq \lambda \leq 1, \\ \dfrac{5+\lambda-\sqrt{(1-\lambda)(25-\lambda)}}{6\lambda}, & 0 < \lambda \leq \dfrac{8}{35}. \end{cases}$$

Proof. Taking logarithmic derivatives in (2) we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{\lambda z w'(z)}{1+\lambda w(z)} + \frac{\lambda z w_1'(z) + 1 + \lambda w(z)}{1-\lambda w_1(z)} \tag{3}$$

By a simple computation we see that $zw_1'(z) = w(z) + w_1(z)$. By using this, we deduce from (3) that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{\lambda z w'(z)}{1+\lambda w(z)} + \frac{\lambda(w(z)+w_1(z)) + 1 + \lambda w(z)}{1-\lambda w_1(z)}$$
$$= 2\,\frac{1+\lambda w(z)}{1-\lambda w_1(z)} - 1 + \frac{\lambda z w'(z)}{1+\lambda w(z)}.$$

Therefore

$$\frac{zf''(z)}{f'(z)} = 2\left(\frac{1+\lambda w(z)}{1-\lambda w_1(z)} - 1\right) + \frac{\lambda z w'(z)}{1+\lambda w(z)} = 2\,\frac{\lambda(w(z)+w_1(z))}{1-\lambda w_1(z)} + \frac{\lambda z w'(z)}{1+\lambda w(z)} \tag{4}$$

Since $w_2(z) = w(z)/z$ is a Schwarz function,

$$|w_2'(z)| \le \frac{1-|w_2(z)|^2}{1-|z|^2}$$

(see [1, p. 136]), which is equivalent to

$$|zw'(z) - w(z)| \le \frac{|z|^2 - |w(z)|^2}{1-|z|^2}. \tag{5}$$

From (4) and (5) we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{2\lambda(|w(z)|+r^2)}{1-\lambda r^2} + \frac{\lambda}{1-\lambda|w(z)|}\,|w(z)| + \frac{r^2-|w(z)|^2}{1-r^2}\,\frac{\lambda}{1-\lambda|w(z)|}.$$

If we let $|w(z)| = x$ (note that $0 \le x \le r^2$), the above inequality becomes

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{2\lambda(x+r^2)}{1-\lambda r^2} + \frac{\lambda x}{1-\lambda x} + \frac{r^2-x^2}{1-r^2}\,\frac{\lambda}{1-\lambda x} = \phi(x) \quad \text{(say)}.$$

By some simple computations, we see that

$$\phi(x) \le \phi(r^2) = \frac{6\lambda r^2}{1-\lambda r^2} \quad \text{if} \quad r^2 < \frac{5+\lambda-\sqrt{(1-\lambda)(25-\lambda)}}{6\lambda} = x_0.$$

Thus

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 \quad \text{if} \quad r^2 < \frac{1}{7\lambda}.$$

Therefore

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1, \quad \text{that is,} \quad f \text{ is convex for } |z| \le r \quad \text{if}$$

$$r^2 \le \min\left\{\frac{1}{7\lambda}, \frac{5+\lambda-\sqrt{(1-\lambda)(25-\lambda)}}{6\lambda}\right\}.$$

However, we have

$$\frac{1}{7\lambda} < \frac{5+\lambda-\sqrt{(1-\lambda)(25-\lambda)}}{6\lambda} \quad \text{for } \lambda \ge 8/35$$

and

$$\frac{5+\lambda-\sqrt{(1-\lambda)(25-\lambda)}}{6\lambda} < \frac{1}{7\lambda} \quad \text{for } \lambda \le 8/35,$$

proving the theorem.

NOTE 2.5. Theorem 2.3 implies that the conjecture proposed in [5] is not true. Indeed, if it were, even for a single λ with $0 < \lambda \leq 1$, then for each fixed $r < 1$ we would have

$$\begin{aligned}\operatorname{Re}\left(1 + \frac{r f_r''(r)}{f_r'(r)}\right) &\geq \inf_{\alpha \in \mathbb{D}} \operatorname{Re}\left(1 + \frac{r f_\alpha''(r)}{f_\alpha'(r)}\right) \\ &\geq \inf_{f \in U_0(\lambda)} \operatorname{Re}\left(1 + \frac{r f''(r)}{f'(r)}\right) \geq 0.\end{aligned}$$

However, this implies that if $\lim_{r\to 1-} \operatorname{Re}(1+r f_r''(r)/f_r'(r))$ exists, it is greater than or equal to zero, contradicting Theorem 2.3.

Theorem 2.4 improves the lower bounds obtained in [6] for the radius of convexity of the family $U_0(\lambda)$. For example if $\lambda = 1$ then $R = 1/\sqrt{7} = 0.377$ whereas for the same value of λ the lower bound in [6] is 0.3489 approximately.

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