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BUTLER GROUPS SPLITTING OVER A BASE ELEMENT

BY

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Abstract. We characterize a particular kind of decomposition of a Butler group that is the general case for Butler $B(1)$-groups; and exhibit a decomposition of a $B(2)$-group which is not of that kind.

Introduction. All groups in the following are torsionfree Abelian of finite rank. A *Butler* $B(n)$*-group* G is a torsionfree Abelian group that is the sum of $m \ge n$ rank 1 groups, $G = \langle g_1\rangle_* + \cdots + \langle g_m\rangle_*$ (where $_*$ indicates pure closure), subject to n independent relations involving all of the m rank 1 groups. $B(0)$ is the class of completely decomposable groups; in the following we suppose $n \ge 1$. $B(1)$-groups have been amply studied (for history, see [1]), using, as a basic equivalence, *quasi-isomorphism* [6] instead of isomorphism; this is also what we do in this paper; in fact, we will write *isomorphic*, *indecomposable*, *direct decomposition*,... instead of "quasi-isomorphic, strongly indecomposable, quasi-direct decomposition,...".

Direct decompositions of $B(1)$-groups were studied in [7], [3], and many other papers; when a $B(1)$-group G splits, it always has a decomposition $G = G' \oplus G''$ such that all but one of the base elements $g_1, \dots, g_m$ belong either to G' or to G''; we call this a decomposition *over a base element*. This is not the case in general: in Section 2 we give a necessary and sufficient condition for a Butler $B(n)$-group to split over a base element. The condition consists of two parts, mirroring the double nature of Butler groups: an order-theoretical one, which is the one that is necessary and sufficient in the $B(1)$ case, and guarantees $G = G' + G''$; and an additional linear one, ensuring that $G' \cap G'' = 0$. In Section 3 we give two examples showing a decomposition of a $B(2)$-group that does not occur over a base element.

1. Notation and first remarks. *Lower case Greek letters will denote rational numbers.* We will use extensively the notation and tools developed in our previous papers on $B(1)$-groups (see in particular [2], [3]); we recall here some of them.

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By *type* we mean the isomorphism type of an additive subgroup of $\mathbb{Q}$. $\mathbb{T}(\vee, \wedge)$ denotes the lattice of all types, with the added maximum ∞ for the type of the 0 group; if w is an element of a group W, $t_W(w)$ denotes the type of w in W.

Throughout, $I = \{1, \dots, m\}$; partitions of I are ordered by "bigger = coarser"; blocks of partitions are nonempty by definition.

If $w_1, \dots, w_m$ are elements of a group W and $E \subseteq I$, we define

$$w_E = \sum \{w_i \mid i \in E\}, \quad \text{with} \quad w_\emptyset = 0,$$

$$W_E = \langle w_i \mid i \in E \rangle_* = \sum \{\langle w_i \rangle_* \mid i \in E\} + \langle w_E \rangle_*$$

(the last equality is proved in [5]).

If $W = \langle w_1 \rangle_* + \dots + \langle w_m \rangle_*$, and $w = \beta_1 w_{C_1} + \dots + \beta_h w_{C_h}$ with $\beta_i \neq \beta_j$ if $i \neq j$, then $\mathcal{C} = \{C_1, \dots, C_h\}$ will be called a *partition of I into equal-coefficient blocks* for w.

In our $B(n)$-group

$$G = \langle g_1 \rangle_* + \dots + \langle g_m \rangle_*$$

the elements $g_1, \dots, g_m$ are the *base elements*, and are fixed throughout; setting $t_i = t_G(g_i)$ for all $i \in I$, the t_i are the *base types* of G; $(t_1, \dots, t_m)$ is the *type-base* of G.

It is not difficult to show (see also [5]) that there is no loss of generality in supposing that the relations are of the form

$$\begin{aligned} g_I = g_1 + \dots + g_m &= 0 \qquad \text{(the \textit{first,} or \textit{diagonal, relation})}, \\ \alpha_{2,1} g_{A_1} + \dots + \alpha_{2,k} g_{A_k} &= 0, \\ \alpha_{3,1} g_{A_1} + \dots + \alpha_{3,k} g_{A_k} &= 0, \\ &\vdots \\ \alpha_{n,1} g_{A_1} + \dots + \alpha_{n,k} g_{A_k} &= 0, \end{aligned}$$

where each $A_j \subseteq I$ collects some indices of generators with equal coefficients, and moreover if $j \neq j' = 1, \dots, k$ there is at least one row $r = 2, \dots, n$ such that $\alpha_{r,j} \neq \alpha_{r,j'}$. Clearly, $n \leq k \leq m$. The ensuing partition

$$\mathcal{A} = \{A_1, \dots, A_k\}$$

of I is the *basic partition* of (the given base of) G; its blocks A_j are called the *sections* of G. Note that if $0 = \beta_1 g_{C_1} + \dots + \beta_h g_{C_h}$ with $\beta_i \neq \beta_j$ if $i \neq j$ then $\mathcal{C} = \{C_1, \dots, C_h\} \geq \mathcal{A}$. Replacing each nondiagonal relation with the difference

$$\alpha_{r,1} g_{A_1} + \dots + \alpha_{r,k} g_{A_k} - \alpha_{r,1}(g_1 + \dots + g_m) = 0$$

we may suppose that $\alpha_{r,1} = 0$ for all $r = 2, \ldots, n$. Then the matrix

$$M = \begin{bmatrix} \alpha_{2,2} & \cdots & \alpha_{2,k} \\ \ldots & \ddots & \ldots \\ \alpha_{n,2} & \cdots & \alpha_{n,k} \end{bmatrix},$$

a *reduced matrix* of G, is of rank $n-1$.

2. Splitting over a base element. For completeness, let us first sketch the situation of a nontrivial splitting $G = G' \oplus G''$ in which *all of the base elements belong either to G' or to G''*; the conditions will turn out to be only linear. Note that this is, in fact, the case when we consider the direct sum of a $B(n')$-group G' and a $B(n'')$-group G'', and choose for a base the union of the two bases and for the linear system the union of the two systems; in this case the $B(n)$-group G will be called *degenerate*.

Examining necessary conditions, setting $E' = \{i \in I \mid g_i \in G'\}$, $E'' = \{i \in I \mid g_i \in G''\}$, we have $G' = G_{E'}$ $(= \sum\{\langle g_i\rangle_* \mid i \in E\} + \langle g_E\rangle_*)$, $G'' = G_{E''}$. The diagonal relation $g_I = g_{E'} + g_{E''} = 0$ provides a common element to the two summands, hence we have the two relations $g_{E'} = -g_{E''} = 0$; in particular, this yields $\{E', E''\} \geq \mathcal{A}$. Analogously, each nondiagonal relation in G yields two relations, one in $G_{E'}$ and one in $G_{E''}$, therefore among the n relations thus obtained there will be r' independent ones in $G_{E'}$, and r'' in $G_{E''}$, with $r' + r'' = n$. Replacing the initial relations with these makes the matrix M block-diagonal, where the columns of one block are indexed by j's such that $A_j \subseteq E'$, and those of the other by j's such that $A_j \subseteq E''$. Conversely, if the matrix M of G is (equivalent to) a block-diagonal one, with E' resp. E'' gathering the indices of base elements involved in the first resp. second block, it is not difficult to prove that $G = G_{E'} \oplus G_{E''}$. Note that this situation cannot occur for $B(1)$-groups; for more about degenerate $B(2)$-groups see [4]. Thus we have

PROPOSITION 2.1. *A decomposition in which all base elements belong to some summand is possible if and only if the matrix is equivalent to a block-diagonal one*, *that is*, *if and only if the group is degenerate.*

An interesting observation can be drawn from the above considerations. Recall that a $B(1)$-group G is decomposable [7] if and only if there exist a subset E of I and an index $i \in I$ with $i \notin E$ such that $t_i \leq t_G(g_E)$. Necessity derives from the following two facts:

I) If $\operatorname{rk} G(t_i) = 1$ for all $i \in I$, then G is indecomposable. (The proof consists in noting that if G did decompose then each base element would have to belong to some summand, an impossibility, as noted above.) Decomposability then follows from the existence of elements of maximal type different from (not proportional to) base elements.

II) The maximal types of G are of the form $t_G(g_E)$ for some $E \subseteq I$.

Neither of these two facts holds any more if G is a $B(n)$-group with $n \geq 2$: I) has been examined above, II) is shown to fail in Example 1. Therefore, while for $n \geq 2$ a condition $t_i \leq t_G(g_E)$ may stand a chance among sufficient conditions, for necessary conditions we must look elsewhere.

Let now G split over a base element, that is, $G = G' \oplus G''$, and for some $i' \in I$ and all $i \in I$, $i \neq i'$, g_i belongs either to G' or to G''. There is no loss of generality in supposing

$$i' = 1 \in A_1;$$

then for

$$E = \{i \in I \mid g_i \in G'\}, \qquad F = \{i \in I \mid g_i \in G''\}$$

we have $I = \{1\} \cup E \cup F$, a disjoint union.

Observe that in the above notation $G_E \leq_* G'$ and $G_F \leq_* G''$; the diagonal relation $g_1 = -g_E - g_F$ yields $(G_E + G_F)_* = G$; then for the splitting we need $G_E = G'$ and $G_F = G''$; that is, $G_E + G_F = G$ and $G_E \cap G_F = 0$.

Lemma 2.2. *If $I = \{1\} \cup E \cup F$ is a disjoint union, then $t_1 = t_G(g_E) \wedge t_G(g_F)$ if and only if $t_1 \leq t_G(g_E)$.*

Proof. One way is obvious; the other follows from the fact that, if $t_1 = t_G(g_1) \leq t_G(g_E)$, then $t_1 \leq t_G(g_E + g_1) = t_G(g_F)$; but $g_1 = -g_E - g_F$ implies $t_1 \geq t_G(g_E) \wedge t_G(g_F)$. ∎

Clearly, a necessary condition for $G = G_E \oplus G_F$ is $t_1 \leq t_G(g_E)$; moreover from Lemma 2.2 we have

Proposition 2.3. *If $I = \{1\} \cup E \cup F$ is a disjoint union, then $t_1 \leq t_G(g_E)$ implies $G = G_E + G_F$.*

Proof. By Lemma 2.2, we have $t_1 = t_G(g_E) \wedge t_G(g_F)$, hence $\langle g_1 \rangle_*$ is contained in $G_E + G_F$, which then equals G. ∎

Consider now the condition $G_E \cap G_F = 0$. It does not involve the diagonal relation; but if we set $E_j = E \cap A_j$ and $F_j = F \cap A_j$ for all $j = 1, \ldots, k$, then any other relation (with zero as the first coefficient)

$$\alpha_{r,2} g_{A_2} + \cdots + \alpha_{r,k} g_{A_k} = 0$$

may yield a nonzero common element $g = \alpha_{r,2} g_{E_2} + \cdots + \alpha_{r,k} g_{E_k} = -(\alpha_{r,2} g_{F_2} + \cdots + \alpha_{r,k} g_{F_k})$, unless

($*$) either E or F is contained in A_1.

If ($*$) holds, then $G_E \cap G_F = 0$: for if, say, $E = E_1$, a common element $0 \neq \sum\{\beta_i g_i \mid i \in E_1\} = \sum\{\gamma_i g_i \mid i \in F\}$ would yield $0 = \sum\{\beta_i g_i \mid i \in E_1\}$

$-\sum\{\gamma_i g_i \mid i \in F\}$, absurd since this element has a partition that is not $\geq \mathcal{A}$ (it splits A_1). Note that $(*)$ obviously always holds in a $B(1)$-group, where $I = A_1$.

If $(*)$ does not hold, the above element $g \in G_E \cap G_F$ is certainly nonzero if there is a $j \in \{2, \ldots, k\}$ such that both E_j and F_j are nonempty, for then g has a partition that is not $\geq \mathcal{A}$. Therefore a first necessary condition for $G_E \cap G_F = 0$ is that, for every $j = 2, \ldots, k$, one of E_j, F_j be empty; that is, *each* A_j *with* $j \neq 1$ *must be contained either in* E *or in* F.

Let then without loss of generality $A_2, \ldots, A_s \subseteq E$, $A_{s+1}, \ldots, A_k \subseteq F$ with $2 \leq s \leq k$. As above, each relation except the diagonal one provides an element $a_r \in G_E \cap G_F$ with $2 \leq r \leq n$,

$$a_r = \sum\{\alpha_{r,j} g_{A_j} \mid 2 \leq j \leq s\} = \sum\{\alpha_{r,j} g_{A_j} \mid s+1 \leq j \leq k\}.$$

To get $G_E \cap G_F = 0$ we need $a_r = 0$ for all $r = 2, \ldots, n$. At this point we can already draw a conclusion for $n = 2$: the unique second relation does not allow $a_2 = 0$ for any choice of E and F, therefore $(*)$ is a necessary condition.

For $n > 2$ the conditions $a_r = 0$ for all $r = 2, \ldots, n$ yield a set of $2(n-1)$ relations, among which there will be $n-1$ independent ones; then the matrix M can be replaced by a block-diagonal matrix

$$M' = \begin{bmatrix} M(E) & 0 \\ 0 & M(F) \end{bmatrix},$$

where (after possibly re-ordering rows) *the columns of* $M(E)$ *are indexed by* j*'s such that* $A_j \subseteq E$*, and those of* $M(F)$ *are indexed by* j*'s such that* $A_j \subseteq F$.

We can now reach a conclusion:

THEOREM 2.4. *Let* $G = \langle g_1 \rangle_* + \cdots + \langle g_m \rangle_*$ *be a* $B(n)$*-group. Then* G *splits over one of its base elements—without loss of generality over* g_1*—if and only if there is a partition* $\{\{1\}, E, F\}$ *of* I *such that*

(a) $t_1 \leq t_G(g_E)$,

(b) *one of the following two conditions holds*:

  $(*)$ *one of* E, F *is contained in* A_1;

  $(**)$ *a reduced matrix of* G *is block-diagonal of the form*

$$M = \begin{bmatrix} M(E) & 0 \\ 0 & M(F) \end{bmatrix},$$

  with the above notation.

In particular: *if* G *is a* $B(1)$*-group, then* (a) *is also sufficient*; *if* G *is* $B(2)$, *then* G *splits over* t_1 *if and only if* (a) *and* $(*)$ *hold.*

Proof. Necessity has been proved above. For sufficiency, note that Proposition 2.3(a) implies $G = G_E + G_F$. Now if $(*)$ holds, we proved above that $G_E \cap G_F = 0$. If $(**)$ holds, let $g \in G_E \cap G_F$; then $g = \sum\{\beta_i g_i \mid i \in E\} = \sum\{\beta_j g_j \mid j \in F\}$, hence $\sum\{\beta_i g_i \mid i \in E\} - \sum\{\beta_j g_j \mid j \in F\} = 0$. This relation is then a linear combination of nondiagonal basic relations, which by $(**)$ separate into those indexed in E and those indexed in F. This in turn implies that each of the two terms is 0, that is, $g = 0$. ∎

3. Examples. With our definitions, not all m-tuples of types are bases of $B(n)$-groups; "regularity" must be imposed [5], and all our examples are checked for it. We will use the notation of [2]–[5], where a type with all zeros but a finite number of infinities is denoted by replacing each infinity by its prime and each zero by a dot, while eliminating infinite tails of zeros [3]; e.g.

$$\sigma = 0\ 0\ \infty\ \infty\ 0\ \ldots\ \text{zeros}\ \ldots\ = \cdot\ \cdot\ 5\ 7\ = \cdot\ \cdot\ p\ q\ .$$

Example 1 (of a type which is maximal but not of the form $t_G(g_E)$). Let $m = 8$, $n = 2$, $k = 4$, $\mathcal{A} = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$, with relations

$$g_1 + \cdots + g_8 = 0, \qquad g_{\{3,4\}} - 2g_{\{5,6\}} + g_{\{7,8\}} = 0.$$

Let the base types be:

A_1	$t_1 = \cdot\ \cdot\ \cdot$
	$t_2 = \cdot\ \cdot\ \cdot$
A_2	$t_3 = p_1\ \cdot\ p_3$
	$t_4 = p_1\ p_2\ \cdot$
A_3	$t_5 = p_1\ \cdot\ p_3$
	$t_6 = \cdot\ p_2\ p_3$
A_4	$t_7 = p_1\ p_2\ \cdot$
	$t_8 = \cdot\ p_2\ p_3$

The element $g = -2g_3 - 3g_4 + 4g_5 - 3g_7 = -g_4 - 4g_6 - g_7 + 2g_8 = g_3 - 2g_5 - 6g_6 + 3g_8$ (obtained by adding $2(g_{\{3,4\}} - 2g_{\{5,6\}} + g_{\{7,8\}})$ resp. $3(g_{\{3,4\}} - 2g_{\{5,6\}} + g_{\{7,8\}})$) is clearly divisible by p_1, p_2 and p_3, hence has maximum type, while a computation shows that no element of the form g_E reaches the maximum type.

Example 2 (showing that, in the above notation, $t_1 \le t_G(g_E)$ does not imply $t_1 \le t_G(g_{E_1})$, which proves that condition (b) cannot be eliminated). Let $m = 7$, $n = 2$, with relations

$$g_1 + \cdots + g_7 = 0,$$
$$\alpha_1 g_{\{1,2,3\}} + \alpha_2 g_4 + \alpha_3 g_5 + \alpha_4 g_6 + \alpha_5 g_7 = 0,$$

with pairwise different coefficients, thus $\mathcal{A} = \{\{1,2,3\},\{4\},\{5\},\{6\},\{7\}\}$; let $E = \{2,4,5\}$ (thus $E_1 = \{2\}$), $F = \{3,6,7\}$;

A_1	$t_1 = p\,q$
	$t_2 = \cdot\ q$
	$t_3 = p\ \cdot$
A_2	$t_4 = \cdot\ q$
A_3	$t_5 = \cdot\ q$
A_4	$t_6 = p\ \cdot$
A_5	$t_7 = p\ \cdot$

Here $t_1 = pq = t_G(g_E) > t_G(g_{E_1}) = t_2$; in fact we have $0 \neq (\alpha_2 - \alpha_1)g_4 + (\alpha_3 - \alpha_1)g_5 = (\alpha_1 - \alpha_4)g_6 + (\alpha_1 - \alpha_5)g_7 \in G_{\{2,4,5\}} \cap G_{\{3,6,7\}}$.

We now give two examples of $B(2)$-group-splittings that do not occur over a single base element, showing that our study does not exhaust the decomposition problem.

Example 3. Let $n = 2$, $I = \{1, \ldots, 8\}$, $\mathcal{A} = \{A_1, A_2, A_3\} = \{\{1,2,3\}, \{4,5\}, \{6,7,8\}\}$; $E = \{3,5,7\}$, $F = \{2,4,6\}$; $E_j = E \cap A_j$, $F_j = F \cap A_j$ for $j = 1,2,3$; with the two relations $g_I = 0$, $g_{A_1} + \alpha_2 g_{A_2} = 0$, with $\alpha_2 \neq 0, 1$.

Let

A_1	$t_1 = p'\,p''\ \cdot\ \ \cdot$
	$t_2 = \cdot\ p''\ \cdot\ q''$
	$t_3 = p'\ \cdot\ q'\ \cdot$
A_2	$t_4 = \cdot\ p''\ \cdot\ \ q''$
	$t_5 = p'\ \cdot\ q'\ \ \cdot$
A_3	$t_6 = \cdot\ p''\ \cdot\ q''$
	$t_7 = p'\ \cdot\ q'\ \cdot$
	$t_8 = \cdot\ \ \cdot\ q'\,q''$

$$g_1' = g_1 + g_2 + \alpha_2 g_4,$$
$$g_1'' = g_1 + g_3 + \alpha_2 g_5,$$
$$g_8' = (\alpha_2 - 1)g_5 - g_7,$$
$$g_8'' = (\alpha_2 - 1)g_4 - g_6,$$

$$G' = \langle g_1' \rangle_* + G_E + \langle g_8' \rangle_*,$$
$$G'' = \langle g_1'' \rangle_* + G_F + \langle g_8'' \rangle_*.$$

Here $G = G' \oplus G''$, where G' and G'' are $B(2)$-groups with relations $g_1' + g_E + g_8' = 0 = g_1'' + g_F + g_8''$ (the diagonal relations) and $g_1' + g_{E_1} + \alpha_2 g_{E_2} = 0 = g_1'' + g_{F_1} + \alpha_2 g_{F_2}$ (the secondary relations). The relations ensure that the sum of the ranks of G' and G'' does not exceed the rank of G; we will have a direct sum if we show that the types t_1 of $g_1 = g_1' + g_1''$ and t_8 of $g_8 = g_8' + g_8''$ are fully recovered in $G' \oplus G''$. But this is the case because $g_1' = g_1 + g_2 + \alpha_2 g_4 = g_1' - (g_{A_1} + \alpha_2 g_{A_2}) = -(g_3 + g_5)$ has the prime p'' in its first form and p' in its second form, thus completing $t_1 = p'p''$; and analogously does g''. The same works for g_8' and g_8'' with respect to q', q'' and $t_8 = q'q''$.

In order to see that G does not split over any of its base elements, it is enough to verify that no t_i is $\leq t_E$ with E a subset of the section A_j containing i, with E disjoint from $\{i\}$. This is clear if E is a singleton. If not, then we are left with the sections A_1 and A_3, whose situation is analogous. Starting with $i = 1$, we have $t_1 = p'p''$; to get p' to divide the type of

$$\begin{aligned} g_E = g_{\{2,3\}} &= g_{\{2,3\}} + \lambda g_I + \mu(g_{\{1,2,3\}} + \alpha_2 g_{\{4,5\}}) \\ &= (\lambda+\mu)g_1 + (\lambda+\mu+1)g_{\{2,3\}} + (\lambda+\mu\alpha_2)g_{\{4,5\}} + \lambda g_{\{6,7,8\}} \end{aligned}$$

we must determine λ, μ in such a way as to eliminate g_2, g_4, g_6, g_8, that is,

$$\lambda + \mu + 1 = \lambda + \mu\alpha_2 = \lambda = 0,$$

which is impossible. For $i = 2$ we have $t_2 = p''q''$; for p'' to divide the type of

$$\begin{aligned} g_E = g_{\{1,3\}} &= g_{\{1,3\}} + \lambda g_I + \mu(g_{\{1,2,3\}} + \alpha_2 g_{\{4,5\}}) \\ &= (\lambda+\mu)g_2 + (\lambda+\mu+1)g_{\{1,3\}} + (\lambda+\mu\alpha_2)g_{\{4,5\}} + \lambda g_{\{6,7,8\}} \end{aligned}$$

we must determine λ, μ in such a way as to eliminate g_3, g_5, g_7, g_8, which is impossible; the same happens for $i = 3$.

Example 4. Let $n = 2$, $m = k = 6$, $\mathcal{A} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$; $E = \{2,3\}$, $F = \{4,5\}$; $E_j = E \cap A_j$, $F_j = F \cap A_j$ for $j = 1,\dots,6$; with the two relations $g_I = 0$, $\alpha_2 g_2 + \alpha_3 g_3 + \alpha_4 g_4 + \alpha_5 g_5 + \alpha_6 g_6 = 0$, with coefficients nonzero and pairwise different.

Let

A_1	$t_1 =$	p_1	q_1	$\cdot$	$\cdot$	
A_2	$t_2 =$	p_1	$\cdot$	p_6	$\cdot$	
A_3	$t_3 =$	p_1	$\cdot$	p_6	$\cdot$	
A_4	$t_4 =$	$\cdot$	q_1	$\cdot$	q_6	
A_5	$t_5 =$	$\cdot$	q_1	$\cdot$	q_6	
A_6	$t_6 =$	$\cdot$	$\cdot$	p_6	q_6	

$$\begin{aligned} g_1' &= \alpha_6^{-1}((\alpha_2-\alpha_6)g_2 + (\alpha_3-\alpha_6)g_3), \\ g_1'' &= \alpha_6^{-1}((\alpha_4-\alpha_6)g_4 + (\alpha_5-\alpha_6)g_5), \\ g_6' &= -\alpha_6^{-1}(\alpha_2 g_2 + \alpha_3 g_3), \\ g_6'' &= -\alpha_6^{-1}(\alpha_4 g_4 + \alpha_5 g_5), \end{aligned}$$

$$\begin{aligned} G' &= \langle g_1'\rangle_* + G_E + \langle g_6'\rangle_*, \\ G'' &= \langle g_1''\rangle_* + G_F + \langle g_6''\rangle_*. \end{aligned}$$

We have $\alpha_6 g_1' = (\alpha_2-\alpha_6)g_2 + (\alpha_3-\alpha_6)g_3 = -\alpha_6(g_2+g_3+g_6) = -\alpha_6(g_1+g_4+g_5)$, hence $t_G(g_1') \geq p_1q_1p_6$; analogously, $t_G(g_1'') \geq p_1q_1q_6$; $t_G(g_6') \geq p_1p_6q_6$; $t_G(g_6'') \geq q_1p_6q_6$. We thus recover the types of $g_1 = g_1' + g_1''$ and of $g_6 = g_6' + g_6''$, hence $G' + G'' = G$. Now, G' and G'' are $B(2)$-groups with relations $g_1' + g_E + g_6' = 0 = g_1'' + g_F + g_6''$ (the diagonal relations) and $\alpha_6 g_1' - (\alpha_2-\alpha_6)g_{E_2} - (\alpha_3-\alpha_6)g_{E_3} = 0 = \alpha_6 g_1'' - (\alpha_4-\alpha_6)g_{F_4} - (\alpha_5-\alpha_6)g_{F_5}$ (the secondary relations). The relations ensure that the sum of the ranks of G' and G'' does not exceed the rank of G, hence the splitting is proved.

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