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PATH COALGEBRAS OF PROFINITE BOUND QUIVERS, COTENSOR COALGEBRAS OF BOUND SPECIES AND LOCALLY NILPOTENT REPRESENTATIONS

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Abstract. We prove that the study of the category C-Comod of left comodules over a K-coalgebra C reduces to the study of K-linear representations of a quiver with relations if K is an algebraically closed field, and to the study of K-linear representations of a K-species with relations if K is a perfect field. Given a field K and a quiver $Q=(Q_0,Q_1)$, we show that any subcoalgebra C of the path K-coalgebra $K^{\square}Q$ containing $K^{\square}Q_0\oplus K^{\square}Q_1$ is the path coalgebra $K^{\square}(Q,\mathfrak{B})$ of a profinite bound quiver $(Q,\mathfrak{B})$, and the category C-Comod of left C-comodules is equivalent to the category $\mathrm{Rep}_K^{\ell n\ell f}(Q,\mathfrak{B})$ of locally nilpotent and locally finite K-linear representations of Q bound by the profinite relation ideal $\mathfrak{B}\subset\widehat{KQ}$.

Given a K-species $\mathcal{M}=(F_j,{}_iM_j)$ and a relation ideal $\mathfrak{B}$ of the complete tensor K-algebra $\widehat{T}(\mathcal{M})=\widehat{T_F(M)}$ of $\mathcal{M}$, the bound species subcoalgebra $T^{\square}(\mathcal{M},\mathfrak{B})$ of the cotensor K-coalgebra $T^{\square}(\mathcal{M})=T_F^{\square}(M)$ of $\mathcal{M}$ is defined. We show that any subcoalgebra C of $T^{\square}(\mathcal{M})$ containing $T^{\square}(\mathcal{M})_0\oplus T^{\square}(\mathcal{M})_1$ is of the form $T^{\square}(\mathcal{M},\mathfrak{B})$, and the category C-Comod is equivalent to the category $\mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M},\mathfrak{B})$ of locally nilpotent and locally finite K-linear representations of $\mathcal{M}$ bound by the profinite relation ideal $\mathfrak{B}$. The question when a basic K-coalgebra C is of the form $T_F^{\square}(M,\mathfrak{B})$, up to isomorphism, is also discussed.

1. Introduction. Throughout this paper, K is a field. Given a K-coalgebra C, we denote by C-Comod and C-comod the categories of left C-comodules and left C-comodules of finite K-dimension, respectively. The problem we study is to view the category C-Comod, for C basic with left Gabriel quiver $Q={}_CQ$, as a full subcategory of the category $\mathrm{Rep}_K^{\ell n\ell f}(Q)$ of locally nilpotent and locally finite K-linear representations of Q that satisfy some relations.

We call C *basic* if $\dim_K S=\dim_K \mathrm{End}_C S$ for any simple left C-comodule S (see [7], [24], [27, p. 404], [34, Lemma 1.2]). It is known that if K is algebraically closed then any basic K-coalgebra is a relation subcoalgebra

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of the path K-coalgebra $K^\square Q$ of the left Gabriel quiver $Q = {}_CQ$ of C, that is, $K^\square Q_0 \oplus K^\square Q_1 \subseteq C \subseteq K^\square Q$.

One of the main results of this paper is Corollary 4.10 asserting that, given an arbitrary field K, any relation subcoalgebra C of $K^\square Q$ is the path coalgebra $K^\square(Q, \mathfrak{B})$ of a profinite bound quiver $(Q, \mathfrak{B})$ in the sense of Definition 4.2, and the category C-Comod is equivalent to the category $\mathrm{Rep}_K^{\ell n\ell f}(Q, \mathfrak{B})$ of locally nilpotent and locally finite K-linear representations of Q bound by a profinite relation ideal $\mathfrak{B}$ of the profinite K-algebra $\widehat{KQ}$ (the completion of the path K-algebra KQ in the finite subquiver topology). If, in addition, Q is intervally finite then $C = K^\square(Q, \Omega)$ and C-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(Q, \Omega)$, where Ω is a relation ideal of KQ. The results complete and generalise those in [31]. As a consequence, we get Corollary 4.12 asserting that, if the field K is algebraically closed, then any basic K-coalgebra C is of the form $K^\square(Q, \mathfrak{B})$ and there is an equivalence of categories C-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(Q, \mathfrak{B})$ with $Q = {}_CQ$. This generalises the well-known result of Gabriel [10] proved for finite-dimensional algebras over an algebraically closed field K.

In Section 5, we extend the above results to a more general class of coalgebras, by applying the K-species technique for coalgebras introduced in [9] and [18]. Following [9], to any K-species $\mathcal{M} = (F_j, {}_iM_j)$, we associate the complete tensor K-algebra $\widehat{T}(\mathcal{M})$. Then we define the cotensor (basic hereditary) K-coalgebra $T^\square(\mathcal{M})$ of $\mathcal{M}$ in such a way that the path K-coalgebra $K^\square Q$ of any quiver Q is of the form $T^\square(\mathcal{M})$ for a suitable $\mathcal{M}$ associated to Q. Moreover, to any profinite relation ideal $\mathfrak{B}$ in $\widehat{T}(\mathcal{M})$, we associate a relation cotensor K-coalgebra $T_F^\square(\mathcal{M}, \mathfrak{B}) \subseteq T_F^\square(\mathcal{M})$, that is, $T_F^\square(\mathcal{M}, \mathfrak{B})$ contains the subcoalgebra $T^\square(\mathcal{M})_0 \oplus T^\square(\mathcal{M})_1$ of $T_F^\square(\mathcal{M})$. By applying [18, Proposition 4.16], we show that any hereditary basic K-coalgebra C is of the form $C \cong T_F^\square(\mathcal{M})$, where $\mathcal{M}$ is the dual to the Ext-species of C, under some acyclicity assumption on the left valued Gabriel quiver of C.

The main results of Section 5 are Theorem 5.20 and Corollary 5.22 asserting that, given a relation subcoalgebra C of $T^\square(\mathcal{M})$, there exists a profinite relation ideal $\mathfrak{B}$ in the complete tensor K-algebra $\widehat{T}(\mathcal{M})$ such that $C = T^\square(\mathcal{M}, \mathfrak{B})$ and the category C-Comod is equivalent to the category $\mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M}, \mathfrak{B}) \subseteq \mathrm{Rep}_K(\mathcal{M})$ of locally nilpotent and locally finite K-linear representations of $\mathcal{M}$ [18] bound by the profinite relation ideal $\mathfrak{B}$. In view of Woodcock [41, (4.6)], the above results imply Corollary 5.23 asserting that, if the centre of the division algebra $\mathrm{End}_C S$ is a separable extension of K for any simple C-comodule S, then C is of the form $C \cong T^\square(\mathcal{M}, \mathfrak{B})$ and there is an equivalence C-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M}, \mathfrak{B})$, where $\mathcal{M}$ is the dual to the Ext-species of C and $\mathfrak{B}$ is a profinite relation ideal of the complete tensor K-algebra $\widehat{T}(\mathcal{M})$. Obviously, this happens if K is a perfect field.

Throughout this paper we assume that C is a basic K-coalgebra, with comultiplication Δ and counit ε. We fix the direct sum decompositions

$$ {}_C C = \bigoplus_{j\in I_C} E(j) \quad \text{and} \quad \operatorname{soc}{}_C C = \bigoplus_{j\in I_C} S(j), \tag{1.1} $$

where I_C is a set, $E(j)$ is an indecomposable injective comodule, $S(j)$ is a simple comodule, and $E(j)$ is the injective envelope of $S(j)$ for each $j \in I_C$; moreover, $E(i) \ncong E(j)$ and $S(i) \ncong S(j)$ for $i \neq j$. We set $F_j = \operatorname{End}_C S(j)$ for each $j \in I_C$. Note that $\{S(j)\}_{j\in I_C}$ is a complete set of pairwise nonisomorphic simple left C-comodules.

Here we use the coalgebra representation theory notation and terminology introduced in [29]–[31]. Given a K-coalgebra C with comultiplication Δ and counit ε, we denote by $C^* = \operatorname{Hom}_K(C,K)$ the K-dual algebra with respect to the convolution product (see [8], [23], [39]). The counit $\varepsilon : C \to K$ of C is the identity element of C^*. We view C^* as a pseudocompact K-algebra, with

$$ C^* \cong \varprojlim_{H_\beta} H_\beta^* \cong \varprojlim_{H_\beta} C^*/H_\beta^{\perp}, $$

where $H_\beta \subseteq C$ runs through all finite-dimensional subcoalgebras of C and $H_\beta^{\perp} = \{\varphi \in \operatorname{Hom}_K(C,K);\ \varphi(H_\beta) = 0\}$ is viewed as a cofinite ideal of C^*.

Let $\langle -,-\rangle : C^* \times C \to K$ be the non-degenerate bilinear form defined by $\langle \varphi, c\rangle = \varphi(c)$. We denote by $C_0 \subseteq C_1 \subseteq \cdots \subseteq C$ the coradical filtration of C, where $C_0 = \operatorname{soc}{}_C C = \operatorname{soc} C_C$. The reader is referred to [5], [8], [23], [39] for the coalgebra and comodule terminology, and to [1], [2], [28], [37], and [38] for the standard representation theory terminology and notation. In particular, given a unitary ring R, we denote by $J(R)$ the Jacobson radical of R, by $\operatorname{Mod}(R)$ the category of all unitary right R-modules, and by $\operatorname{mod}(R)$ the full subcategory of $\operatorname{Mod}(R)$ formed by finitely generated R-modules.

2. Preliminaries on quivers and path coalgebras. To make the paper self-contained, we recall briefly the terminology and notation introduced in [29] and [31], and some facts we need throughout this paper. A *quiver* $Q = (Q_0, Q_1)$ is an oriented graph (in general infinite) with the set Q_0 of vertices and the set Q_1 of arrows. We denote by Q_m the set of all oriented paths $\omega = \beta_1\beta_2\cdots\beta_m \equiv (a = i_0 \xrightarrow{\beta_1} i_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} i_m = b)$ of length $m \geq 1$ in $Q = (Q_0, Q_1)$ from a vertex $a = i_0$ to a vertex $b = i_m$. To any vertex $a \in Q_0$, we attach a *stationary path* e_a starting and ending at a. Given $a, b \in Q_0$, we denote by $Q(a,b)$ the set of all oriented paths ω in Q from a to b. A quiver Q is called *acyclic* if there is no oriented cycle in Q, that is, there is no path ω of positive length $m \geq 1$ with $a = b$. The vector space spanned by Q_m is denoted by KQ_m, and $KQ_m(a,b)$ is the subspace of $KQ(a,b)$ generated by all paths of length m. The *path K-algebra* of the quiver Q is the graded

K-vector space

$$KQ = KQ_0 \oplus KQ_1 \oplus \cdots \quad \text{with} \quad KQ_m = \bigoplus_{\omega \in Q_m} K\omega, \tag{2.1}$$

equipped with the obvious addition and multiplication (see [10, 4.2], [1], [2], [28]). It is clear that KQ is a graded K-algebra (with local units e_a), the stationary paths e_a, $a \in Q_0$, form a complete set of primitive orthogonal idempotents of KQ, and there is a right ideal decomposition $KQ = \bigoplus_{a \in Q_0} e_a KQ$. If Q_0 is finite then the element $\sum_{a \in Q_0} e_a$ is the identity of KQ; if Q_0 is infinite the algebra KQ has no identity element. It is clear that the dimension of KQ is finite if and only if Q is finite and acyclic. The two-sided ideal of KQ generated by all paths of length $m \geq 1$ has the form $KQ_{\geq m} = \bigoplus_{j \geq m} KQ_j$.

The *path K-coalgebra* of Q is the graded K-coalgebra

$$K^{\square}Q \equiv (KQ, \varDelta, \varepsilon), \tag{2.2}$$

where KQ is the graded K-vector space (path algebra) (2.1) endowed with the comultiplication $\varDelta : KQ \to KQ \otimes KQ$ and the counit $\varepsilon : KQ \to K$ defined as follows. Given a stationary path e_a at a, we set $\varDelta(e_a) = e_a \otimes e_a$ and $\varepsilon(e_a) = 1$. Given any path $\omega = \beta_1\beta_2\cdots\beta_m$ of length $m \geq 1$ from $a = i_0$ to $b = i_m$, we set

$$\varDelta(\omega) = e_a \otimes \omega + \omega \otimes e_b + \sum_{s=1}^{m-1} (\beta_1\beta_2\cdots\beta_s) \otimes (\beta_{s+1}\cdots\beta_m) \quad \text{and} \quad \varepsilon(\omega) = 0,$$

where $\otimes = \otimes_K$. Obviously, the K-vector space $K^{\square}Q_{\leq m} = KQ_0 \oplus KQ_1 \oplus \cdots \oplus KQ_m$ is a subcoalgebra of $K^{\square}Q$ for each $m \geq 0$. The coalgebra $K^{\square}Q$ is hereditary, basic, $K^{\square}Q_0 = KQ_0 = \operatorname{soc} K^{\square}Q = \bigoplus_{a \in Q_0} Ke_a$, $S(a) = Ke_a$ is a simple subcoalgebra of $K^{\square}Q$ for each $a \in Q_0$, and the subcoalgebra chain $K^{\square}Q_0 \subseteq K^{\square}Q_{\leq 1} \subseteq \cdots \subseteq K^{\square}Q_{\leq m} \subseteq \cdots$ is the coradical filtration of $K^{\square}Q$ (see [29, Proposition 8.1]). The notation $K^{\square}Q$ is inspired by the fact that $K^{\square}Q$ is isomorphic to the cotensor coalgebra

$$T^{\square}_{KQ_0}(KQ_1) = KQ_0 \oplus KQ_1 \oplus KQ_1 \square KQ_1 \oplus KQ_1 \square KQ_1 \square KQ_1 \oplus \ldots,$$

where $KQ_0 = \bigoplus_{a \in Q_0} Ke_a$ is viewed as a direct sum of the one-dimensional simple coalgebras Ke_a, and $KQ_1 = \bigoplus_{a,b \in Q_0} KQ_1(a,b)$ is viewed as a KQ_0-KQ_0-bicomodule in a natural way (see [7, Remark 4.2], [25], and [41]).

The following simple lemma is very useful.

LEMMA 2.3. *Let Q be a finite quiver and K a field.*

(a) *If Q is acyclic then $\dim_K K^{\square}Q$ is finite and there is a K-algebra isomorphism $KQ \cong (K^{\square}Q)^*$ defined by $\omega \mapsto \omega^*$ for any path ω in Q.*

(b) *If $m \geq 2$ then the subcoalgebra*

$$K^{\square}Q_{\leq m-1} = KQ_0 \oplus KQ_1 \oplus \cdots \oplus KQ_{m-1}$$

of $K^{\square}Q$ is finite-dimensional and there is a K-algebra isomorphism $(K^{\square}Q_{\leq m-1})^* \cong KQ/KQ_{\geq m}$.

Proof. (a) Since Q is finite and acyclic, the set Q_m is finite for each $m \geq 0$, and $Q_m = 0$ for m sufficiently large. Consequently, $\dim_K K^{\square}Q$ is finite. It follows from the definition of the convolution product in $(K^{\square}Q)^* = \mathrm{Hom}_K(K^{\square}Q, K)$ that the K-linear map $KQ \to (K^{\square}Q)^*$, defined on the paths ω (the elements of the standard K-basis of $K^{\square}Q$) by setting $\omega \mapsto \omega^*$, is a K-algebra isomorphism. Statement (b) follows in a similar way. ∎

Given a quiver Q and a field K, we define the *finite subquiver topology* (K-linear and Hausdorff) on KQ as follows (see [29, p. 133] and [31, p. 475]). First we note that there is a canonical algebra embedding $KQ \hookrightarrow (K^{\square}Q)^*$ defined by $\omega \mapsto \omega^*$ for any path ω in Q, and the non-degenerate bilinear form $\langle -,- \rangle : (K^{\square}Q)^* \times K^{\square}Q \to K$ defined by $\langle \varphi, c \rangle = \varphi(c)$ restricts to the non-degenerate bilinear form $\langle -,- \rangle_\delta : KQ \times K^{\square}Q \to K$ defined by $\langle \omega, \omega' \rangle_\delta = \delta_{\omega,\omega'}$ (the Kronecker delta) for any paths ω, ω' in KQ.

Given $m \geq 2$ and a finite subquiver $Q^{(x)}$ of Q, we consider the finite-dimensional subcoalgebra

$$K^{\square}Q^{(x)}_{\leq m-1} = KQ^{(x)}_0 \oplus KQ^{(x)}_1 \oplus \cdots \oplus KQ^{(x)}_{m-1}$$

of $K^{\square}Q$. By Lemma 2.3, the finite-dimensional algebra K-dual to $K^{\square}Q^{(x)}_{\leq m-1}$ has the form

$$(K^{\square}Q^{(x)}_{\leq m-1})^* \cong KQ^{(x)}/KQ^{(x)}_{\geq m}.$$

Let $U^{(x)}_m = \mathrm{Ker}\,\psi^{(x)}_m$ be the kernel of the composite algebra surjection

$$\psi^{(x)}_m = [KQ \hookrightarrow (K^{\square}Q)^* \xrightarrow{(u^{(x)}_m)^*} (K^{\square}Q^{(x)}_{\leq m-1})^* \cong KQ^{(x)}/KQ^{(x)}_{\geq m}],$$

where $u^{(x)}_m : K^{\square}Q^{(x)}_{\leq m-1} \hookrightarrow K^{\square}Q$ is the coalgebra embedding and $KQ \hookrightarrow (K^{\square}Q)^*$ is the canonical algebra embedding. It is clear that the K-linear topology defined by the ideals $U^{(x)}_m = \mathrm{Ker}\ \psi^{(x)}_m$ is Hausdorff; we call it the *finite subquiver topology* of KQ (cf. [11, Section 10]; see also [20] and [21]). The completion

$$\widehat{KQ} = \varprojlim_{Q^{(x)},\, m \geq 2} KQ/U^{(x)}_m \tag{2.4}$$

of KQ is obviously a pseudocompact K-algebra, where $Q^{(x)}$ runs through the finite subquivers of Q. We have the following improvement of Proposition 8.1 of [29].

PROPOSITION 2.5. *Let Q be an arbitrary quiver and K a field.*

(a) *For each $a \in Q_0$, the indecomposable left ideal $E(a) = (KQ)e_a$ of the path K-algebra KQ, spanned by all oriented paths in Q ending at a, is an indecomposable injective left coideal of $K^\square Q$, $\operatorname{soc} E(a) = S(a)$, and $K^\square Q = \bigoplus_{a \in Q_0} E(a)$.*

(b) *The left Gabriel quiver ${}_CQ$ of $C = K^\square Q$ is isomorphic to Q.*

(c) *The path coalgebra $K^\square Q$ is the directed union of the finite-dimensional subcoalgebras $K^\square Q^{(x)}_{\le m}$, where $m \ge 2$ and $Q^{(x)}$ runs through the finite subquivers of Q.*

(d) *Let $(K^\square Q)^*$ be the pseudocompact K-algebra K-dual to $K^\square Q$ and let $\widehat{KQ}$ be the completion (2.4) of the path algebra KQ in the finite subquiver topology. Then there are isomorphisms of pseudocompact K-algebras*

$$(K^\square Q)^* \cong \widehat{KQ} \cong \widehat{T}_{\widehat{KQ_0}}(\widehat{KQ_1}),$$

where

$$\begin{aligned}\widehat{T}_{\widehat{KQ_0}}(\widehat{KQ_1}) &= \widehat{KQ_0} \amalg \widehat{KQ_1} \amalg \widehat{KQ_1}^{\widehat{\otimes}^2} \amalg \cdots \amalg \widehat{KQ_1}^{\widehat{\otimes}^m} \amalg \cdots \\ &= \widehat{KQ_0} \oplus \prod_{m=1}^{\infty} \widehat{KQ_1}^{\widehat{\otimes}^m}\end{aligned}$$

is the complete tensor K-algebra [9, p. 96] *of the topological vector space $\widehat{KQ_1} = \prod_{a,b \in Q_0} K\widehat{Q_1(a,b)}$ viewed as a $\widehat{KQ_0}$-$\widehat{KQ_0}$-bimodule over the topological product $\widehat{KQ_0} = \prod_{a \in Q_0} Ke_a$ of Q_0 copies of the field K.*

(e) *$J((K^\square Q)^*) = (K^\square Q_0)^\perp$ is closed in $(K^\square Q)^*$ and the isomorphism $(K^\square Q)^* \cong \widehat{KQ}$ restricts to isomorphisms*

(e1) *$J((K^\square Q)^*) \cong J(\widehat{KQ}) = \widehat{J(KQ)} = \widehat{KQ_{\ge 1}}$,*

(e2) *$(K^\square Q_0 \oplus K^\square Q_1)^\perp = \overline{J((K^\square Q)^*)^2} \cong \overline{J(\widehat{KQ})^2} = \overline{(\widehat{J(KQ)})^2} = \widehat{KQ_{\ge 2}}$.*

Proof. (a) and (b) are consequences of the proof of [29, Prop. 8.1 and 8.13].

(c) It follows from the definition of the comultiplication in $K^\square Q$ that, given a finite subset $\mathcal{X}$ of $K^\square Q$, there exist a finite subquiver $Q^{(x)}$ of Q and an integer $m \ge 2$ such that $\mathcal{X}$ is contained in the finite-dimensional subcoalgebra $K^\square Q^{(x)}_{\le m-1}$ of $K^\square Q$. Hence (c) follows.

(d) We follow the proof given in [29, p. 134]. By (c), $K^\square Q$ is the directed union of the finite-dimensional subcoalgebras

$$H^{(x)}_m = K^\square (Q^{(x)})_{\le m-1} \subseteq K^\square Q^{(x)},$$

where $m \ge 2$ and $Q^{(x)}$ is a finite subquiver of Q. By Lemma 2.3, for any such

$H_m^{(x)}$, there is an isomorphism of finite-dimensional K-algebras $(H_m^{(x)})^* \cong KQ^{(x)}/KQ^{(x)}_{\geq m}$, and the restriction

$$\psi_m^{(x)} : KQ \to (H_m^{(x)})^* \cong KQ^{(x)}/KQ^{(x)}_{\geq m}$$

of the canonical algebra surjection $(K^\square Q)^* \to (H_m^{(x)})^*$ to the subalgebra KQ of $(K^\square Q)^*$ is surjective. Hence we get isomorphisms

$$\begin{aligned}(2.6)\qquad (K^\square Q)^* &= \Big(\bigcup_{Q^{(x)},\, m\geq 2} H_m^{(x)}\Big)^* \cong \varprojlim_{Q^{(x)},\, m\geq 2} (K^\square Q)^*/(H_m^{(x)})^\perp \\ &\cong \varprojlim_{Q^{(x)},\, m\geq 2} (H_m^{(x)})^* \cong \varprojlim_{Q^{(x)},\, m\geq 2} KQ/U_m^{(x)} = \widehat{KQ}\end{aligned}$$

of pseudocompact K-algebras, where $Q^{(x)}$ runs through finite subquivers of Q. Now (e) follows from [23, Proposition 5.2.9] (and its proof), by applying the foregoing definitions (see also [11, Section 10], [20], [21], [29] and [41]). The details are left to the reader. Here we only recall that there is a duality $\mathcal{C}oalg_K \cong \mathcal{PC}_K^{\mathrm{op}}$ between the categories of K-coalgebras and of pseudocompact K-algebras (see [29, Theorem 3.6]), and we note that the coalgebra embeddings $K^\square Q_0 \hookrightarrow K^\square Q_0 \oplus K^\square Q_1 \hookrightarrow K^\square Q$ induce topological algebra surjections

$$(K^\square Q)^* \xrightarrow{f_1} (K^\square Q_0 \oplus K^\square Q_1)^* \xrightarrow{f_0} (K^\square Q_0)^* = \widehat{KQ_0} = \prod_{a\in Q_0} Ke_a$$

with $\operatorname{Ker} f_0 f_1 = (K^\square Q_0)^\perp = J((K^\square Q)^*)$ and $\operatorname{Ker} f_1 = (K^\square Q_0 \oplus K^\square Q_1)^\perp = \overline{J((K^\square Q)^*)^2}$.

To get an isomorphism $\widehat{T}_{\widehat{KQ_0}}(\widehat{KQ_1}) \cong \widehat{KQ}$ required in (d), we note that the embedding $\widehat{KQ_0} \oplus \widehat{KQ_1} \hookrightarrow \widehat{KQ}$ uniquely extends to a continuous homomorphism $\Phi : \widehat{T}_{\widehat{KQ_0}}(\widehat{KQ_1}) \to \widehat{KQ}$ of pseudocompact K-algebras, which is obviously surjective (see [9, p. 96]). By routine arguments, Φ is injective, and hence an isomorphism. This finishes the proof. ∎

Remark 2.7. In [11, Section 10], a description of the pseudocompact K-algebra $\widehat{KQ}$ in terms of Cauchy nets is given, where the path algebra KQ is equipped with a K-precompact topology, which is obviously equivalent to our finite subquiver topology (see also [20] and [21]).

3. Comodules and representations of quivers. Given a quiver $Q = (Q_0, Q_1)$, we denote by $\operatorname{Rep}_K(Q)$ the category of K-linear representations of Q, and by $\operatorname{rep}_K(Q)$ the full subcategory of $\operatorname{Rep}_K(Q)$ whose objects are the finitely generated representations. Given a representation $X =$

$(X_a, \varphi_\beta^X)_{a\in Q_0,\,\beta\in Q_1}$ in $\mathrm{Rep}_K(Q)$, we define its *support* to be the subquiver

$$Q^X = (Q_0^X, Q_1^X)$$

of Q, with $Q_0^X = \{a \in Q_0;\ X_a \neq 0\}$ and $Q_1^X = \{\beta \in Q_1;\ \varphi_\beta^X \neq 0\}$. We call X *of finite length* if Q^X is a finite subquiver of Q and $\dim_K X_a$ is finite for any $a \in Q_0^X$. We denote by $\mathrm{rep}_K^{\ell f}(Q) \supseteq \mathrm{nilrep}_K^{\ell f}(Q)$ the full subcategories of $\mathrm{rep}_K(Q)$ whose objects are the finite length representations and nilpotent representations of finite length, respectively. We denote by $\mathrm{Rep}_K^{\ell f}(Q)$ the category of locally finite length representations, that is, directed unions of finite length representations in $\mathrm{rep}_K^{\ell f}(Q)$. Finally, $\mathrm{Rep}_K^{\ell n\ell f}(Q)$ is the full subcategory of $\mathrm{Rep}_K^{\ell f}(Q)$ whose objects are the locally finite locally nilpotent representations in $\mathrm{nilrep}_K^{\ell f}(Q)$, that is, directed unions of finite length representations that are nilpotent.

Now we give an explicit description of a correspondence between left $K^\square Q$-comodules and K-linear representations of the quiver Q. To define it, we recall that given a K-coalgebra C, any left C-comodule $X = (X, \delta_X)$, with $\delta_X : X \to C \otimes X$, can be viewed as a right rational (= discrete) C^*-module via the action $x \cdot \varphi^* = \sum_{(x)} \varphi(c_{(1)})x_{(2)}$, where $\varphi \in C^*$, $x \in X$, and $\delta_M(x) = \sum_{(x)} c_{(1)} \otimes x_{(2)}$ (see [29], [39], [40]). It is shown in [39] that by associating to any left C-comodule X the underlying vector space X endowed with rational right C^*-module structure, we define a categorical isomorphism $C\text{-Comod} \cong \mathrm{Rat}(C^*) = \mathrm{Dis}(C^*)$, where $\mathrm{Rat}(C^*)$ is the category of rational right C^*-modules and $\mathrm{Dis}(C^*)$ is the category of discrete right C^*-modules. The reader is referred to [15], [29, Theorem 4.3(a)] and [40] for the equality $\mathrm{Rat}(C^*) = \mathrm{Dis}(C^*)$. By using the right action of $(K^\square Q)^* \cong \widehat{KQ}$ on left $K^\square Q$-comodules X, we define the K-linear functor

$$F : K^\square Q\text{-Comod} \to \mathrm{Rep}_K(Q) \tag{3.1}$$

as in [29], [41]. Given a comodule X in $K^\square Q$-Comod (viewed as a rational right module over $(K^\square Q)^* \cong \widehat{KQ}$) we define $F(X)$ in $\mathrm{Rep}_K(Q)$ by setting

$$F(X) = (X_a, \varphi_\beta^X)_{a\in Q_0,\,\beta\in Q_1}, \tag{3.2}$$

where $X_a = X \cdot e_a = X \cdot e_a^*$ and $e_a \in KQ \subseteq \widehat{KQ}$ is the stationary path at a (see Example 3.6 below). Here the algebra embedding $KQ \subseteq \widehat{KQ}$ is defined by $\omega \mapsto \omega^*$ for any path ω in Q. For any arrow $\beta : a \to b$, the K-linear map $\varphi_\beta^X : X_a \to X_b$ is defined by $\varphi_\beta^X(x \cdot e_a) = (x \cdot e_a) \cdot \beta = x \cdot e_a \beta e_b$, where $x \in X$ and $x \cdot e_a \in X_a = X \cdot e_a$. Since $\beta \in Q_1$ is viewed as an element of KQ, we have $\beta = e_a\beta = \beta e_b$ in KQ. Given a $K^\square Q$-comodule homomorphism $f : X \to Y$, we set $F(f) = (f_a)_{a\in Q_0}$, where $f_a : X_a \to Y_a$ is the restriction of f to X_a (see the proof of Theorem III.1.6 in [1]). It is clear that $F(f) : F(X) \to F(Y)$ is a morphism in $\mathrm{Rep}_K(Q)$ and that we have defined a covariant K-linear exact faithful functor $F : K^{\square}Q\text{-Comod} \to \mathrm{Rep}_K(Q)$ that restricts to a functor $F : K^{\square}Q\text{-comod} \to \mathrm{rep}_K(Q)$.

PROPOSITION 3.3. *Let Q be an arbitrary quiver and K a field.*

(a) *The functor F of* (3.1) *commutes with arbitrary direct sums and directed unions, and restricts to two equivalences of categories*

$$\begin{array}{ccc} K^{\square}Q\text{-Comod} & \xrightarrow{F} & \mathrm{Rep}_K^{\ell n\ell f}(Q) \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ K^{\square}Q\text{-comod} & \xrightarrow{F} & \mathrm{nilrep}_K^{\ell f}(Q) \end{array} \tag{3.4}$$

making the diagram commutative.

(b) *The functor F^{-1} inverse to F associates to any representation $X = (X_a, \varphi_\beta^X)_{a\in Q_0,\,\beta\in Q_1}$ in $\mathrm{Rep}_K^{\ell n\ell f}(Q)$ the vector space $F^{-1}(X) = \bigoplus_{a\in Q_0} X_a$ equipped with the left $K^{\square}Q$-comodule structure induced by the natural discrete right module structure on the profinite K-algebra $(K^{\square}Q)^* \cong \widehat{KQ}$.*

Proof. That F commutes with arbitrary direct sums and directed unions follows immediately from the definition (3.2) of F. To prove the second part of (a), we note that if $Q^{(x)}$ is a finite subquiver of Q and $m \geq 2$ then, by Lemma 2.3, the subcoalgebra $K^{\square}Q_{\leq m-1} = KQ_0 \oplus KQ_1 \oplus \cdots \oplus KQ_{m-1}$ of $K^{\square}Q$ is finite-dimensional and there is a K-algebra isomorphism $(K^{\square}Q_{\leq m-1})^* \cong KQ/KQ_{\geq m}$.

Then $K^{\square}Q^{(x)}_{\leq m-1}\text{-comod} \subseteq K^{\square}Q\text{-comod}$ and, by the definition of F, given a comodule X in $K^{\square}Q^{(x)}_{\leq m-1}$-comod, the support $Q^{F(X)}$ of the representation $F(X)$ is a finite subquiver of $Q^{(x)} \subseteq Q$. Hence it follows easily that F restricts to the K-linear functor

$$F_m^{(x)} : K^{\square}Q^{(x)}_{\leq m-1}\text{-comod} \to \mathrm{rep}_K^{\ell f}(Q^{(x)}, KQ^{(x)}_{\geq m}) \cong \mathrm{mod}\, KQ^{(x)}/KQ^{(x)}_{\geq m},$$

where $\mathrm{rep}_K^{\ell f}(Q^{(x)}, KQ^{(x)}_{\geq m})$ is the full subcategory of $\mathrm{rep}_K^{\ell f}(Q^{(x)})$ consisting of the representations $X = (X_a, \varphi_\beta^X)_{a\in Q_0^{(x)},\beta\in Q_1^{(x)}}$ such that the composite K-linear map

$$\varphi_\omega^X = [X_a \xrightarrow{\varphi_{\beta_1}^X} X_{i_1} \xrightarrow{\varphi_{\beta_2}^X} \cdots \xrightarrow{\varphi_{\beta_m}^X} X_b]$$

is zero for any oriented path $\omega \equiv (a = i_0 \xrightarrow{\beta_1} i_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} i_m = b)$ of length m in $Q^{(x)}$. This shows that, given a finite subquiver $Q^{(x)}$ of Q, $m \geq 2$, and a comodule M in $K^{\square}Q^{(x)}_{\leq m-1}$-comod, the representation $F_m^{(x)}(M) = F(M)$ is nilpotent, that is, $F_m^{(x)}(X) = F(X)$ lies in $\mathrm{nilrep}_K^{\ell f}(Q^{(x)})$. Since, by Proposition 2.5(c), $K^{\square}Q$ is the directed union of the finite-dimensional subcoalgebras $K^{\square}Q^{(x)}_{\leq m-1}$, where $m \geq 2$ and $Q^{(x)}$ runs through the finite subquivers of Q, it follows that $F(X)$ lies in $\operatorname{nilrep}_K^{\ell f}(Q)$ for any X in $K^\square Q$-comod. Hence, given X in $K^\square Q$-Comod, the representation $F(X)$ lies in $\operatorname{Rep}_K^{\ell n\ell f}(Q)$.

Conversely, assume that $X = (X_a, \varphi_\beta^X)_{a\in Q_0,\, \beta\in Q_1}$ is a representation of Q lying in $\operatorname{nilrep}_K^{\ell f}(Q)$. Then its support Q^X is a finite subquiver of Q, X is nilpotent, and hence there is an $m \geq 2$ such that the composite K-linear map $\varphi_\omega^X = \varphi_{\beta_m}^X \cdots \varphi_{\beta_1}^X : X_a \to X_b$ is zero for any oriented path $\omega \equiv (a = i_0 \xrightarrow{\beta_1} i_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} i_m = b)$ of length m in Q. If we set $Q^{(x)} = Q^X$ and $m \geq 2$ as above, then X lies in $\operatorname{nilrep}_K^{\ell f}(Q^{(x)}) \subseteq \operatorname{nilrep}_K^{\ell f}(Q)$ and the definition of F shows that X lies in the image of the composite functor

$$F_m^{(x)} : K^\square Q_{\leq m-1}^{(x)}\text{-comod} \to \operatorname{rep}_K^{\ell f}(Q^{(x)}, KQ_{\geq m}^{(x)}) = \operatorname{nilrep}_K^{\ell f}(Q^{(x)}),$$

because the vector space $\widetilde{X} = \bigoplus_{a\in Q_0} X_a$, viewed as a right module over $(K^\square Q)^*$ in a natural way, is a finite-dimensional module over the finite-dimensional quotient $(K^\square Q_{\leq m-1}^{(x)})^* \cong KQ^{(x)}/KQ_{\geq m}^{(x)}$ of $(K^\square Q)^* \cong \widehat{KQ}$ (see the proof of Th. III.1.6 in [1]). This shows that $\widetilde{X}$ lies in $\operatorname{dis}((K^\square Q_{\leq m-1}^{(x)})^*)$ $= \operatorname{rat}((K^\square Q_{\leq m-1}^{(x)})^*)$, that is, $\widetilde{X}$ has a natural structure of a left $K^\square Q_{\leq m-1}$-comodule such that $F(\widetilde{X}) \cong X$. It follows that every X in $\operatorname{nilrep}_K^{\ell f}(Q)$ lies in the image of F. Similarly, we show that any morphism $f : X \to Y$ in $\operatorname{nilrep}_K^{\ell f}(Q)$ lies in the image of F. Consequently, $F : K^\square Q$-comod $\to$ $\operatorname{nilrep}_K^{\ell f}(Q)$ is an equivalence of categories and, by standard limit arguments, so is $F : K^\square Q$-Comod $\to \operatorname{Rep}_K^{\ell n\ell f}(Q)$, because it commutes with directed unions. ∎

The following corollary is a consequence of the above proof.

COROLLARY 3.5. *Let K be a field and Q a quiver. For any representation X in* $\operatorname{nilrep}_K^{\ell f}(Q)$ *there exists a finite subquiver $Q^{(x)}$ of Q and an integer $m \geq 2$ such that X lies in* $\operatorname{rep}_K^{\ell f}(Q^{(x)}, KQ_{\geq m}^{(x)}) \subseteq \operatorname{nilrep}_K^{\ell f}(Q)$.

Now we illustrate the definition of the functor $F : K^\square Q$-Comod $\to$ $\operatorname{Rep}_K^{\ell n\ell f}(Q)$ of (3.4) by an example.

EXAMPLE 3.6. Let Q be an arbitrary quiver, $b \in Q_0$ a fixed vertex in Q and $X = E(b) = (KQ)e_b$ the left indecomposable direct summand of the path K-coalgebra $K^\square Q$. Note that $(KQ)e_b$ is the left ideal of the path algebra KQ generated by the stationary path e_b at b. Obviously, the vector space $(KQ)e_b$ is generated by all paths in Q that terminate at b. We view $(KQ)e_b$ as a right rational module over the algebra $(K^\square Q)^*$, K-dual to $K^\square Q$, and we define the canonical K-algebra embedding $KQ \subseteq (K^\square Q)^* \cong \widehat{KQ}$ by $\omega \mapsto \omega^*$ for any path ω in Q. Then the left $K^\square Q$-comodule $X = E(b) = (KQ)e_b$ has a right module structure over KQ. It is easy to see that, given a stationary path e_c of KQ and a path

$$\omega = \beta_1\beta_2\cdots\beta_m \equiv (a = i_0 \xrightarrow{\beta_1} i_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} i_m = b)$$

of length $m \geq 0$ in $Q = (Q_0, Q_1)$, we have $\omega\cdot e_c = \omega\cdot e_c^* = e_c\omega$, where $e_c\omega$ is a product in KQ, and $\omega \cdot e_c = \omega \cdot e_c^*$ means the rational action of e_c (identified with $e_c^* : K^\square Q \to K$) on the path $\omega \in X = E(b) = (KQ)e_b$. Moreover, given an arrow $\beta : c \to c'$ in Q_1 and a path $\omega = \beta_1\beta_2\cdots\beta_m \in X = E(b) = (KQ)e_b$ of length $m \geq 1$, we have

$$\omega \cdot \beta = \omega \cdot \beta^* = \begin{cases} \beta_2\cdots\beta_m & \text{if } \beta = \beta_1, \\ 0 & \text{if } \beta \neq \beta_1. \end{cases}$$

It follows that the K-linear representation $F(X) \in \mathrm{Rep}_K^{\ell n\ell f}(Q)$ (see (3.2)) of the quiver Q has the form $F(X) = (X_a, \varphi_\beta^X)_{a\in Q_0,\,\beta\in Q_1}$, where

$$X_a = X \cdot e_a = X \cdot e_a^* = e_a(KQ)e_b = KQ(a, b)$$

is the K-vector space spanned by all oriented paths from a to b.

Given an arrow $\beta : a \to c$, the K-linear map $\varphi_\beta^X : X_a = KQ(a, b) \to X_c = KQ(c, b)$ is defined by the formula

$$\varphi_\beta^X(\omega) = \begin{cases} \beta_2\cdots\beta_m & \text{if } c = j_1 \text{ and } \beta = \beta_1, \\ 0 & \text{if } \beta \neq \beta_1, \end{cases}$$

for any path $\omega = \beta_1\beta_2\cdots\beta_m \equiv (a = i_0 \xrightarrow{\beta_1} i_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} i_m = b)$ in $Q(a, b)$.

An important role in the study of basic K-coalgebras C is played by the full subcategory C-Comp of C-Comod (introduced in [33]) consisting of all *computable comodules*, that is, the C-comodules X such that $\dim_K \mathrm{Hom}_C(X, E(j))$ is finite for any indecomposable left C-comodule $E(j)$ (see (1.1)). In view of the functors (3.4), given a quiver Q, we define the category $\mathrm{Comp}_K(Q)$ of computable representations of Q to be the full subcategory of $\mathrm{Rep}_K(Q)$ formed by all representations $X = (X_a, \varphi_\beta^X)_{a\in Q_0,\,\beta\in Q_1}$ of Q such that $\dim_K X_a$ is finite for each $a \in Q_0$. We call such a representation X *computable.*

COROLLARY 3.7. *Let K be a field and Q a quiver. The functor* (3.4) *restricts to a K-linear equivalence of categories*

$$F : K^\square Q\text{-Comp} \to \mathrm{Comp}_K^{\ell n\ell f}(Q) = \mathrm{Comp}_K(Q) \cap \mathrm{Rep}_K^{\ell n\ell f}(Q). \tag{3.8}$$

Proof. We recall from Proposition 2.5(a) that $E(a) = (KQ)e_a$ is an indecomposable summand of $K^\square Q$. We show that the representation $F(X)$ is computable if and only if X is a computable $K^\square Q$-comodule, that is, $\dim_K \mathrm{Hom}_{K^\square Q}(X, E(j))$ is finite for any $j \in Q_0$. In this case, we show that, for each $j \in Q_0$, there is a K-linear isomorphism

$$X_j = X \cdot e_j \xrightarrow[\cong]{\theta_j} \mathrm{Hom}_{K^\square Q}(X, E(j))^* \tag{3.9}$$

(in the notation of (3.2)), that is functorial in $K^{\square}Q$-comodule homomorphisms $f : X \to Y$. Given $j \in Q_0$, we set $\theta_j = \xi_j^*$, where ξ_j is the unique K-linear isomorphism making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{K^{\square}Q}(X, E(j)) & \xrightarrow{\xi_j} & (X \cdot e_j)^* \\ \big\downarrow{\scriptstyle \widetilde{u}_j} & & \big\downarrow{\scriptstyle \pi_j^*} \\ \mathrm{Hom}_{K^{\square}Q}(X, K^{\square}Q) & \xrightarrow{\widetilde{\varepsilon}_X} & X^* \end{array}$$

Here $\widetilde{\varepsilon}_X$ is the Yoneda isomorphism given by $f \mapsto \varepsilon f$ (see [39] and [29, Lemma 4.9]), $\widetilde{u}_j = \mathrm{Hom}_{K^{\square}Q}(X, u_j)$ with $u_j : E(j) \to K^{\square}Q$ the inclusion, and $\pi_j : X \to X_j = X \cdot e_j$ is the retraction given by $x \mapsto x \cdot e_j$. Hence, $\dim_K \mathrm{Hom}_{K^{\square}Q}(X, E(j))$ is finite if and only if $\dim_K X_j$ is finite. It follows that a comodule X is computable if and only if the representation $F(X)$ of Q is computable. Since, by Proposition 3.3, $F(X)$ is locally nilpotent and locally finite, the corollary follows. ■

4. Bound quiver coalgebras. We recall from [29] and [31, Definition 3.5] that a *bound quiver* (or a *quiver with relations*) is a pair (Q, Ω), where Q is a quiver (in general infinite) and Ω is a two-sided ideal of the path K-algebra KQ such that $\Omega \subseteq KQ_{\geq 2}$. Every such ideal Ω is called an *ideal of relations*, or a *relation ideal* of KQ. It is easy to see that any relation ideal Ω of KQ has a decomposition $\Omega = \bigoplus_{a,b \in Q_0} \Omega(a,b)$, where $\Omega(a,b) = \Omega \cap KQ(a,b)$.

If (Q, Ω) is a quiver with relations, we define $\mathrm{rep}_K(Q, \Omega) \supseteq \mathrm{rep}_K^{\ell f}(Q, \Omega) \supseteq \mathrm{nilrep}_K^{\ell f}(Q, \Omega)$ to be the corresponding full subcategories of $\mathrm{rep}_K(Q) \supseteq \mathrm{rep}_K^{\ell f}(Q) \supseteq \mathrm{nilrep}_K^{\ell f}(Q)$ formed by the K-linear representations of Q satisfying all relations in Ω (see [10, 4.2] and [28, Section 14.1]). Analogously, we define the categories $\mathrm{Rep}_K(Q, \Omega) \supseteq \mathrm{Rep}_K^{\ell n}(Q, \Omega) \supseteq \mathrm{Rep}_K^{\ell n \ell f}(Q, \Omega)$.

DEFINITION 4.1.

(a) A *profinite bound quiver* (or a *quiver with profinite relations*) is a pair $(Q, \mathfrak{B})$, where Q is a quiver and $\mathfrak{B}$ is a closed two-sided ideal (called a *profinite relation ideal*) of the profinite K-algebra $\widehat{KQ}$ of (2.4) such that $\mathfrak{B} \subseteq \widehat{KQ_{\geq 2}}$.

(b) Let $\mathfrak{B} \subseteq \widehat{KQ_{\geq 2}} \subseteq \widehat{KQ}$ be a profinite relation ideal, $Q^{(x)}$ a finite subquiver of Q, and

$$\widehat{\psi}_m^{(x)} : \widehat{KQ} \to KQ/U_m^{(x)} \cong KQ^{(x)}/KQ_{\geq m}^{(x)},$$

with $m \geq 2$, the canonical algebra surjection. Fix a relation ideal $\mathfrak{B}_m^{(x)}$ of $KQ^{(x)}$ such that

$$\widehat{\psi}_m^{(x)}(\mathfrak{B}) = \mathfrak{B}_m^{(x)}/KQ_{\geq m}^{(x)} \subseteq KQ^{(x)}/KQ_{\geq m}^{(x)}.$$

The bound quiver $(Q^{(x)}, \mathfrak{B}_m^{(x)} + KQ_{\geq m}^{(x)})$ is called the *projection* of $(Q, \mathfrak{B})$ on $(Q^{(x)}, KQ_{\geq m}^{(x)})$.

By Corollary 3.5, given a representation X in $\mathrm{nilrep}_K^{\ell f}(Q)$, there exist an $m \geq 2$ and a finite subquiver $Q^{(x)}$ of Q such that X lies in $\mathrm{rep}_K^{\ell f}(Q^{(x)}, Q_{\geq m}^{(x)}) \subseteq \mathrm{nilrep}_K^{\ell f}(Q)$. We define X to *satisfy the relations* in $\mathfrak{B}$ if X satisfies the relations in $\mathfrak{B}_m^{(x)} + Q_{\geq m}^{(x)}$, that is, X lies in $\mathrm{rep}_K(Q^{(x)}, \mathfrak{B}_m^{(x)} + Q_{\geq m}^{(x)}) \subseteq \mathrm{rep}_K^{\ell f}(Q^{(x)}, Q_{\geq m}^{(x)})$. A representation Y in $\mathrm{Rep}_K^{\ell n \ell f}(Q)$ is defined to *satisfy the relations* in $\mathfrak{B}$ if any subrepresentation X of Y lying in $\mathrm{nilrep}_K^{\ell f}(Q)$ satisfies the relations in $\mathfrak{B}$.

Given a profinite bound quiver $(Q, \mathfrak{B})$, we denote by $\mathrm{Rep}_K^{\ell n \ell f}(Q, \mathfrak{B}) \supseteq \mathrm{nilrep}_K^{\ell f}(Q, \mathfrak{B})$ the full subcategories of $\mathrm{Rep}_K^{\ell n \ell f}(Q)$ whose objects are the representations in $\mathrm{Rep}_K^{\ell n \ell f}(Q)$ (resp. in $\mathrm{nilrep}_K^{\ell f}(Q)$) that satisfy the relations in $\mathfrak{B}$.

The following definition is very useful (see [29, p. 135], [30] and [31, Definition 3.8]).

Definition 4.2. Let K be a field, (Q, Ω) a bound quiver, and $(Q, \mathfrak{B})$ a profinite bound quiver.

(a) The *path K-coalgebra of the bound quiver* (Q, Ω) is the subcoalgebra

$$K^{\square}(Q, \Omega) = C(Q, \Omega) = \{c \in K^{\square}Q;\ \langle \Omega, c\rangle_\delta = 0\} \tag{4.3}$$

of $K^{\square}Q$, where $\langle -, - \rangle_\delta : KQ \times K^{\square}Q \to K$ is the standard non-degenerate symmetric K-bilinear form defined by $\langle u, w\rangle_\delta = \delta_{u,w}$ for all paths u, w in Q.

(b) The *path K-coalgebra of the profinite bound quiver* $(Q, \mathfrak{B})$ is the subcoalgebra

$$K^{\square}(Q, \mathfrak{B}) = \{c \in K^{\square}Q;\ \langle \mathfrak{B}', c\rangle = 0\} \tag{4.4}$$

of $K^{\square}Q$, where $\langle -, - \rangle : (K^{\square}Q)^* \times K^{\square}Q \to K$ is the non-degenerate symmetric K-bilinear form defined by $\langle \varphi, c\rangle = \varphi(c)$, and $\mathfrak{B}'$ is the image of $\mathfrak{B} \subseteq \widehat{KQ}$ under the isomorphism $\widehat{KQ} \cong (K^{\square}Q)^*$ (see (2.6)) of pseudocompact K-algebras.

(c) A K-coalgebra C is defined to be a *bound quiver coalgebra* if there are a bound quiver (Q, Ω) and a coalgebra isomorphism $C \cong K^{\square}(Q, \Omega)$; and C is defined to be a *profinite bound quiver coalgebra* if there is a profinite bound quiver $(Q, \mathfrak{B})$ and a coalgebra isomorphism $C \cong K^{\square}(Q, \mathfrak{B})$.

We recall from [31, Definition 3.13] that $H \subseteq K^{\square}Q$ is a *relation subcoalgebra* if H contains $KQ_0 \oplus KQ_1$. In this case $H = \bigoplus_{a,b \in Q_0} H(a, b)$, where $H(a, b) = H \cap KQ(a, b)$ (see [13] and [14]). A quiver Q is said to be *intervally* *finite* if, for each pair a, b of vertices of Q, the set $Q(a,b)$ of all paths from a to b in Q is finite.

The following result follows from [31, Theorem 3.14] and its proof.

THEOREM 4.5. *Let K be a field and Q a quiver.*

(a) *Given a relation ideal Ω of KQ, the subspace $K^{\square}(Q,\Omega)$ (see (4.3)) of $K^{\square}Q$ is a basic relation subcoalgebra of $K^{\square}Q$ whose Gabriel quiver is isomorphic to Q.*

(b) *The K-linear category equivalences* (3.4) *restrict to the category equivalences*

$$\begin{array}{ccc} K^{\square}(Q,\Omega)\text{-Comod} & \xrightarrow{F} & \mathrm{Rep}_K^{\ell n\ell f}(Q,\Omega) \\ \bigcup\!\!\!\uparrow & & \bigcup\!\!\!\uparrow \\ K^{\square}(Q,\Omega)\text{-comod} & \xrightarrow{F} & \mathrm{nilrep}_K^{\ell f}(Q,\Omega) \end{array} \tag{4.6}$$

(c) *If C is a relation subcoalgebra of $K^{\square}Q$, then the vector space $C^{\perp} = \{c \in KQ;\ \langle c, C\rangle_{\delta} = 0\}$ is a two-sided relation ideal of KQ.*

(d) *If Q is an intervally finite quiver then the map $\Omega \mapsto K^{\square}(Q,\Omega)$ defines a bijection between the set of relation ideals Ω of KQ and the set of relation subcoalgebras C of $K^{\square}Q$. The inverse map is given by $C \mapsto C^{\perp}$.*

REMARKS 4.7. (a) Statement (d) of Theorem 4.5 is proved in [31, Theorem 3.14(c), (d)] under the assumption that Q is locally finite. Unfortunately, this assumption is not sufficient (see [13]). Under the assumption that Q is intervally finite, made in Theorem 4.5, the proof given in [31, pp. 477–478] goes through.

(b) A proof of (d) can also be found in [13], where a criterion is given to decide whether or not a relation subcoalgebra of $K^{\square}Q$ is of the form $K^{\square}(Q,\Omega)$. It is shown in [14] that if Q is acyclic, then any tame relation subcoalgebra of $K^{\square}Q$ is of the form $K^{\square}(Q,\Omega)$.

(c) The problem whether or not any tame relation subcoalgebra of $K^{\square}Q$ is of the form $K^{\square}(Q,\Omega)$ remains open.

Now we study profinite bound quiver coalgebras and their comodules. One of the main results of this section is the following theorem, which extends Theorem 4.5 to profinite bound quiver coalgebras.

THEOREM 4.8. *Let K be a field, Q a quiver, and $(Q,\mathfrak{B})$ a profinite bound quiver with a profinite relation ideal $\mathfrak{B} \subseteq \widehat{KQ_{\geq 2}} \subseteq \widehat{KQ}$.*

(a) *The subspace $K^{\square}(Q,\mathfrak{B})$ (see (4.4)) of the path K-coalgebra $K^{\square}Q$ is a basic relation subcoalgebra of $K^{\square}Q$ whose Gabriel quiver is isomorphic to Q.*

(b) *The coalgebra* $H = K^{\square}(Q, \mathfrak{B})$ *is the directed union of the finite-dimensional subcoalgebras of the form*

$$H_m^{(x)} = K^{\square}Q_{\leq m-1}^{(x)} \cap K^{\square}(Q^{(x)}, \mathfrak{B}_m^{(x)}),$$

where $Q^{(x)}$ *runs through the finite subquivers of* Q, $m \geq 2$, *and* $\mathfrak{B}_m^{(x)} \subseteq KQ^{(x)}$ *is a relation ideal such that* $(Q^{(x)}, \mathfrak{B}_m^{(x)} + Q_{\geq m}^{(x)})$ *is the projection of* $(Q, \mathfrak{B})$ *on* $(Q^{(x)}, Q_{\geq m}^{(x)})$ (*see* (4.1)).

(c) *There are isomorphisms*

$$(H_m^{(x)})^* \cong KQ_{\geq m}^{(x)}/(\mathfrak{B}_m^{(x)} + KQ_{\geq m})$$

of finite-dimensional K*-algebras that are functorial with respect to coalgebra embeddings* $H_m^{(x)} \hookrightarrow H_{m'}^{(x')}$ *and induce isomorphisms*

$$H^* \cong \widehat{KQ}/\mathfrak{B} \cong \varprojlim_{Q^{(x)},\, m \geq 2} KQ^{(x)}/(\mathfrak{B}_m^{(x)} + KQ_{\geq m})$$

of pseudocompact K*-algebras, where* $Q^{(x)}$ *runs through the finite subquivers of* Q.

(d) *The* K*-linear category equivalences* (3.4) *restrict to category equivalences*

$$\begin{array}{ccc} K^{\square}(Q, \mathfrak{B})\text{-Comod} & \xrightarrow{F} & \mathrm{Rep}_K^{\ell n \ell f}(Q, \mathfrak{B}) \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ K^{\square}(Q, \mathfrak{B})\text{-comod} & \xrightarrow{F} & \mathrm{nilrep}_K^{\ell f}(Q, \mathfrak{B}) \end{array} \tag{4.9}$$

Proof. (a) We recall that the profinite topology on the profinite K-algebra $\widehat{KQ}$ is defined by the kernels $\widehat{U}_m^{(x)} = \mathrm{Ker}\, \widehat{\psi}_m^{(x)}$ of the canonical algebra surjections

$$\widehat{\psi}_m^{(x)} : \widehat{KQ} \to KQ/U_m^{(x)} \cong KQ^{(x)}/KQ_{\geq m}^{(x)},$$

where $Q^{(x)}$ runs through the finite subquivers of Q and $m \geq 2$. Since $\mathfrak{B} \subseteq \widehat{KQ_{\geq 2}}$ is a profinite relation ideal of $\widehat{KQ}$, $\mathfrak{B}$ is a closed ideal in $\widehat{KQ}$, and the K-linear topology on the quotient algebra

$$\Lambda_{\mathfrak{B}} = \widehat{KQ}/\mathfrak{B},$$

induced by that of $\widehat{KQ}$, is given by the ideals

$$(\widehat{U}_m^{(x)} + \mathfrak{B})/\mathfrak{B} \cong \widehat{U}_m^{(x)}/(\mathfrak{B} \cap \widehat{U}^{(x)})_m.$$

It is clear that the topology on $\Lambda_{\mathfrak{B}}$ is Hausdorff. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{B}\cap \widehat{U}_m^{(x)} & \longrightarrow & \mathfrak{B} & \longrightarrow & (\mathfrak{B}_m^{(x)} + KQ^{(x)}_{\geq m})/KQ^{(x)}_{\geq m} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widehat{U}_m^{(x)} & \longrightarrow & \widehat{KQ} & \xrightarrow{\widehat{\psi}_m^{(x)}} & KQ^{(x)}/KQ^{(x)}_{\geq m} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\widehat{U}_m^{(x)}+\mathfrak{B})/\mathfrak{B} & \longrightarrow & \widehat{KQ}/\mathfrak{B} & \xrightarrow{\widetilde{\psi}_m^{(x)}} & KQ^{(x)}/(\mathfrak{B}_m^{(x)} + KQ^{(x)}_{\geq m}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\widetilde{\psi}_m^{(x)}$ is the algebra surjection induced by $\widehat{\psi}_m^{(x)}$. The rows and columns of the diagram are exact. By the definitions of $\widehat{U}_m^{(x)}$, $\widehat{\psi}_m^{(x)}$ and $\mathfrak{B}_m^{(x)}$ the upper two rows are exact. The left hand column is exact, because there is an isomorphism $(\widehat{U}_m^{(x)}+\mathfrak{B})/\mathfrak{B}\cong \widehat{U}^{(x)}/(\mathfrak{B}\cap\widehat{U}^{(x)})$. Since the exactness of the remaining two columns is obvious, the lower row is also exact. Hence, as

$$\widehat{KQ} = \varprojlim_{Q^{(x)},\, m\geq 2} KQ/U_m^{(x)} \cong \varprojlim_{Q^{(x)},\, m\geq 2} KQ^{(x)}/KQ^{(x)}_{\geq m}$$

(see (2.4)), we have

$$\Lambda_{\mathfrak{B}} = \widehat{KQ}/\mathfrak{B} \cong \varprojlim_{Q^{(x)},\, m\geq 2} KQ^{(x)}/(\mathfrak{B}_m^{(x)} + KQ^{(x)}_{\geq m}),$$

and $\Lambda_{\mathfrak{B}}$ is a pseudocompact K-algebra. In view of the duality $\mathcal{C}oalg_K \cong \mathcal{PC}_K^{\mathrm{op}}$ between the categories $\mathcal{C}oalg_K$ of K-coalgebras and $\mathcal{PC}_K$ of pseudocompact K-algebras (see [29, Theorem 3.6]), the surjection $(K^{\square}Q)^*\cong \widehat{KQ}\to\Lambda_{\mathfrak{B}}$ induces a coalgebra injection $\Lambda_{\mathfrak{B}}^{\circ}\hookrightarrow((K^{\square}Q)^*)^{\circ} = K^{\square}Q$, where $V^{\circ} = \hom_K(V,K)$ is the vector space of all continuous K-linear maps from the topological K-vector space V to the field K. Moreover, there is an isomorphism of pseudocompact K-algebras $\Lambda_{\mathfrak{B}}\cong(\Lambda_{\mathfrak{B}}^{\circ})^*\cong(K^{\square}Q)^*/(\Lambda_{\mathfrak{B}}^{\circ})^{\perp}$, where

$$(\Lambda_{\mathfrak{B}}^{\circ})^{\perp} = \{\varphi\in(K^{\square}Q)^*;\ \varphi(\Lambda_{\mathfrak{B}}^{\circ}) = 0\} = \{\varphi\in(K^{\square}Q)^*;\ \langle\varphi,\Lambda_{\mathfrak{B}}^{\circ}\rangle = 0\}.$$

Since $\Lambda_{\mathfrak{B}}^{\circ} = ((\Lambda_{\mathfrak{B}}^{\circ})^{\perp})^{\perp}$ (see [23, p. 57]) and, obviously, the isomorphism $\widehat{KQ}\cong(K^{\square}Q)^*$ carries the ideal $\mathfrak{B}$ of $\widehat{KQ}$ to the ideal $\mathfrak{B}' = (\Lambda_{\mathfrak{B}}^{\circ})^{\perp}$ of $(K^{\square}Q)^*$, we get

$$\begin{aligned} K^{\square}(Q,\mathfrak{B}) &= \{c\in K^{\square}Q;\ \langle\mathfrak{B}',c\rangle = 0\} = \{c\in K^{\square}Q;\ \langle(\Lambda_{\mathfrak{B}}^{\circ})^{\perp},c\rangle = 0\}\\ &= ((\Lambda_{\mathfrak{B}}^{\circ})^{\perp})^{\perp} = \Lambda_{\mathfrak{B}}^{\circ}. \end{aligned}$$

This shows that $K^{\square}(Q,\mathfrak{B})$ is a subcoalgebra of $K^{\square}Q$ such that $(K^{\square}(Q,\mathfrak{B}))^*\cong\widehat{KQ}/\mathfrak{B} = \Lambda_{\mathfrak{B}}$. Since $\mathfrak{B}$ is a profinite relation ideal of $\widehat{KQ}$, we have $\mathfrak{B}'\subseteq$

$\overline{J((K^{\square}Q)^{*})^{2}} = (KQ_0 \oplus KQ_1)^{\perp}$, by Proposition 2.5(e2). It follows that

$$\begin{aligned} K^{\square}(Q,\mathfrak{B}) &= \{c \in K^{\square}Q;\ \langle \mathfrak{B}', c\rangle = 0\} = (\mathfrak{B}')^{\perp} = (KQ_0 \oplus KQ_1)^{\perp\perp} \\ &\supseteq KQ_0 \oplus KQ_1 \end{aligned}$$

(see [23, p. 57]). Hence $H = K^{\square}(Q,\mathfrak{B})$ is a relation subcoalgebra of $K^{\square}Q$ and there is an isomorphism $J(H^*)/\overline{J((H)^*)^2} \cong J((K^{\square}Q)^*)/\overline{J((K^{\square}Q)^*)^2}$. Thus, by a description given in [29, p. 136] (see also the proof of [18, Proposition 4.10]), the left Gabriel quiver of H is isomorphic to that of $K^{\square}Q$, that is, to Q. This finishes the proof of (a).

To prove the remaining statements of the theorem, given a finite subquiver $Q^{(x)}$ of Q and $m \geq 2$, we consider the finite-dimensional subcoalgebra

$$H_m^{(x)} = K^{\square}Q^{(x)}_{\leq m-1} \cap K^{\square}(Q^{(x)}, \mathfrak{B}_m^{(x)})$$

of $H = K^{\square}(Q,\mathfrak{B})$. It is easy to see that the algebra isomorphism $(K^{\square}Q_m^{(x)})^* \cong KQ^{(x)}/KQ^{(x)}_{\geq m}$ of Lemma 2.3(b) restricts to the isomorphism $(H_m^{(x)})^* \cong KQ_m^{(x)}/(\mathfrak{B}_m^{(x)} + KQ_{\geq m})$ of finite-dimensional K-algebras such that the coalgebra embedding $H_m^{(x)} \hookrightarrow H$ induces the algebra surjection

$$H^* \cong \widehat{KQ}/\mathfrak{B} \xrightarrow{\widetilde{\psi}_m^{(x)}} KQ^{(x)}/(\mathfrak{B}_m^{(x)} + KQ^{(x)}_{\geq m}).$$

Moreover, it is easy to see by the definition of the equivalence of categories $F_m^{(x)} : K^{\square}Q_m^{(x)}\text{-comod} \to \mathrm{nilrep}^{\ell f}(KQ^{(x)})$ (see (3.4)) that $F_m^{(x)}$ restricts to an equivalence of categories $F_m^{(x)} : H_m^{(x)}\text{-comod} \to \mathrm{nilrep}^{\ell f}(KQ^{(x)}, \mathfrak{B}_m^{(x)})$. Hence, (b)–(d) follow, and the proof is complete. ■

Corollary 4.10. *Let K be a field, Q a quiver and $(Q,\mathfrak{B})$ a profinite bound quiver.*

(a) *If H is a relation subcoalgebra of $K^{\square}Q$, then the vector space $H^{\perp} = \{\varphi \in (K^{\square}Q)^*;\ \langle \varphi, H\rangle = 0\}$ is a two-sided relation ideal of the profinite algebra $(K^{\square}Q)^* \cong \widehat{KQ}$.*

(b) *The map $\mathfrak{B} \mapsto K^{\square}(Q,\mathfrak{B})$ defines a bijection between the set of profinite relation ideals $\mathfrak{B}$ of the profinite K-algebra $\widehat{KQ}$ and the set of relation subcoalgebras H of the path coalgebra $K^{\square}(Q,\mathfrak{B})$. The inverse map is given by $H \mapsto H^{\perp}$.*

Proof. (a) In view of the duality $\mathcal{C}oalg_K \cong \mathcal{PC}_K^{\mathrm{op}}$, the coalgebra injection $H \hookrightarrow K^{\square}Q$ induces a surjection $(K^{\square}Q)^* \cong \widehat{KQ} \to H^*$ of profinite K-algebras and isomorphisms $H^* \cong (K^{\square}Q)^*/H^{\perp} \cong \widehat{KQ}/\mathfrak{B}$ of algebras, where

$$H^{\perp} = \{\varphi \in (K^{\square}Q)^*;\ \varphi(H) = 0\} = \{\varphi \in (K^{\square}Q)^*;\ \langle \varphi, H\rangle = 0\}.$$

It follows that the isomorphism $(K^{\square}Q)^* \cong \widehat{KQ}$ of pseudocompact algebras carries the ideal $H^{\perp}$ of $(K^{\square}Q)^*$ to an ideal $\mathfrak{B}$ of $\widehat{KQ}$. Since H is a relation subcoalgebra of $K^{\square}Q$, Proposition 2.5 shows that $\mathfrak{B}$ is a profinite relation ideal of $\widehat{KQ}$. It is easy to see, by applying the arguments used in the proof of Theorem 4.8(a) to the profinite algebra $\Lambda_{\mathfrak{B}} = \widehat{KQ}/\mathfrak{B}$, that $H \cong \Lambda_{\mathfrak{B}}^{\circ} \cong K^{\square}(Q, \mathfrak{B})$. Hence (a) follows.

(b) Apply (a) and Theorem 4.8. ∎

COROLLARY 4.11. *Assume that K is arbitrary and C is a pointed K-coalgebra, that is, every simple C-comodule is one-dimensional. Let $Q = {}_CQ$ be the left Gabriel quiver of C.*

(a) *There exist a profinite relation ideal $\mathfrak{B}$ of the profinite K-algebra $\widehat{KQ} \cong (K^{\square}Q)^*$, a coalgebra isomorphism $C \cong K^{\square}(Q, \mathfrak{B})$ and K-linear equivalences of categories C-comod $\cong \mathrm{nilrep}_K^{\ell f}(Q, \mathfrak{B})$ and C-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(Q, \mathfrak{B})$.*

(b) *If Q is intervally finite then $C \cong K^{\square}(Q, \Omega)$ and there are K-linear equivalences C-comod $\cong \mathrm{nilrep}_K^{\ell f}(Q, \Omega)$ and C-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(Q, \Omega)$, where Ω is a relation ideal of KQ.*

Proof. (a) Since C is pointed, by [41, Theorem 4.13], C is isomorphic to a relation subcoalgebra H of $K^{\square}Q$. Hence, by applying Corollary 4.10(b) and Theorem 4.8, statement (a) follows.

(b) Apply Theorem 4.5 to a relation subcoalgebra H of $K^{\square}Q$ such that $H \cong C$. ∎

COROLLARY 4.12. *Assume that K is an algebraically closed field and C is a basic K-coalgebra. Let $Q = {}_CQ$ be the left Gabriel quiver of C.*

(a) *If C is hereditary then $C \cong K^{\square}Q$ and C-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(Q)$.*

(b) *If C is arbitrary, then there exist a profinite relation ideal $\mathfrak{B}$ of $\widehat{KQ} \cong (K^{\square}Q)^*$, a coalgebra isomorphism $C \cong K^{\square}(Q, \mathfrak{B})$ and K-linear equivalences of categories C-comod $\cong \mathrm{nilrep}_K^{\ell f}(Q, \mathfrak{B})$ and C-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(Q, \mathfrak{B})$.*

If, in addition, Q is intervally finite then $C \cong K^{\square}(Q, \Omega)$ and there are K-linear equivalences C-comod $\cong \mathrm{nilrep}_K^{\ell f}(Q, \Omega)$ and C-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(Q, \Omega)$, where Ω is a relation ideal of KQ.

Proof. (a) Apply [4] and [12] (see also [31, Theorem 4.9]).

(b) By applying [7, Theorem 4.2], [41, Section 4], and the arguments used in the proof of [31, Theorem 4.9], we show that any basic coalgebra C is isomorphic to a relation subcoalgebra of $K^{\square}Q$, because K is algebraically closed. Then Theorems 4.8 and 4.5(d) apply. ∎

Note that the second part of (b) corrects [31, Theorem 4.9(c)].

REMARK 4.13. It follows from the definition (2.4) of $\widehat{KQ}$ that a profinite relation ideal $\mathfrak{B}$ of $\widehat{KQ} \cong (K^{\square}Q)^*$ has the form $\mathfrak{B} = \prod_{a,b\in Q_0} \mathfrak{B}(a,b)$, where

$\mathfrak{B}(a,b) = \mathfrak{B} \cap \widehat{KQ(a,b)}$. This means that $\mathfrak{B}$ is generated, as a topological ideal, by the elements in $\mathfrak{B}(a,b)$. It is easy to see (by using the Cauchy nets technique) that every element $f \in \mathfrak{B}(a,b)$ is a formal power series

$$f = \sum_{\omega \in Q(a,b)} \lambda_\omega \omega,$$

where ω runs through all paths in $Q(a,b)$ and $\lambda_\omega \in K$. The restriction of f to any finite-dimensional algebra $KQ^{(x)}/KQ^{(x)}_{\geq m}$, with a finite subquiver $Q^{(x)}$ of Q, is the finite sum

$$f^{(x)} = \sum_{\omega \in Q^{(x)}(a,b)} \lambda_\omega \omega$$

and it belongs to the ideal $\mathfrak{B}^{(x)}_m$ (see 4.1). Below, we illustrate it by an example.

EXAMPLE 4.14. Let K be a field and let Q be the infinite quiver

$$\begin{tikzcd}
1 \arrow[loop above, "\beta_1"] & 2 \arrow[l, "h_1"'] \arrow[loop above, "\beta_2"] & \cdots \arrow[l, "h_2"'] & s{-}1 \arrow[l] \arrow[loop above, "\beta_{s-1}"] & s \arrow[l, "h_{s-1}"'] \arrow[loop above, "\beta_s"] & s{+}1 \arrow[l, "h_s"'] \arrow[loop above, "\beta_{s+1}"] & \cdots \arrow[l, "h_{s+1}"'] \\
1' \arrow[r, "g_1"] & 2' \arrow[u, "f_2"] \arrow[r, "g_2"] & \cdots \arrow[r] & (s{-}1)' \arrow[u, "f_{s-1}"] \arrow[r, "g_{s-1}"] & s' \arrow[u, "f_s"] \arrow[r, "g_s"] & (s{+}1)' \arrow[u, "f_{s+1}"] \arrow[r, "g_{s+1}"] & \cdots .
\end{tikzcd}$$

We fix scalars $\lambda_1, \lambda_2, \ldots \in K$ and, given $s \geq 2$, we set

$$w^{(s)} = \sum_{j=s}^{\infty} \lambda_s \beta_s h_s \cdots h_{j-1} \beta_j f_j g_{j-1} \cdots g_s \in \widehat{KQ(s', s)}.$$

Then the ideal $\mathfrak{B}$ of the pseudocompact K-algebra $\widehat{KQ}$ generated (as a topological ideal) by the formal power series $w^{(1)}, w^{(2)}, \ldots$ is a profinite relation ideal of $\widehat{KQ} \cong (K^\square Q)^*$.

5. Cotensor coalgebras of species with profinite relations. Assume that K is an arbitrary field, C is a basic K-coalgebra with fixed decompositions (1.1), and set $F_j = \mathrm{End}_C S(j)$ for each $j \in I_C$. Denote by $({}_CQ, {}_C\mathbf{d})$ the left Gabriel valued quiver of C (see [31, Definition 4.3] and [32]).

The aim of this section is to show that, for the class of basic K-coalgebras C, the study of the category C-Comod reduces to the study of locally nilpotent locally finite representations of the K-species associated to C. To formulate the results, we need some notation.

Following [9], a *K-species* is a system

$$\mathcal{M} = (F_i, {}_iM_j)_{i,j \in I_{\mathcal{M}}}, \tag{5.1}$$

where F_i is a finite-dimensional division K-algebra for each $i \in I_{\mathcal{M}}$, ${}_iM_j$ is an F_i-F_j-bimodule such that ${}_iM_j$ is a K-vector space for all $i \neq j$ in $I_{\mathcal{M}}$, and the field K acts centrally on each F_i and on ${}_iM_j$ for each pair $i, j \in I_{\mathcal{M}}$. We set ${}_jM_j = F_j$ for each $i \in I_{\mathcal{M}}$.

We call $\mathcal{M}$ *locally finite* if every F_i-F_j-bimodule ${}_iM_j$ is a directed union of finite-dimensional F_i-F_j-subbimodules. If the index set $I_{\mathcal{M}}$ is finite and each bimodule ${}_iM_j$ is of finite K-dimension, $\mathcal{M}$ is called *finite*. Throughout this section, we freely use the terminology and notation introduced in [18]. In particular, we denote by $(Q^{\mathcal{M}}, \mathbf{d}^{\mathcal{M}})$ the valued quiver of the species $\mathcal{M}$, where $Q_0^{\mathcal{M}} = I_{\mathcal{M}}$ is the set of vertices of the quiver $Q^{\mathcal{M}}$.

To any K-species $\mathcal{M} = (F_i, {}_iM_j)_{i,j \in I_{\mathcal{M}}}$, we associate the tensor K-algebra

$$(5.2) \qquad T(\mathcal{M}) = T_F(M) = F \oplus M \oplus M^{\otimes^2} \oplus M^{\otimes^3} \oplus \cdots$$

of $\mathcal{M}$, where

- $F = \bigoplus_{j \in I_{\mathcal{M}}} F_j$ (direct sum of the division rings F_j, viewed as a ring with the local units $e_j = 1_{F_j} \in F_j$),
- $M = \bigoplus_{i,j \in I_{\mathcal{M}}} {}_iM_j$ is a K-vector space viewed as a unitary F-F-bimodule in the obvious way, and
- $M^{\otimes^s} = M \otimes_F \cdots \otimes_F M$ is the tensor product of s copies of M, for each $s \geq 1$, and we set $M^{\otimes^0} = F$ (see [26] and [18]).

Note that the local units e_j of F are primitive pairwise orthogonal idempotents of $T(\mathcal{M})$ and, given $s \geq 1$, the vector subspace

$$T(\mathcal{M})_{\geq s} = \bigoplus_{j \geq s} M^{\otimes^j}$$

of $T(\mathcal{M})$ is the two-sided ideal generated by $M^{\otimes^s}$. Obviously, $T(\mathcal{M})$ has an identity element if and only if the index set $I_{\mathcal{M}}$ is finite.

We note that, given $s \geq 1$, there is a decomposition

$$(5.3) \qquad M^{\otimes^s} = M \otimes_F \cdots \otimes_F M = \bigoplus_{a,b \in I_{\mathcal{M}}} {}_aM_b^{(s)}$$

where ${}_aM_b^{(1)} = {}_aM_b$,

$${}_aM_b^{(s)} = \bigoplus_{j_1, j_2, \ldots, j_s} {}_aM_{j_1} \otimes {}_{j_1}M_{j_2} \otimes \cdots \otimes {}_{j_s}M_b \quad \text{for } s \geq 2,$$

and the sum is taken over all paths $a = j_0 \to j_1 \to \cdots \to j_s \to b$ of length $s \geq 1$ from a to b in the quiver $Q^{\mathcal{M}}$.

To any K-species $\mathcal{M} = (F_i, {}_iM_j)_{i,j \in I_{\mathcal{M}}}$, we associate the graded cotensor K-coalgebra

$$(5.4) \qquad T^{\square}(\mathcal{M}) = (T(\mathcal{M}), \Delta, \varepsilon),$$

endowed with the comultiplication $\Delta : T(\mathcal{M}) \to T(\mathcal{M}) \otimes T(\mathcal{M})$ and the counit $\varepsilon : T(\mathcal{M}) \to K$ defined as follows. Given a local unit $e_a \in F$ at a, we set $\Delta(e_a) = e_a \otimes e_a$ and $\varepsilon(e_a) = 1$. Given $s \geq 1$ and any element ${}_a\overline{m}_b \in {}_aM_b^{(s)} \subseteq M^{\otimes^s}$ of the form

$$ {}_a\overline{m}_b = {}_am_{j_1} \otimes {}_{j_1}m_{j_2} \otimes \cdots \otimes {}_{j_s}m_b \in {}_aM_{j_1} \otimes {}_{j_1}M_{j_2} \otimes \cdots \otimes {}_{j_s}M_b $$

(see (5.3)), we set

$$ \begin{aligned} \Delta({}_a\overline{m}_b) &= e_a \otimes {}_a\overline{m}_b + {}_a\overline{m}_b \otimes e_b + \sum_{r=1}^{s-1} ({}_a\overline{m}_{j_r}) \otimes ({}_{j_r}\overline{m}_b), \\ \varepsilon({}_a\overline{m}_b) &= 0, \end{aligned} $$

where ${}_a\overline{m}_{j_r} = {}_am_{j_1} \otimes \cdots \otimes {}_{j_{r-1}}m_{j_r}$ and ${}_{j_r}\overline{m}_b = {}_{j_r}m_{j_{r+1}} \otimes \cdots \otimes {}_{j_s}m_b$.

It follows that, given $s \geq 1$, the vector subspace

$$ T^{\square}(\mathcal{M})_{\leq s-1} = \bigoplus_{j \leq s-1} M^{\otimes^j} $$

of $T^{\square}(\mathcal{M})$ is a K-subcoalgebra of $T^{\square}(\mathcal{M})$, and $T^{\square}(\mathcal{M})_{\leq s-1}$ is the $(s-1)$th term of the coradical filtration of $T^{\square}(\mathcal{M})$ [41, Lemma 4.4]. Note also that $T^{\square}(\mathcal{M})$ is isomorphic to the cotensor coalgebra

$$ T_F^{\square}(M) = F \oplus M \oplus M\square M \oplus M\square M\square M \oplus \cdots $$

where $F = \bigoplus_{a\in I_{\mathcal{M}}} F_a$ is viewed as the direct sum of the simple coalgebras F_a, and $M = \bigoplus_{a,b\in I_{\mathcal{M}}} {}_aM_b$ is viewed as an F-F-bicomodule in a natural way (see [25] and [41]). We denote by $T^{\square}(\mathcal{M})^*$ the algebra $(T^{\square}(\mathcal{M}))^*$ K-dual to $T^{\square}(\mathcal{M})$ with respect to the convolution product.

The following lemma extends Lemma 2.3 to the K-species case.

LEMMA 5.5. *Let $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in I_{\mathcal{M}}}$ be a finite K-species and K a field.*

(a) *If the quiver $Q^{\mathcal{M}}$ is acyclic then* $\dim_K T^{\square}(\mathcal{M})$ *is finite and there is a K-algebra isomorphism $T^{\square}(\mathcal{M})^* \cong T(\mathcal{M})$.*
(b) *If $s \geq 2$ then the subcoalgebra $T^{\square}(\mathcal{M})_{\leq s-1} = \bigoplus_{j\leq s-1} M^{\otimes^j}$ of $T^{\square}(\mathcal{M})$ is finite-dimensional and there is a K-algebra isomorphism $(T^{\square}(\mathcal{M})_{\leq s-1})^* \cong T(\mathcal{M})/T(\mathcal{M})_{\geq s}$.*

Proof. Apply the arguments used in the proof of Lemma 2.3. ∎

Imitating the concept of the finite subquiver topology on KQ defined in Section 2, given a locally finite K-species $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in I_{\mathcal{M}}}$ and a field K, we define the *finite subspecies topology* (K-linear and Hausdorff) on $T(\mathcal{M})$ as follows. First we recall that there is a canonical algebra embedding $T(\mathcal{M}) \hookrightarrow T^{\square}(\mathcal{M})^*$ and the non-degenerate bilinear form $\langle -, - \rangle : T^{\square}(\mathcal{M})^* \times T^{\square}(\mathcal{M}) \to K$ defined by $\langle \varphi, c\rangle = \varphi(c)$.

Given $s \geq 2$ and a finite K-subspecies $\mathcal{M}^{(x)}$ of $\mathcal{M}$, we consider the finite-dimensional subcoalgebra $T^{\square}(\mathcal{M}^{(x)})_{\leq s-1} = \bigoplus_{j\leq s-1} M^{\otimes^j}$ of $T^{\square}(\mathcal{M})$. By Lemma 5.5, the finite-dimensional algebra K-dual to $T^{\square}(\mathcal{M}^{(x)})_{\leq s-1}$ has the form $(T^{\square}(\mathcal{M}^{(x)})_{\leq s-1})^* \cong T(\mathcal{M}^{(x)})/T(\mathcal{M}^{(x)})_{\geq s}$. Let $U_s^{(x)} = \operatorname{Ker} \psi_s^{(x)} \subseteq T(\mathcal{M})$ be the kernel of the composite algebra surjection

$$\begin{aligned}(5.6)\qquad \psi_s^{(x)} = [T(\mathcal{M}) \hookrightarrow T^{\square}(\mathcal{M})^* \xrightarrow{(u_m^{(x)})^*} &(T^{\square}(\mathcal{M}^{(x)})_{\leq s-1})^* \\ &= T(\mathcal{M}^{(x)})/T^{\square}(\mathcal{M}^{(x)})_{\geq s}],\end{aligned}$$

where $u_m^{(x)} : T^{\square}(\mathcal{M}^{(x)})_{\leq s-1} \hookrightarrow T^{\square}(\mathcal{M})$ is the coalgebra embedding and $T^{\square}(\mathcal{M}) \hookrightarrow T^{\square}(\mathcal{M})^*$ is the canonical algebra embedding. It is clear that the K-linear topology on $T(\mathcal{M})$ defined by the ideals $U_s^{(x)} = \operatorname{Ker} \psi_s^{(x)}$ is Hausdorff; we call it the finite subspecies topology of $T(\mathcal{M})$ (see [20] and [21]). The completion

$$(5.7)\qquad \widehat{T}(\mathcal{M}) = \widehat{T(\mathcal{M})} = \varprojlim_{\mathcal{M}^{(x)},\, s\geq 2} T(\mathcal{M})/U_s^{(x)}$$

of $T(\mathcal{M})$ is obviously a pseudocompact K-algebra, where $\mathcal{M}^{(x)}$ runs through the finite K-subspecies of $\mathcal{M}$, because we assume that the K-species $\mathcal{M}$ is locally finite.

The following result extends Proposition 2.5 to the K-species case.

PROPOSITION 5.8. *Let $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in I_{\mathcal{M}}}$ be a locally finite K-species and K a field.*

(a) *The tensor K-algebra $T(\mathcal{M})$ is left and right hereditary. For each $a \in I_{\mathcal{M}}$, the indecomposable left ideal $E(a) = T(\mathcal{M})e_a$ of $T(\mathcal{M})$ is an indecomposable injective left coideal of $T^{\square}(\mathcal{M})$, $\operatorname{soc} E(a) = S(a) = Fe_a$ and $T^{\square}(\mathcal{M}) = \bigoplus_{a\in I_{\mathcal{M}}} E(a)$.*

(b) *The left Gabriel valued quiver $({}_CQ, {}_C\mathbf{d})$ of $C = T^{\square}(\mathcal{M})$ is isomorphic to the valued quiver $(Q^{\mathcal{M}}, \mathbf{d}^{\mathcal{M}})$ of the species $\mathcal{M}$ (see* [18]*).*

(c) *The cotensor K-coalgebra $T^{\square}(\mathcal{M})$ is hereditary and it is the directed union of the finite-dimensional subcoalgebras $T^{\square}(\mathcal{M}^{(x)})_{\leq s-1}$, where $s \geq 2$ and $\mathcal{M}^{(x)}$ runs through the finite K-subspecies of $\mathcal{M}$.*

(d) *Let $T^{\square}(\mathcal{M})^*$ be the pseudocompact K-algebra K-dual to $T^{\square}(\mathcal{M})$ and let $\widehat{T}(\mathcal{M})$ be the completion* (5.7) *of $T(\mathcal{M})$ in the finite subspecies topology. Then there are isomorphisms of pseudocompact K-algebras*

$$T^{\square}(\mathcal{M})^* \cong \widehat{T}(\mathcal{M}) \cong \widehat{T}_{\widehat{F}}(\widehat{M}),$$

where

$$\widehat{T}_{\widehat{F}}(\widehat{M}) = \widehat{F} \,\Pi\, \widehat{M} \,\Pi\, \widehat{M}^{\widehat{\otimes}^2} \,\Pi \cdots \Pi\, \widehat{M}^{\widehat{\otimes}^m} \,\Pi \cdots = \widehat{F} \oplus \prod_{s=1}^{\infty} \widehat{M}^{\widehat{\otimes}^s}$$

is the complete tensor K-algebra [9, p. 96] *of the topological vector space* $\widehat{M} = \prod_{a,b \in I_{\mathcal{M}}} \widehat{{}_aM_b^{(s)}}$ *(see* (5.3)*), viewed as an* $\widehat{F}$-$\widehat{F}$*-bimodule over the topological product* $\widehat{F} = \prod_{a \in I_{\mathcal{M}}} F_a$ *of the division rings* F_a *with* $a \in I_{\mathcal{M}}$.

(e) $J(T^{\square}(\mathcal{M})^*) = (T^{\square}(\mathcal{M})_0)^{\perp} = F^{\perp}$ *is closed in* $T^{\square}(\mathcal{M})^*$ *and the isomorphism* $T^{\square}(\mathcal{M})^* \cong \widehat{T}(\mathcal{M})$ *restricts to isomorphisms*

(e1) $J(T^{\square}(\mathcal{M})^*) \cong J(\widehat{T}(\mathcal{M})) = J(\widehat{T(\mathcal{M})}) = \overline{\widehat{T(\mathcal{M})}_{\geq 1}}$, *and*

(e2) $(F \oplus M)^{\perp} = \overline{J(T(\mathcal{M})^*)^2} \cong \overline{J(\widehat{T}(\mathcal{M}))^2} = \overline{(J(\widehat{T(\mathcal{M})}))^2} = \widehat{T(\mathcal{M})}_{\geq 2}$.

Proof. By a slight modification of the arguments in [3, Appendix], one can prove that $T_F(M)$ is left and right hereditary. To prove that $T^{\square}(\mathcal{M})$ is hereditary, we show that, for any simple left $T^{\square}(\mathcal{M})$-comodule $S(b) = Fe_b$, there exists an exact sequence

$$0 \to S(b) \xrightarrow{u_b} E(b) \to \bigoplus_{j \in I_{\mathcal{M}}} E(j)^{(d_{jb})} \to 0 \tag{5.9}$$

in $T^{\square}(\mathcal{M})$-Comod, where $E(j) = T(\mathcal{M})e_j$ is the injective envelope of $S(j)$, $d_{jb} = \dim_{F_j} {}_jM_b$, and $U^{(m)}$ denotes the direct sum of m copies of U for any cardinal number m. For the proof, we note that (in the notation of (5.3)) the $T^{\square}(\mathcal{M})$-comodule $\operatorname{Coker} u_b$ has the form

$$\begin{aligned}
&\operatorname{Coker} u_b \\
&= E(b)/S(b) \cong T_F(M) \cdot e_b / F \cdot e_b \\
&\cong (M \oplus M^{\otimes^2} \oplus M^{\otimes^3} \oplus \cdots) \cdot e_b \\
&\cong \bigoplus_{a \in I_{\mathcal{M}}} ({}_aM_b \oplus {}_aM_b^{(2)} \oplus {}_aM_b^{(3)} \oplus {}_aM_b^{(4)} \oplus \cdots) \\
&\cong \bigoplus_{a \in I_{\mathcal{M}}} \bigoplus_{j \in I_{\mathcal{M}}} (F_j \otimes {}_jM_b \oplus {}_aM_j \otimes {}_jM_b \oplus {}_aM_j^{(2)} \otimes {}_jM_b \oplus {}_aM_j^{(3)} \otimes {}_jM_b \oplus \cdots) \\
&\cong \bigoplus_{j \in I_{\mathcal{M}}} \bigoplus_{a \in I_{\mathcal{M}}} (F_j \oplus {}_aM_j \oplus {}_aM_j^{(2)} \oplus {}_aM_j^{(3)} \oplus \cdots) \otimes {}_jM_b \\
&\cong \bigoplus_{j \in I_{\mathcal{M}}} (F_j \oplus M \oplus M^{\otimes^2} \oplus M^{\otimes^3} \oplus \cdots) \cdot e_j \otimes {}_jM_b \ldots \\
&\cong \bigoplus_{j \in I_{\mathcal{M}}} (F \oplus M \oplus M^{\otimes^2} \oplus M^{\otimes^3} \oplus \cdots) \cdot e_j \otimes {}_jM_b \ldots \\
&\cong \bigoplus_{j \in I_{\mathcal{M}}} T_F(M) \cdot e_j \otimes {}_jM_b \cong \bigoplus_{j \in I_{\mathcal{M}}} E(j) \otimes {}_jM_b \cong \bigoplus_{j \in I_{\mathcal{M}}} E(j)^{(d_{jb})}
\end{aligned}$$

(see also [19]). This proves that the sequence (5.9) of left $T^{\square}(\mathcal{M})$-comodules is exact, for any $b \in I_{\mathcal{M}}$. Hence, it follows easily that the coalgebra $T^{\square}(\mathcal{M})$ is hereditary (see also [12]).

Now, the arguments given in the proof of Proposition 2.5 extend to the K-species case almost verbatim. The details are left to the reader. We only note that, by the assumption that the species $\mathcal{M}$ is locally finite, the definition of the comultiplication in $T^{\square}(\mathcal{M})$ implies that, for any finite-dimensional subcoalgebra H of $T^{\square}(\mathcal{M})$, there exists a finite K-subspecies $\mathcal{M}^{(x)}$ of $\mathcal{M}$ such that H is contained in $T^{\square}(\mathcal{M}^{(x)})_{\leq s}$ for some $s \geq 2$. It follows that $T^{\square}(\mathcal{M})$ is the directed union of the finite-dimensional subcoalgebras $H_s^{(x)} = T^{\square}(\mathcal{M}^{(x)})_{\leq s-1} \subseteq T^{\square}(\mathcal{M}^{(x)})$, where $s \geq 2$ and $\mathcal{M}^{(x)}$ runs through the finite K-subspecies of $\mathcal{M}$. Then (c) follows and we get isomorphisms

$$\begin{aligned}(5.10)\quad T^{\square}(\mathcal{M})^* = \Big(\bigcup_{\mathcal{M}^{(x)},\, s\geq 2} H_s^{(x)}\Big)^* &\cong \varprojlim_{\mathcal{M}^{(x)},\, s\geq 2} T^{\square}(\mathcal{M})^*/(H_s^{(x)})^{\perp} \\ &\cong \varprojlim_{\mathcal{M}^{(x)},\, s\geq 2} (H_s^{(x)})^* \cong \varprojlim_{\mathcal{M}^{(x)},\, s\geq 2} T(\mathcal{M})/U_s^{(x)} = \widehat{T}(\mathcal{M})\end{aligned}$$

of pseudocompact K-algebras, where $\mathcal{M}^{(x)}$ runs through the finite K-subspecies of $\mathcal{M}$. Hence, (d) and (e) follow as in the proof of Proposition 2.5. ■

Let K be a field and

$$\mathcal{M} = (F_i, {}_iM_j)_{i,j\in I_{\mathcal{M}}}$$

a locally finite K-species. We denote by $\mathrm{Rep}_K(\mathcal{M})$ the category of all K-linear representations of $\mathcal{M}$ (see [18] for a precise definition). Following [18] and the definition (3.2), we define the K-linear functor

$$(5.11)\qquad F : T^{\square}(\mathcal{M})\text{-Comod} \to \mathrm{Rep}_K(\mathcal{M})$$

as follows. Given a comodule X in $T^{\square}(\mathcal{M})$-Comod (viewed as a rational right module over $T^{\square}(\mathcal{M})^* \cong \widehat{T}^{\square}(\mathcal{M})$) we define $F(X)$ in $\mathrm{Rep}_K(\mathcal{M})$ (see [18, Definition 2.1]), by setting

$$(5.12)\qquad F(X) = (X_a, \varphi_{ab}^X)_{a,b\in I_{\mathcal{M}}},$$

where $X_a = X\cdot e_a = X\cdot e_a^*$ and $e_a \in T(\mathcal{M}) \subseteq T(\mathcal{M})^* \cong \widehat{T}(\mathcal{M})$ is the local unit at a.

For any bimodule ${}_aM_b$, the K-linear map $\varphi_{ab}^X : X_a \otimes {}_aM_b \to X_b$ is defined by the formula $\varphi_{ab}^X(x\cdot e_a) = (x\cdot e_a)\cdot m = x\cdot me_b$, where $x \in X$, $m \in {}_aM_b$, and $x\cdot e_a = x\cdot e_a^* \in X_a = X\cdot e_a$ (see Example 3.6). Since $m \in {}_aM_b$ is viewed as an element of $T(\mathcal{M})$, we have $m = e_am = me_b$ in $T(\mathcal{M})$. Given a $T^{\square}(\mathcal{M})$-comodule homomorphism $f : X \to Y$, we set $F(f) = (f_a)_{a\in I_{\mathcal{M}}}$, where $f_a : X_a \to Y_a$ is the restriction of f to X_a. It is clear that $F(f) : F(X) \to F(Y)$ is a morphism in $\mathrm{Rep}_K(\mathcal{M})$ defining a covariant K-linear exact faithful functor such that the following proposition holds.

PROPOSITION 5.13. *Let* $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in I_{\mathcal{M}}}$ *be a locally finite* K*-species.*

(a) *The functor* F (5.11) *commutes with arbitrary direct sums and directed unions, and restricts to two equivalences of categories*

$$\begin{array}{ccc} T^{\square}(\mathcal{M})\text{-Comod} & \xrightarrow{F} & \mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M}) \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ T^{\square}(\mathcal{M})\text{-comod} & \xrightarrow{F} & \mathrm{nilrep}_K^{\ell f}(\mathcal{M}) \end{array} \tag{5.14}$$

making the diagram commutative.

(b) *The functor* F^{-1} *inverse to* F *associates to any representation* $X = (X_a, \varphi^X_{ab})_{a,b\in I_{\mathcal{M}}}$ *in* $\mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M})$ *the vector space* $F^{-1}(X) = \bigoplus_{a\in I_{\mathcal{M}}} X_a$ *equipped with the left* $T^{\square}(\mathcal{M})$*-comodule structure on induced by the natural discrete right module structure over the profinite* K*-algebra* $T^{\square}(\mathcal{M})^* \cong \widehat{T}(\mathcal{M})$.

(c) *For any representation* X *in* $\mathrm{nilrep}_K^{\ell f}(\mathcal{M})$ *there exists a finite subspecies* $\mathcal{M}^{(x)}$ *of* $\mathcal{M}$ *and an integer* $s \geq 2$ *such that* X *lies in* $\mathrm{rep}_K^{\ell f}(\mathcal{M}^{(x)}, T(\mathcal{M}^{(x)})_{\geq s}) \subseteq \mathrm{nilrep}_K^{\ell f}(\mathcal{M})$.

Proof. In view of Lemma 5.5 and Proposition 5.8, the arguments used in the proof of Proposition 3.3 extend to the K-species case. ■

By applying previous results, we are able to prove the following important theorem extending Corollary 4.12(a) from the case of K algebrically closed to arbitrary K.

THEOREM 5.15. *Assume that* K *is an arbitrary field and* C *is a basic indecomposable hereditary* K*-coalgebra such that the* Ext*-species* ${}_C\mathcal{E}xt$ *of* C *is locally finite and the left valued Gabriel quiver* $({}_CQ, {}_C\mathbf{d})$ *of* C *is a valued tree and contains no infinite path of the form*

$$\bullet \xleftarrow{(d'_{\beta_1}, d''_{\beta_1})} \bullet \xleftarrow{(d'_{\beta_2}, d''_{\beta_2})} \bullet \leftarrow \cdots \leftarrow \bullet \xleftarrow{(d'_{\beta_m}, d''_{\beta_m})} \bullet \leftarrow \cdots$$

(a) *There is a coalgebra isomorphism* $C \cong T^{\square}(\mathcal{M})$, *where* $\mathcal{M} = {}_C\mathcal{E}xt^{\#}$ *is the* $\#$*-dual to* ${}_C\mathcal{E}xt$ (*see* [18, (4.11)]).

(b) *There is an equivalence of* K*-categories* C-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M})$ *that restricts to an equivalence* C-comod $\cong \mathrm{nilrep}_K^{\ell f}(\mathcal{M})$.

Proof. It follows from [18, Proposition 4.16] that there is an equivalence of K-categories C-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M})$. On the other hand, by Proposition 5.8, there is an equivalence of K-categories $T^{\square}(\mathcal{M})$-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M})$, and consequently, C-Comod $\cong T^{\square}_F(M)$-Comod. Since the coalgebras C and $T^{\square}(\mathcal{M})$ are basic, there is a coalgebra isomorphism as in (a) (see [27, p. 404], [5], and [7]). ■

Following [29] and Section 4, we introduce the following definition.

DEFINITION 5.16. Given a locally finite K-species $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in I_{\mathcal{M}}}$, a K-coalgebra H is defined to be a *relation subcoalgebra* of the cotensor (hereditary) coalgebra $T^{\square}(\mathcal{M})$ if $T^{\square}(\mathcal{M})_0 \oplus T^{\square}(\mathcal{M})_1 = F \oplus M \subseteq H \subseteq T^{\square}(\mathcal{M})$.

We end this section by showing that any basic indecomposable K-coalgebra C over a perfect field K is a relation subcoalgebra of $T^{\square}(\mathcal{M})$, where $\mathcal{M} = {}_C\mathcal{E}xt^{\#}$. Moreover, there is an equivalence of categories C-Comod $\cong \mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M}, \mathfrak{B})$, where $\mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M}, \mathfrak{B})$ is the full subcategory of $\mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M})$, consisting of the representations that satisfy the relations of a profinite relation ideal $\mathfrak{B}$ of the profinite K-algebra $T^{\square}(\mathcal{M})^* \cong \widehat{T(\mathcal{M})}$, defined as follows.

DEFINITION 5.17.

(a) A *profinite bound K-species* (or a *K-species with profinite relations*) is a pair $(\mathcal{M}, \mathfrak{B})$, where $\mathcal{M}$ is a K-species and $\mathfrak{B}$ is a closed two-sided ideal (called a *profinite relation ideal*) of the profinite K-algebra $\widehat{T(\mathcal{M})}$ (see (5.7)) such that $\mathfrak{B} \subseteq \widehat{T(\mathcal{M})_{\geq 2}}$.

(b) Let $\mathfrak{B} \subseteq \widehat{T(\mathcal{M})_{\geq 2}} \subseteq \widehat{T(\mathcal{M})}$ be a profinite relation ideal, $\mathcal{M}^{(x)}$ a finite subspecies of $\mathcal{M}$, and

$$\widehat{\psi}_s^{(x)} : \widehat{T(\mathcal{M})} \to T(\mathcal{M})/U_s^{(x)} \cong T(\mathcal{M}^{(x)})/T(\mathcal{M}^{(x)})_{\geq s},$$

with $s \geq 2$, the canonical algebra surjection. Fix a relation ideal $\mathfrak{B}_s^{(x)}$ of $T(\mathcal{M}^{(x)})$ such that

$$\widehat{\psi}_s^{(x)}(\mathfrak{B}) = \mathfrak{B}_s^{(x)}/T(\mathcal{M}^{(x)})_{\geq s} \subseteq T(\mathcal{M}^{(x)})/T(\mathcal{M}^{(x)})_{\geq s}.$$

The bound K-species $(\mathcal{M}^{(x)}, \mathfrak{B}_s^{(x)} + T(\mathcal{M}^{(x)})_{\geq s})$ is called the *projection* of $(\mathcal{M}, \mathfrak{B})$ on $(\mathcal{M}^{(x)}, T(\mathcal{M}^{(x)}_{\geq s}))$, for $s \geq 2$ and a finite subspecies $\mathcal{M}^{(x)}$ of $\mathcal{M}$.

By Proposition 5.13(c), given a representation X in $\mathrm{nilrep}_K^{\ell f}(\mathcal{M})$, there exist an $s \geq 2$ and a finite subspecies $\mathcal{M}^{(x)}$ of $\mathcal{M}$ such that X lies in

$$\mathrm{rep}_K^{\ell f}(\mathcal{M}^{(x)}, T(\mathcal{M}^{(x)})_{\geq s}) \subseteq \mathrm{nilrep}_K^{\ell f}(\mathcal{M}).$$

We define X to *satisfy the relations in* $\mathfrak{B}$ if X satisfies the relations in $\mathfrak{B}_s^{(x)} + T(\mathcal{M}^{(x)})_{\geq s}$, that is, X lies in

$$\mathrm{rep}_K(\mathcal{M}^{(x)}, \mathfrak{B}_s^{(x)} + T(\mathcal{M}^{(x)})_{\geq s}) \subseteq \mathrm{rep}_K^{\ell f}(\mathcal{M}^{(x)}, T(\mathcal{M}^{(x)})_{\geq s}).$$

A representation Y in $\mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M})$ is defined to *satisfy the relations in* $\mathfrak{B}$ if any subrepresentation X of Y lying in $\mathrm{nilrep}_K^{\ell f}(\mathcal{M})$ satisfies the relations in $\mathfrak{B}$.

Given a profinite bound species $(\mathcal{M}, \mathfrak{B})$, we denote by $\mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M}, \mathfrak{B}) \supseteq \mathrm{nilrep}_K^{\ell f}(\mathcal{M}, \mathfrak{B})$ the full subcategories of $\mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M})$ whose objects are the representations in $\mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M})$ (resp. in $\mathrm{nilrep}_K^{\ell f}(\mathcal{M})$) that satisfy the relations in $\mathfrak{B}$.

DEFINITION 5.18. Let K be a field, $\mathcal{M}$ a locally finite K-species, and $(\mathcal{M}, \mathfrak{B})$ a profinite bound species.

(a) The *cotensor K-coalgebra of the profinite bound species* $(\mathcal{M}, \mathfrak{B})$ is the subcoalgebra

$$T^{\square}(\mathcal{M}, \mathfrak{B}) = \{c \in T^{\square}(\mathcal{M});\ \langle \mathfrak{B}', c\rangle = 0\} \tag{5.19}$$

of the cotensor hereditary K-coalgebra $T^{\square}(\mathcal{M})$, where $\langle -,-\rangle$: $T^{\square}(\mathcal{M})^* \times T^{\square}(\mathcal{M}) \to K$ is the non-degenerate symmetric K-bilinear form defined by $\langle \varphi, c\rangle = \varphi(c)$, and $\mathfrak{B}'$ is the image of $\mathfrak{B} \subseteq \widehat{T^{\square}(\mathcal{M})}$ under the isomorphism $\widehat{T(\mathcal{M})} \cong T^{\square}(\mathcal{M})^*$ (see (5.10)) of pseudocompact K-algebras.

(b) A K-coalgebra C is defined to be a *profinite bound species coalgebra* if there are a profinite bound species $(\mathcal{M}, \mathfrak{B})$ and a coalgebra isomorphism $C \cong T^{\square}(\mathcal{M}, \mathfrak{B})$. If the ideal $\mathfrak{B}$ is contained in the tensor algebra $T(\mathcal{M})$, the coalgebra $C \cong T^{\square}(\mathcal{M}, \mathfrak{B})$ is called a *bound species coalgebra.*

The following result is a K-species analogy of Theorem 4.8.

THEOREM 5.20. *Let K be a field, $\mathcal{M}$ a locally finite species and $(\mathcal{M}, \mathfrak{B})$ a profinite bound species with a profinite relation ideal $\mathfrak{B} \subseteq \widehat{T(\mathcal{M})_{\geq 2}} \subseteq \widehat{\mathcal{M}}$.*

(a) *The subspace $T^{\square}(Q, \mathfrak{B})$ (see (5.19)) of $T^{\square}(\mathcal{M})$ is a basic relation subcoalgebra of $T^{\square}(\mathcal{M})$ such that the Gabriel valued quiver $({}_HQ, {}_H\mathbf{d})$ of $H = T^{\square}(\mathcal{M}, \mathfrak{B})$ is isomorphic to the valued quiver $(Q^{\mathcal{M}}, \mathbf{d}^{\mathcal{M}})$ of the species $\mathcal{M}$.*

(b) *The coalgebra $H = T^{\square}(\mathcal{M}, \mathfrak{B})$ is the directed union of the finite-dimensional subcoalgebras of the form*

$$H_s^{(x)} = T^{\square}(\mathcal{M}^{(x)})_{\leq s-1} \cap T^{\square}(\mathcal{M}^{(x)}, \mathfrak{B}_s^{(x)}),$$

where $\mathcal{M}^{(x)}$ runs through the finite subspecies of $\mathcal{M}$, $s \geq 2$, and $\mathfrak{B}_s^{(x)} \subseteq T(\mathcal{M}^{(x)})$ is a relation ideal such that $(\mathcal{M}^{(x)}, \mathfrak{B}_s^{(x)} + T(\mathcal{M}^{(x)})_{\geq s})$ is the projection of $(\mathcal{M}, \mathfrak{B})$ on $(\mathcal{M}^{(x)}, T(\mathcal{M}^{(x)})_{\geq s})$ (see (5.17)).

(c) *There are functorial isomorphisms*

$$(H_s^{(x)})^* \cong T(\mathcal{M}^{(x)})_{\geq s}/(\mathfrak{B}_s^{(x)} + T(\mathcal{M})_{\geq s})$$

of finite-dimensional K-algebras that are functorial with respect to coalgebra embeddings $H_s^{(x)} \hookrightarrow H_{s'}^{(x')}$ and induce isomorphisms

$$H^* \cong \widehat{T(\mathcal{M})}/\mathfrak{B} \cong \varprojlim_{\mathcal{M}^{(x)},\, s \geq 2} T(\mathcal{M}^{(x)})/(\mathfrak{B}_s^{(x)} + T(\mathcal{M})_{\geq s})$$

of pseudocompact K-algebras, where $\mathcal{M}^{(x)}$ runs through the finite subspecies of $\mathcal{M}$.

(d) *The K-linear category equivalences* (5.14) *restrict to category equivalences*

$$\begin{array}{ccc} T^{\square}(\mathcal{M},\mathfrak{B})\text{-Comod} & \xrightarrow{F} & \mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M},\mathfrak{B}) \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ T^{\square}(\mathcal{M},\mathfrak{B})\text{-comod} & \xrightarrow{F} & \mathrm{nilrep}_K^{\ell f}(\mathcal{M},\mathfrak{B}) \end{array} \tag{5.21}$$

Proof. The arguments used in the proof of Theorem 4.8 generalize almost verbatim. The details are left to the reader. ∎

COROLLARY 5.22. *Let K be a field, $\mathcal{M}$ a species and $(\mathcal{M},\mathfrak{B})$ a profinite bound species.*

(a) *If H is a relation subcoalgebra of $T^{\square}(\mathcal{M})$, then the vector space $H^{\perp}=\{\varphi\in T^{\square}(\mathcal{M})^*;\ \langle\varphi,H\rangle=0\}$ is a two-sided relation ideal of the profinite algebra $T^{\square}(\mathcal{M})^*\cong\widehat{T(\mathcal{M})}$.*

(b) *The map $\mathfrak{B}\mapsto T^{\square}(\mathcal{M},\mathfrak{B})$ defines a bijection between the set of profinite relation ideals $\mathfrak{B}$ of the profinite K-algebra $\widehat{T(\mathcal{M})}$ and the set of relation subcoalgebras H of the cotensor coalgebra $T^{\square}(\mathcal{M},\mathfrak{B})$. The inverse map is given by $H\mapsto H^{\perp}$.*

Proof. Apply Theorem 5.20 and the arguments used in the proof of Corollary 4.10. ∎

COROLLARY 5.23. *Assume that C is a basic K-coalgebra and K is a field such that the centre of the division algebra* $\mathrm{End}_C S$ *is a separable extension of K for any simple C-comodule S. Let $\mathcal{M}={}_C\mathcal{E}xt^{\#}$ be the $\#$-dual to the left* Ext*-species ${}_C\mathcal{E}xt$ of C (see* [18, (4.11)]*).*

(a) *There exist a profinite relation ideal $\mathfrak{B}$ of the complete tensor K-algebra $\widehat{T}(\mathcal{M})$ and a coalgebra isomorphism $C\cong T^{\square}(\mathcal{M},\mathfrak{B})$.*

(b) *There exist two K-linear equivalences of categories C-*Comod $\cong$ $\mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M},\mathfrak{B})$ *and C-*comod $\cong\mathrm{nilrep}_K^{\ell f}(\mathcal{M},\mathfrak{B})$.

Proof. (a) By [41, Proposition 4.5], there exists a relation subcoalgebra H of $T^{\square}(\mathcal{M})$ and a coalgebra isomorphism $C\cong H$. On the other hand, by Theorem 5.20 and Corollary 5.22, there exist a profinite relation ideal $\mathfrak{B}$ of $\widehat{T}(\mathcal{M})$ and a coalgebra isomorphism $H\cong T^{\square}(\mathcal{M},\mathfrak{B})$. Hence there exists a coalgebra isomorphism $C\cong T^{\square}(\mathcal{M},\mathfrak{B})$.

(b) Apply (a) and Theorem 5.20(d). ∎

The following corollary is an immediate consequence of Corollary 5.23.

COROLLARY 5.24. *If C is a basic K-coalgebra, K is a perfect field, and $\mathcal{M}={}_C\mathcal{E}xt^{\#}$ then there exist a profinite relation ideal $\mathfrak{B}$ of $\widehat{T}(\mathcal{M})$, a coal-*

gebra isomorphism $C \cong T^{\square}(\mathcal{M}, \mathfrak{B})$, *and* K*-linear equivalences of categories* $C\text{-comod} \cong \mathrm{nilrep}_K^{\ell f}(\mathcal{M}, \mathfrak{B})$ *and* $C\text{-Comod} \cong \mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M}, \mathfrak{B})$.

6. Species with bimodule relations. The aim of this section is to show that the study of K-species with bimodule relations (considered in [26, Section 3]) leads to an interesting class of K-coalgebras, including a class of $c\ell$-hereditary coalgebras and piecewise prime coalgebras in the sense of the following definition (see [16, Section 3], [22], [26, Section 3], [36]).

DEFINITION 6.1. Let K be an arbitrary field and C a K-coalgebra.

(a) A left C-comodule X is called *colocal* if X contains a unique simple subcomodule, or equivalently, X is isomorphic to a subcomodule of an indecomposable injective comodule.
(b) A K-coalgebra C is defined to be *$c\ell$-hereditary* if every colocal epimorphic image of an injective left C-comodule is injective (see [36]).
(c) C is defined to be a *piecewise prime* coalgebra if C satisfies the following two conditions:
   - $\mathrm{End}_C E$ is a division K-algebra, for any indecomposable injective left C-comodule E,
   - given a triple E, E', E'' of indecomposable injective left C-comodules, the equality $\mathrm{Hom}_C(E', E'') \circ \mathrm{Hom}_C(E, E') = 0$ implies $\mathrm{Hom}_C(E', E'') = 0$ or $\mathrm{Hom}_C(E, E') = 0$, where $\mathrm{Hom}_C(E', E'') \circ \mathrm{Hom}_C(E, E')$ is the image of the K-bilinear map

$$\circ : \mathrm{Hom}_C(E', E'') \otimes \mathrm{Hom}_C(E, E') \to \mathrm{Hom}_C(E, E''), \qquad f \otimes g \mapsto f \circ g.$$

Note that any hereditary coalgebra is $c\ell$-hereditary.

LEMMA 6.2. *Let K be an arbitrary field and C a K-coalgebra.*

(a) *The following three conditions are equivalent:*
   (a1) *C is $c\ell$-hereditary;*
   (a2) *every non-zero homomorphism $f : E \to E'$ between indecomposable injective left C-comodules E and E' is surjective;*
   (a3) *if E, E', E'' are indecomposable injective left C-comodules and $f \in \mathrm{Hom}_C(E', E'')$ and $g \in \mathrm{Hom}_C(E, E')$ are such that $f \circ g = 0$ then $f = 0$ or $g = 0$.*
(b) *The definitions of a $c\ell$-hereditary coalgebra and a piecewise prime coalgebra are left-right symmetric.*
(c) *If C is $c\ell$-hereditary then C is piecewise prime.*

Proof. The equivalence of (a1)–(a3) is fairly easy; the proof is left to the reader. Statement (c) follows from (a). For the proof of (b), we recall from

[6, Proposition 3.1(c)] that there is a K-duality

$$D : C\text{-inj} \to \text{inj-}C$$

between the categories of socle-finite injective left C-comodules and socle-finite injective right C-comodules. Hence, in view of (a), statement (b) easily follows. ∎

Following [26, Section 3], [16], and [17, Section 3], we introduce the following definition.

Definition 6.3. Let K be an arbitrary field and $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in I_{\mathcal{M}}}$ a K-species with ${}_jM_j = F_j$ for each $j \in I_{\mathcal{M}}$.

(a) A *bimodule relation on* $\mathcal{M}$ is defined to be a system

$$\mathbf{c} = (c_{ijr} : {}_iM_j \otimes {}_jM_r \to {}_iM_r)_{i,j,r\in I_{\mathcal{M}}} \tag{6.4}$$

of F_i-F_r-bimodule homomorphisms such that

- $c_{iir} : {}_iM_i \otimes {}_iM_r \to {}_iM_r$ and $c_{irr} : {}_iM_r \otimes {}_rM_r \to {}_iM_r$ are the canonical bimodule isomorphisms $F_i \otimes {}_iM_r \xrightarrow{\simeq} {}_iM_r$ and ${}_iM_r \otimes F_r \xrightarrow{\simeq} {}_iM_r$,
- $c_{irs} \circ (c_{ijr} \otimes 1) = c_{ijs} \circ (1 \otimes c_{jrs})$ for all $i, j, r, s \in I_{\mathcal{M}}$.

(b) A bimodule relation $\mathbf{c} = (c_{ijr})_{i,j,r\in I_{\mathcal{M}}}$ on $\mathcal{M}$ is defined to be *piecewise prime* if the equality $c_{ijr}({}_iM_j \otimes {}_jM_r) = 0$ holds if and only if ${}_iM_j = 0$ or ${}_jM_r = 0$.

(c) A bimodule relation $\mathbf{c} = (c_{ijr})_{i,j,r\in I_{\mathcal{M}}}$ on $\mathcal{M}$ is defined to be a *piecewise domain relation* if the equality $c_{ijr}(x \otimes y) = 0$ implies ${}_iM_j = 0$ or ${}_jM_r = 0$, for all $i, j, r \in I_{\mathcal{M}}$ and all $x \in {}_iM_j$, $y \in {}_jM_r$.

(d) A *K-species with a bimodule relation* is a pair $(\mathcal{M}, \mathbf{c})$, where $\mathcal{M}$ is a species and $\mathbf{c} = (c_{ijr})_{i,j,r\in I_{\mathcal{M}}}$ is a bimodule relation on $\mathcal{M}$.

(e) A K-linear representation $X = (X_a, \varphi^X_{ab})_{a,b\in I_{\mathcal{M}}}$ in $\mathrm{Rep}_K(\mathcal{M})$ is defined to *satisfy the relation* $\mathbf{c} = (c_{ijr})_{i,j,r\in I_{\mathcal{M}}}$ on $\mathcal{M}$ if $\varphi^X_{jr} \circ (\varphi^X_{ij} \otimes 1) = \varphi^X_{ir} \circ (1 \otimes c_{ijr})$ for all $i, j, r \in I_{\mathcal{M}}$.

We denote by $\mathrm{Rep}_K(\mathcal{M}, \mathbf{c})$ the full subcategory of $\mathrm{Rep}_K(\mathcal{M})$ whose objects are the representations of $\mathcal{M}$ that satisfy $\mathbf{c}$ (see [26, Section 3]).

Obviously, any piecewise domain bimodule relation $\mathbf{c}$ is piecewise prime.

Following [26, Section 3], we associate to any K-species $(\mathcal{M}, \mathbf{c})$ with a bimodule relation the K-algebra

$$T(\mathcal{M}, \mathbf{c}) = T(\mathcal{M})/\mathfrak{A}_{\mathbf{c}}, \tag{6.5}$$

where $\mathfrak{A}_{\mathbf{c}}$ is the two-sided ideal of the tensor K-algebra $T(\mathcal{M}) = T_F(M)$ generated by the elements $c_{ijr}(x \otimes y) - x \otimes y \in {}_iM_r \oplus {}_iM_j \otimes {}_jM_r \subseteq M \oplus M \otimes M \subseteq T_F(M)$ with $i, j, r, s \in I_{\mathcal{M}}$, $x \in {}_iM_j$, and $y \in {}_jM_r$. It is clear that the local units e_a of $T(\mathcal{M})$ with $a \in I_{\mathcal{M}}$ form a complete set of pairwise orthogonal primitive idempotents of $T(\mathcal{M}, \mathbf{c})$, and the following proposition holds.

PROPOSITION 6.6. *Let K be an arbitrary field, $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in I_\mathcal{M}}$ a K-species* (5.1) *with ${}_jM_j = F_j$ for each $j \in I_\mathcal{M}$, and $\mathbf{c}$ a bimodule relation on $\mathcal{M}$.*

(a) *There is a K-linear equivalence of categories* $\mathrm{Rep}_K(\mathcal{M}, \mathbf{c}) \cong \mathrm{Mod}\, T(\mathcal{M}, \mathbf{c})$.
(b) $\mathbf{c}$ *is a piecewise prime relation on $\mathcal{M}$ if and only if the K-algebra $T(\mathcal{M}, \mathbf{c})$ is piecewise prime, that is, given three pairwise different indices i, j, r in $I_\mathcal{M}$, the equality $e_iT(\mathcal{M}, \mathbf{c})e_jT(\mathcal{M}, \mathbf{c})e_r = 0$ holds if and only if $e_iT(\mathcal{M}, \mathbf{c})e_j = 0$ or $e_jT(\mathcal{M}, \mathbf{c})e_r = 0$ (see* [17, Section 3]*).*
(c) $\mathbf{c}$ *is a piecewise domain relation on $\mathcal{M}$ if and only if $T(\mathcal{M}, \mathbf{c})$ is ℓ-hereditary, that is, every local right ideal of $T(\mathcal{M}, \mathbf{c})$ is projective (see* [22]*).*

Proof. Statement (a) follows from [26, Lemma 3.1], and (c) is a consequence of [26, Proposition 3.1]. Statement (b) follows from the fact that $\mathrm{End}_{T(\mathcal{M},\mathbf{c})}(e_aT(\mathcal{M}, \mathbf{c})) \cong F_a$ and there is a bimodule isomorphism $\mathrm{Hom}_{T(\mathcal{M},\mathbf{c})}(e_bT(\mathcal{M}, \mathbf{c}), e_aT(\mathcal{M}, \mathbf{c})) \cong {}_aM_b$ for all $a, b \in I_\mathcal{M}$. The details are left to the reader. ∎

Now we associate to any $(\mathcal{M}, \mathbf{c})$ the cotensor K-coalgebra

$$T^\square(\mathcal{M}, \mathbf{c}) = T^\square(\mathcal{M}, \mathfrak{A}_\mathbf{c}) = \{x \in T^\square(\mathcal{M});\ \langle \mathfrak{A}_\mathbf{c}, x\rangle = 0\} \tag{6.7}$$

(see (5.19)). By applying the technique of Section 5, one can show that there are K-linear equivalences of categories

$$\begin{aligned} T^\square(\mathcal{M}, \mathbf{c})\text{-Comod} &\cong \mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M}, \mathbf{c}), \\ T^\square(\mathcal{M}, \mathbf{c})\text{-comod} &\cong \mathrm{nilrep}_K^{\ell f}(\mathcal{M}, \mathbf{c}). \end{aligned} \tag{6.8}$$

Now we give an equivalent description of the coalgebra $T^\square(\mathcal{M}, \mathbf{c})$. Let $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in I_\mathcal{M}}$ be a K-species with ${}_jM_j = F_j$ for each $j \in I_\mathcal{M}$. We denote by $\mathbb{M}_{I_\mathcal{M}}(\mathcal{M})$ the K-vector space of all square I by I matrices $\mathbf{m} = [m_{pq}]_{p,q\in I_\mathcal{M}}$ with $m_{pq} \in {}_pM_q$. Note that the diagonal entry $m_{pp} \in {}_pM_p = F_p$ is an element of the division algebra F_p.

Let $(\mathcal{M}, \mathbf{c})$ be a K-species with a bimodule relation. We define the *incidence K-algebra* of $(\mathcal{M}, \mathbf{c})$ to be the K-vector subspace

$$\begin{aligned} &K(\mathcal{M}, \mathbf{c}) \\ &\quad = \{\mathbf{m} = [m_{pq}] \in \mathbb{M}_{I_\mathcal{M}}(\mathcal{M});\ m_{pq} = 0 \text{ for almost all } p, q \in I_\mathcal{M}\} \end{aligned} \tag{6.9}$$

of $\mathbb{M}_{I_\mathcal{M}}(\mathcal{M})$ equipped with the matrix multiplication defined by the formula

$$\mathbf{m}' \cdot \mathbf{m}'' = \mathbf{m} = [m_{pq}], \quad \text{where} \quad m_{pq} = \sum_{j\in I_\mathcal{M}} c_{pjq}(m'_{pq} \otimes m''_{jq}).$$

Since $\mathbf{c}$ is a bimodule relation, $K(\mathcal{M},\mathbf{c})$ is an associative K-algebra with local units $e_p = e_{pp}$, where e_{pp} is the matrix with the identity element 1_{F_p} of the division algebra F_p in the (p,p)-entry, and zeros elsewhere. Note that $K(\mathcal{M},\mathbf{c})$ has an identity element if and only if $I_{\mathcal{M}}$ is a finite set.

PROPOSITION 6.10. *Let $\mathcal{M}$ be a K-species with a bimodule relation $\mathbf{c}$.*

(a) *There is an isomorphism $T(\mathcal{M},\mathbf{c}) \cong K(\mathcal{M},\mathbf{c})$ of K-algebras.*
(b) $\mathbf{c}$ *is a piecewise prime relation on $\mathcal{M}$ if and only if the K-algebra $K(\mathcal{M},\mathbf{c})$ is piecewise prime.*
(c) $\mathbf{c}$ *is a piecewise domain relation on $\mathcal{M}$ if and only if $K(\mathcal{M},\mathbf{c})$ is ℓ-hereditary.*

Proof. (a) In view of the definition of the ideal $\mathfrak{A}_{\mathbf{c}}$ of $T(\mathcal{M})$, any element

$$\begin{aligned} {}_a\overline{m}_b &= {}_am_{j_1} \otimes {}_{j_1}m_{j_2} \otimes \cdots \otimes {}_{j_s}m_b \in {}_aM_{j_1} \otimes {}_{j_1}M_{j_2} \otimes \cdots \otimes {}_{j_s}M_b \subseteq M^{\otimes^s} \\ &= \bigoplus_{a,b\in I_{\mathcal{M}}} {}_aM_b^{(s)} \end{aligned}$$

with $s \geq 2$ (see (5.3)) can be reduced (by applying the bimodule homomorphisms $c_{ijr} : {}_iM_j \otimes {}_jM_r \to {}_iM_r$ of (6.4)) to a unique element $\mathbf{c}({}_a\overline{m}_b)$ of the bimodule ${}_aM_b$ that is congruent to ${}_a\overline{m}_b$ modulo $\mathfrak{A}_{\mathbf{c}}$. It follows that any element $z \in T(\mathcal{M})$ can be reduced to a unique element

$$\mathbf{c}(z) \in F \oplus M = \bigoplus_{j\in I_{\mathcal{M}}} F_j \oplus \bigoplus_{a,b\in I_{\mathcal{M}}} {}_aM_b$$

that is congruent to z modulo $\mathfrak{A}_{\mathbf{c}}$. Obviously, $\mathbf{c}(z)$ can be viewed as a matrix $\mathbf{m}_z \in K(\mathcal{M},\mathbf{c})$. It is easy to see that the composite map $z \mapsto \mathbf{c}(z) \mapsto \mathbf{m}_z$ defines an isomorphism $T(\mathcal{M},\mathbf{c}) \cong K(\mathcal{M},\mathbf{c})$ of K-algebras.

Statements (b) and (c) follow from (a) and Proposition 6.6. ∎

It follows from Propositions 6.6 and 6.10 that there is a K-coalgebra structure (Δ,ε) on the incidence K-algebra $K(\mathcal{M},\mathbf{c})$ such that the coalgebra $K^{\Box}(\mathcal{M},\mathbf{c}) = (K(\mathcal{M},\mathbf{c}),\Delta,\varepsilon)$ (called the *incidence coalgebra of* $(\mathcal{M},\mathbf{c})$) is isomorphic to the cotensor coalgebra $T^{\Box}(\mathcal{M},\mathbf{c})$ and there are K-linear equivalences of categories

$$\begin{aligned} K^{\Box}(\mathcal{M},\mathbf{c})\text{-Comod} &\cong \mathrm{Rep}_K^{\ell n\ell f}(\mathcal{M},\mathbf{c}), \\ K^{\Box}(\mathcal{M},\mathbf{c})\text{-comod} &\cong \mathrm{nilrep}_K^{\ell f}(\mathcal{M},\mathbf{c}), \end{aligned} \tag{6.11}$$

induced by (6.8). The structure of the coalgebra $K^{\Box}(\mathcal{M},\mathbf{c})$ will be discussed in a subsequent paper. Here we describe it under the additional assumption that $(\mathcal{M},\mathbf{c})$ is *epi-special*, that is, the following four conditions are satisfied:

(c1) if ${}_aM_b \neq 0$ and ${}_bM_a \neq 0$ then $a = b$ and ${}_aM_a = F_a$,
(c2) $\mathbf{c} = (c_{ijr})_{i,j,r\in I_{\mathcal{M}}}$ is an epi-bimodule relation on $\mathcal{M}$, that is, each bimodule homomorphism c_{ijr} is surjective,

(c3) each non-zero F_a-F_b-bimodule ${}_aM_b$ is simple and is generated by an element ${}_a\xi_b \in {}_jM_r$ such that $c_{ijr}({}_i\xi_j \otimes {}_j\xi_r) = {}_i\xi_r$ for all $i, j, r \in I_{\mathcal{M}}$,

(c4) the valued quiver of $\mathcal{M}$ is intervally finite.

If $(\mathcal{M}, \mathbf{c})$ epi-special, we associate to $(\mathcal{M}, \mathbf{c})$ the *incidence K-coalgebra*

$$K^{\square}(\mathcal{M}, \mathbf{c}) = (K(\mathcal{M}, \mathbf{c}), \Delta, \varepsilon), \tag{6.12}$$

where $K(\mathcal{M}, \mathbf{c})$ is the incidence K-algebra of $(\mathcal{M}, \mathbf{c})$, endowed with the comultiplication $\Delta : K(\mathcal{M}, \mathbf{c}) \to K(\mathcal{M}, \mathbf{c}) \otimes K(\mathcal{M}, \mathbf{c})$ and the counit $\varepsilon : K(\mathcal{M}, \mathbf{c}) \to K$ defined as follows. Given a local unit $e_a = e_{aa} \in K(\mathcal{M}, \mathbf{c})$ at a, we set $\Delta(e_a) = e_a \otimes e_a$ and $\varepsilon(e_a) = 1$. Given the matrix ${}_a\xi_b e_{ab} \in K(\mathcal{M}, \mathbf{c})$, with ${}_a\xi_b$ in the (a, b)th entry and zeros elsewhere, we set

$$\begin{aligned} &\Delta({}_a\xi_b e_{ab}) = e_a \otimes {}_a\xi_b e_{ab} + {}_a\xi_b e_{ab} \otimes e_b + \sum_{r \in I_{\mathcal{M}}} {}_a\xi_r e_{ar} \otimes {}_r\xi_b e_{rb}, \\ &\varepsilon({}_a\xi_b e_{ab}) = 0. \end{aligned}$$

One can show that there is a coalgebra isomorphism $K^{\square}(\mathcal{M}, \mathbf{c}) \cong T^{\square}(\mathcal{M}, \mathbf{c})$. Following [16], to any such coalgebra we associate a valued poset $(I_{\mathcal{M}}, \preceq, \mathbf{d})$ as follows. We set $i \preceq j$ iff ${}_iM_j \neq 0$. In this case we write a valued dashed arrow

$$i \overset{(d'_{ij}, d''_{ij})}{\dashrightarrow} j,$$

where $d'_{ij} = \dim ({}_iM_j)_{F_j}$ and $d''_{ij} = \dim_{F_i}({}_iM_j)$. If there is no r such that $i \prec r \prec j$, the dashed arrow is replaced by a continuous one.

It is clear that the valued Hasse quiver of the valued poset $(I_{\mathcal{M}}, \preceq, \mathbf{d})$ is just the left valued Gabriel quiver $({}_HQ, {}_H\mathbf{d})$ of the coalgebra $H = K^{\square}(\mathcal{M}, \mathbf{c})$.

We illustrate the definition by an example.

Example 6.13. Let $\mathbb{R} \subset \mathbb{C}$ be the real and complex number fields, respectively. Consider the $\mathbb{R}$-species $\mathcal{M} = (F_i, {}_iM_j)_{i,j \in I_{\mathcal{M}}}$, where

- $I_{\mathcal{M}} = \mathbb{Z}$ is the set of integers,
- $F_a = \begin{cases} \mathbb{R} & \text{for } a = 0, \\ \mathbb{C} & \text{for } a \neq 0, \end{cases} \qquad {}_aM_b = \begin{cases} \mathbb{C} & \text{for } a < b, \\ 0 & \text{for } b < a. \end{cases}$

We define a bimodule relation $\mathbf{c} = (c_{iir} : {}_iM_i \otimes {}_iM_r \to {}_iM_r)_{i,j,r \in \mathbb{Z}}$ on $\mathcal{M}$:

- $c_{iir} : {}_iM_i \otimes {}_iM_r \to {}_iM_r$ and $c_{irr} : {}_iM_r \otimes {}_rM_r \to {}_iM_r$ are the canonical bimodule isomorphisms $F_i \otimes {}_iM_r \xrightarrow{\cong} {}_iM_r$ and ${}_iM_r \otimes F_r \xrightarrow{\cong} {}_iM_r$ for all $i, j, r \in \mathbb{Z}$,
- if $i < j < r$, we define $c_{ijr} : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ to be the multiplication $x \otimes y \mapsto x \cdot y$,
- if ${}_iM_j = 0$ or ${}_jM_r = 0$, we set $c_{ijr} = 0$.

Note that $\mathbf{c}$ is a piecewise domain bimodule relation and the incidence $\mathbb{R}$-coalgebra $\mathbb{R}^{\square}(\mathcal{M}, \mathbf{c})$ has the upper triangular matrix form

$$\mathbb{R}^{\square}(\mathcal{M},\mathbf{c}) = \begin{array}{cc}
\begin{array}{ccccccccccccc} & & & & -1 & 0 & & & & & & & \\ & & & & \downarrow & \downarrow & & & & & & & \end{array} & \\
\left[\begin{array}{ccccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\
\cdots & 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\
\cdots & 0 & 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\
\cdots & 0 & 0 & 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \mathbb{R} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{C} & \mathbb{C} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{C} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right] & \begin{array}{c} \\ \\ \leftarrow -3 \\ \leftarrow -2 \\ \leftarrow -1 \\ \leftarrow 0 \\ \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \\ \vdots \\ \\ \\ \\ \end{array}
\end{array}$$

and consists of all $\mathbb{Z}\times\mathbb{Z}$ matrices $\lambda = [\lambda_{pq}]$ with $\lambda_{pq} \in \mathbb{C}$ such that $\lambda_{pq} = 0$ for almost all $p, q \in \mathbb{Z}$. The multiplication in the incidence $\mathbb{R}$-algebra $\mathbb{R}(\mathcal{M},\mathbf{c})$ is the usual matrix multiplication. The comultiplication Δ and the counit ε in the incidence $\mathbb{R}$-coalgebra $\mathbb{R}^{\square}(\mathcal{M},\mathbf{c})$ are defined as in (6.12).

By (6.11), there are $\mathbb{R}$-linear equivalences of categories

$$\mathbb{R}^{\square}(\mathcal{M},\mathbf{c})\text{-Comod} \cong \operatorname{Rep}_K^{\ell n\ell f}(\mathcal{M},\mathbf{c}), \qquad \mathbb{R}^{\square}(\mathcal{M},\mathbf{c})\text{-comod} \cong \operatorname{nilrep}_K^{\ell f}(\mathcal{M},\mathbf{c}).$$

The left valued Gabriel quiver $({}_HQ, {}_H\mathbf{d})$ of the coalgebra $H = \mathbb{R}^{\square}(\mathcal{M},\mathbf{c})$ has the form

$$\cdots \longrightarrow -2 \longrightarrow -1 \xrightarrow{(2,1)} 0 \xrightarrow{(1,2)} 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots .$$

Note also that the partial order relation $\preceq$ in the valued poset $(I_{\mathcal{M}}, \preceq, \mathbf{d})$ associated to the $\mathbb{R}$-coalgebra $H = K^{\square}(\mathcal{M},\mathbf{c})$ is just the linear order of $\mathbb{Z}$, and we have

- $-i \xrightarrow{(2,1)} 0 \xrightarrow{(1,2)} j$ for all $i \geq 1$ and $j \geq 1$,
- $-i \xrightarrow{(1,1)} j$ for all $i \geq 1$ and $j \geq 1$.

Note that the valued Hasse quiver of the valued poset $(I_{\mathcal{M}}, \preceq, \mathbf{d})$ is just the left valued Gabriel quiver $({}_HQ, {}_H\mathbf{d})$.

One can check that the incidence $\mathbb{R}$-coalgebra $H = \mathbb{R}^{\square}(\mathcal{M},\mathbf{c})$ has the following properties:

(a) H is basic, indecomposable, $c\ell$-hereditary, and locally left (and right) artinian.

(b) $\operatorname{gl.dim} H = 2$.

(c) H is a left Euler coalgebra (in the sense of [33]) and the Euler quadratic form $q_H : \mathbb{Z}^{(\mathbb{Z})} \to \mathbb{Z}$ of H is positive definite.
(d) H is representation-directed in the sense of [33].
(e) The endomorphism $\mathbb{R}$-algebra $\mathrm{End}_H X$ is isomorphic to $\mathbb{R}$ or to $\mathbb{C}$, for any indecomposable finitely copresented H-comodule X.

PROBLEM 6.14. Give a characterisation of representation-directed incidence coalgebras $K^{\square}(\mathcal{M}, \mathbf{c})$ of K-species $\mathcal{M}$ with a piecewise prime bimodule relation $\mathbf{c}$.

To solve the problem, one can apply the technique developed in [33, Section 6], [35], [36], and the results of [17, Section 3] (see also [16]).

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